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Explicit solutions for a network control problem in the large deviation regime *

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Abstract

We consider optimal control in a stochastic network, where service is controlled to prevent buffer overflow. We use a risk-sensitive escape time criterion, which in comparison to the ordinary escape time criteria penalizes exits which occur on short time intervals more heavily. In [2] we showed that, for a large class of networks, the asymptotic cost agrees with the value of a differential game, played between a player that selects whether to serve and who to serve and one that perturbs the arrival and service rates. Moreover, the game's value is characterized as the unique solution to a Hamilton-Jacobi-Bellman Partial Differential Equation (PDE). In the current paper we treat in detail the general case of a network of queues in tandem. Our main result is the explicit solution to the PDE for such a network. A particular aspect of controlling a network so as to prevent buffer overflow manifests itself clearly in such networks. There is a natural competition between two tendencies, namely, to serve a queue, so as to prevent its buffer overflow, and to cease to serve it, so as to assist preventing overflow in the next buffer in line. The solution to the PDE indicates the optimal choice, specifying the parts of the state space where each queue must be served (so as not to loose optimality), or could idle. Referring to those queues which must be served as bottlenecks, one can use the PDE solution to explicitly calculate the bottleneck queues as a function of the system's state, in terms of a simple set of equations. In addition, we derive some identities relating the perturbed rates with the unperturbed ones.

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1 Introduction

In a previous work [2] we considered a stochastic control problem for a Markovian queueing network with deterministic routing, where the service stations may provide service to one or more queues, each queue being limited by a finite buffer. The control refers to the service discipline at the service stations, and the cost involves the time till one of the queues first reaches its buffer limit. Such a problem can be regarded as the control of a Markov process up to the time it exits a domain, the domain being the rectangle associated with the buffer sizes. The cost is chosen so as to obtain a risk-sensitive (or rare-event) control problem: one considers $E_x e^{-c\sigma}$ as a criterion to be minimized, where $c > 0$ is fixed and σ denotes the exit time (the time when any one of the buffers first overflows). Such a criterion penalizes short exit times more heavily than ordinary escape time criteria (such as $E_x\sigma$, a criterion to be maximized). While the main result of [2] is the characterization of the problem's asymptotics in terms of a limiting problem, the current work focuses on finding explicit solutions to this limit problem and on the interpretation thereof.

There are at least two motivations for the use of risk-sensitive criteria when designing policies for the control of a network. The first is that in many communication networks performance is measured in terms of the occurrence of rare but critical events. Buffer overflow is a principal example of such an event. The second motivation follows from the connection between risk-sensitive controls and robust controls. Indeed, as discussed in [7], the optimization of a single fixed stochastic network with respect to a risk-sensitive cost criteria automatically produces controls with specific and predictable robust properties. In particular, these controls give good performance for a family of perturbed network models (where the perturbation is around the design model and the size of the perturbation is measured by relative entropy), and with respect to a corresponding ordinary (i.e., not risk-sensitive) cost.

In many problems, one considers the limit of the risk-sensitive problem as a scaling parameter of the system converges, in the hope that the limit model is more tractable. This is the path we followed in [2], in the asymptotic regime where time is accelerated and buffer lengths are enlarged by a factor n. The limit of the cost was characterized as the value of a differential game on one hand, and as the solution to a PDE on the other hand. In this paper we concentrate on PDE methods. In particular, we treat in detail the general case of tandem queues, and solve the PDE explicitly. The solution turns out to have a simple structure and at the same time to provide insight regarding bottlenecks in the system.

In a system of queues in tandem, each server offers service to exactly one queue, and therefore the service control refers simply to whether to provide service or to idle, at each of the stations. Naturally, it is important to provide service to a queue in order to keep it from reaching the buffer limit. On the other hand, if the next buffer in line is nearly full, it might be necessary to idle the first queue in order to keep the second from overflowing. Although the optimal control problem for the Markovian queueing system is fully described by a dynamic programming equation [2], such equations are typically solved numerically, and it is hard to extract any global structural information therefrom. The solution to the PDE (representing the limiting problem) turns out to simplify things significantly. Roughly speaking, the PDE indicates the following structure of the asymptotic optimally controlled network. In the interior, and depending on the state of the system, service must be provided at certain service stations, while other stations may idle without causing loss of optimality. In that sense, the limit problem sharpens the control policy in emphasizing the importance of serving those 'bottleneck' buffers (it is crucial to serve the bottleneck buffers, and completely unimportant to serve the others). The interpretation of the bottlenecks is interesting, as it turns out that a queue with the smallest service rate is not necessarily a bottleneck. An explicit method to calculate the bottlenecks is derived. Another interesting phenomenon is that the fact that queues cannot become negative does not affect the solution structure.

There is relatively little work on risk-sensitive and robust control of networks. Ball et. al. have considered a robust formulation for network problems arising in vehicular traffic [4], where the cost structure is qualitatively different. Dupuis studies a robust control problem for networks in a deterministic setting and obtains explicit solutions for the value function [6]. The cost there is, in a sense, antipodal to the one considered in the current work, namely the time till the system becomes empty (a criterion to be minimized). For other recent work on queueing control in a large deviation regime see Stolyar and Ramanan [13], where a single server non Markovian system (with quite general stationary increments input flows) is studied, and a particularly simple scheduling control policy is shown to be asymptotically optimal (see also an extension of the work in Stolyar [12]).

The paper is organized as follows. Section 2 introduces the model and the PDE and states the main result. Section 3 contains discussion and interpretation. In Section 4 we prove the main result.

Notation: We use boldface to denote vectors. The symbol $\vee$ stands for maximum, while $\wedge$ stands for minimum. We denote scalar product between two vectors as $\boldsymbol{x} \cdot \boldsymbol{y}$.

2 Model and preliminaries

The queueing network control problem. We consider the following tandem network. There are J queues. Customers arrive to queue 1 according to a Poisson process of rate λ. Service times at queue i are exponential with parameter μ_i, mutually independent and independent of the arrival process. After a customer of class i is served, it turns into a customer of class $r(i) = i+1$, $i = 1, \dots, J-1$, where $i = 0$ is used to denote the "outside" and we set $r(J) = 0$. The state of the network is the vector of queue lengths, denoted by $\boldsymbol{X}$. We let $\{\boldsymbol{e}_j; j = 1, \dots, J\}$ denote the unit vectors and set $\boldsymbol{e}_0 = 0$ so that following service to class j the state changes by $\boldsymbol{e}_{r(j)} - \boldsymbol{e}_j$. The control is specified by the vector $\boldsymbol{u} = (u_1, ..., u_J)$, where $u_i = 1$ if customers in queue i are given service and $u_i = 0$ otherwise. We next consider the scaled process $\boldsymbol{X}^n$ under the scaling which accelerates time by a factor of n and shrinks space by the same factor. We are interested in a risk-sensitive cost functional that is associated with exit from a bounded set. Let G be the rectangle defined through

$$G = \{(x_1, \dots, x_J) : 0 \le x_1 < z_1;\ 0 \le x_j \le z_j,\, j = 2, \dots, J\}, \tag{1}$$

for some $z_i > 0$, $i = 1, \dots, J$. Note that G includes parts, but not all of its boundary. Let

$$\sigma^n \doteq \inf\{t : \boldsymbol{X}^n(t) \notin G\}.$$

The control problem is to minimize the cost $E_x e^{-nc\sigma^n}$, where E_x denotes expectation starting from $\boldsymbol{x}$, and $c > 0$ is a constant. With this cost structure "risk-sensitivity" means that atypically short exit times are weighted heavily by the cost. A "good" control will avoid such an event with high probability. Note that this criterion balances the probability of an event against its criticality: very short exit times, even if they occur with small probability, may have a noticeable effect on the cost. The significance from the point of view of stabilization of the system is clear.

The stochastic control problem is as follows. The state space of the scaled process is $G^n \doteq n^{-1}\mathbb{Z}_+^J \cap G$. The control (action) space is given by

$$U \doteq \{(u_i), i = 1, \ldots, J : 0 \le u_i \le 1,\ i = 1, \ldots, J\}.$$

The scaled queueing network is defined in the same way as the original queueing network, except that the size of each customer is $1/n$, and each rate (arrival, service) is multiplied by the scaling parameter n. The value function for the stochastic control problem is defined by

$$V^n(\boldsymbol{x}) \doteq -\inf n^{-1} \log E_x^{u,n} e^{-nc\sigma_n}, \quad \boldsymbol{x} \in G^n. \tag{2}$$

In this definition the infimum is over all (non anticipative) controls. For a formal definition of nonanticipative controls and their corresponding controlled processes see [2]. We are interested in the value and optimal control for n large, and our approach is to study the limit of the value as $n \to \infty$.

The domain and its boundary. It is possible for the controller to prevent any but the first queue from exceeding z_i, simply by turning off service to the preceding queue. However, the controller cannot prevent overflow of the first queue. Although it is in principle possible that the dynamics could exit through the portion of the boundary defined by queues $2, \ldots, J$, it is always optimal for the controller to not allow this. Consider the simple two-dimensional network illustrated in Figure 1. The controller can prevent exit through the dashed portion of the boundary simply by stopping service at the first queue. As a consequence, there are in general three different types of boundary–the constraining boundary due to non-negativity constraints on queue length, the part of the boundary where exit can be blocked, and the remainder. These three types of boundary behavior result in the PDE in three types of boundary conditions. We now define the three portions of the boundary. Let

$$\partial_c G = \{(x_1, \ldots, x_J) : 0 \le x_1 < z_1, \text{ and } x_j = z_j \text{ for some } j > 1\}$$

$$\partial_o G = \{(x_1, \ldots, x_J) : x_1 = z_1, \text{ and } 0 \le x_j \le z_j \text{ for all } j > 1\}$$

$$\partial_+ G = \{(x_1, \ldots, x_J) : x_i < z_i \text{ for all } i, \text{ and } x_j = 0 \text{ for some } j\}.$$

Note that $\partial_c G$, $\partial_o G$ and $\partial_+ G$ partition ∂G. Also, $\partial_c G \subset G$ while $\partial_o G \subset G^c$. As usual, we will denote $G^o = G \setminus \partial G$ and $\bar{G} = G \cup \partial G$.

The Hamiltonian, PDE and viscosity solutions. In [2] we show that the upper and lower values of a certain differential game are the unique Lipschitz viscosity solutions (defined below) of the PDE (9), that this game has a value and that the value agrees with the limit of the values of the controlled processes. To describe the PDE, we need the following definitions. Let $l : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ be defined as

$$l(x) \doteq \begin{cases} x \log x - x + 1 & x \ge 0, \\ +\infty & x < 0, \end{cases}$$

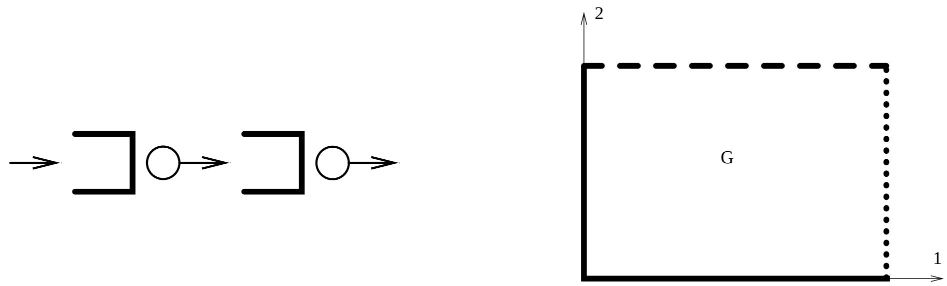

Figure 1: A simple queueing network, a rectangular domain and three types of boundary. Full line: $\partial_+ G$, dashed line: $\partial_c G$, and dotted line: $\partial_o G$

where $0 \log 0 \doteq 0$. Define

$$M = \{\boldsymbol{m} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_J, \bar{\mu}_1, \ldots, \bar{\mu}_J) : \bar{\lambda}_i \geq 0, \ \bar{\mu}_i \geq 0\}. \tag{3}$$

For $\boldsymbol{u} \in U$ and $\boldsymbol{m} \in M$ define

$$\boldsymbol{v}(\boldsymbol{u}, \boldsymbol{m}) \doteq \sum_{j=1}^{J} \bar{\lambda}_j \boldsymbol{v}_j + \sum_{i=1}^{J} u_i \bar{\mu}_i \tilde{\boldsymbol{v}}_i,$$

$$\rho(\boldsymbol{u}, \boldsymbol{m}) \doteq \sum_{i=1}^{J} \lambda_i l\left(\frac{\bar{\lambda}_i}{\lambda_i}\right) + \sum_{i=1}^{J} u_i \mu_i l\left(\frac{\bar{\mu}_i}{\mu_i}\right)$$

where $\boldsymbol{v}_j = \boldsymbol{e}_j$, and $\tilde{\boldsymbol{v}}_i = \boldsymbol{e}_{r(i)} - \boldsymbol{e}_i$. Let

$$H(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{m}) = \boldsymbol{q} \cdot \boldsymbol{v}(\boldsymbol{u}, \boldsymbol{m}) + \rho(\boldsymbol{u}, \boldsymbol{m}) + c, \tag{4}$$

$$H(\boldsymbol{q}, \boldsymbol{u}) = \sup_{\boldsymbol{m} \in M} H(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{m}), \tag{5}$$

and let the Hamiltonian be defined as

$$H(\boldsymbol{q}) = \inf_{\boldsymbol{u} \in U} H(\boldsymbol{q}, \boldsymbol{u}) = \inf_{\boldsymbol{u} \in U} \sup_{\boldsymbol{m} \in M} H(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{m}) = \sup_{\boldsymbol{m} \in M} \inf_{\boldsymbol{u} \in U} H(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{m}), \tag{6}$$

where the last identity, expressing the *Isaacs Condition* is proved in [2]. To state the PDE of interest we need some terminology. Note that the value (or asymptotic optimal cost) for this problem is not in general smooth. As we show below, the solution is piecewise linear, and as a consequence it is not differentiable at certain points. Moreover, it is clear that near the boundary $\partial_c G$ (where $x_i = z_i$ for some $i > 1$) the cost does not vanish, since exit can be prevented. However, just outside this boundary the cost is, by definition, 0. Therefore, we need a definition of a PDE which allows the solution to be nonsmooth. The right framework for such problems is that of viscosity solutions [5]: see references in [2].

We shall use the well known fact (see [5], Lemma II.1.7) that the definition of viscosity solutions in terms of smooth test functions (used in [2], Section 4) can equivalently be stated in terms of sub- and superdifferentials. Recall that for $\boldsymbol{x} \in G$, the set of superdifferentials $D^+V(\boldsymbol{x})$ and the set of subdifferentials $D^-V(\boldsymbol{x})$ are defined as

$$D^+V(\boldsymbol{x}) = \left\{\boldsymbol{p} : \limsup_{\boldsymbol{y} \to \boldsymbol{x}} \frac{V(\boldsymbol{y}) - V(\boldsymbol{x}) - \boldsymbol{p} \cdot (\boldsymbol{y} - \boldsymbol{x})}{|\boldsymbol{y} - \boldsymbol{x}|} \leq 0\right\}, \tag{7}$$

$$D^{-}V(\boldsymbol{x}) = \left\{ \boldsymbol{p} : \liminf_{\boldsymbol{y}\to\boldsymbol{x}} \frac{V(\boldsymbol{y}) - V(\boldsymbol{x}) - \boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{x})}{|\boldsymbol{y}-\boldsymbol{x}|} \geq 0 \right\}. \tag{8}$$

Define $I(\boldsymbol{x}) = \{i : x_i = 0\}$. The PDE of interest can be written formally as

$$\begin{cases} H(DV(\boldsymbol{x})) = 0, & \boldsymbol{x} \in G^o, \\ DV(\boldsymbol{x})\cdot\boldsymbol{\gamma}_i = 0, & i \in I(\boldsymbol{x}),\ \boldsymbol{x} \in \partial_+ G, \\ V(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \partial_o G, \end{cases} \tag{9}$$

where $\boldsymbol{\gamma}_i = -\tilde{\boldsymbol{v}}_i$: $\boldsymbol{\gamma}_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ for $i < J$, and $\boldsymbol{\gamma}_J = \boldsymbol{e}_J$.

Definition 1 *V is a viscosity solution to (9) if*

$$H(\boldsymbol{p}) \vee \max_{i\in I(\boldsymbol{x})} \boldsymbol{p}\cdot\boldsymbol{\gamma}_i \geq 0, \quad \boldsymbol{p} \in D^{+}V(\boldsymbol{x}),\ \boldsymbol{x} \in G, \tag{10}$$

$$H(\boldsymbol{p}) \wedge \min_{i\in I(\boldsymbol{x})} \boldsymbol{p}\cdot\boldsymbol{\gamma}_i \leq 0, \quad \boldsymbol{p} \in D^{-}V(\boldsymbol{x}),\ \boldsymbol{x} \in G \setminus \partial_c G, \tag{11}$$

and $V(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial_o G$.

The following result is proved in [2] (for more general networks and domains).

Theorem 1 *If $\boldsymbol{x}_n \in G^n$, $n \in \mathbb{N}$ are such that $\boldsymbol{x}_n \to \boldsymbol{x} \in G$, then $\lim_{n\to\infty} V^n(\boldsymbol{x}_n) = V(\boldsymbol{x})$, where V is the unique Lipschitz viscosity solution to (9).*

As an illustration, consider a single server that provides service to J classes (each customer requires service once). Service rate to queue j is exponential with rate μ_i. Assume that all arrival rates are positive: $\lambda_i > 0$ for all i and that arrivals are rates are mutually independent. For $i = 1, \ldots, J$, define α_i by

$$e^{\alpha_i} = \frac{(c+\lambda+\mu_i) + \sqrt{(c+\lambda+\mu_i)^2 - 4\lambda\mu_i}}{2\lambda}. \tag{12}$$

This α_i is the unique positive solution to

$$c + \lambda_i(1 - e^{\alpha_i}) + \mu_i(1 - e^{-\alpha_i}) = 0.$$

In this case, the PDE solution is provided in the following result from [2], for all large enough values of the parameter c.

Theorem 2 *If c is large enough then the viscosity solution to the PDE (9) is given as*

$$V(\boldsymbol{x}) = \min_{i=1,\ldots,J} \alpha_i(z_i - x_i). \tag{13}$$

Note that in this case, $V(\boldsymbol{x})$ is continuous, but not differentiable on certain parts of the boundary of G, as well as parts of the interior.

Recall now our queueing network, which consists of J queues in tandem, a server per queue. The parameters are $\lambda, \mu_1, \ldots, \mu_J > 0$. For $i = 1, \ldots, J$, let β_i denote the unique positive solution to

$$c + \lambda(1 - e^{\beta_i}) + \mu_i(1 - e^{-\beta_i}) = 0. \tag{14}$$

Set $\boldsymbol{\beta}^i = \beta_i \sum_{j=1}^{i} \boldsymbol{e}_j$. Our main result is the following.

Theorem 3 *Assume $c > 0$. Then the viscosity solution to the PDE (9) is given by*

$$V(\boldsymbol{x}) = \min_{i=1,\ldots,J} \boldsymbol{\beta}^i \cdot (\boldsymbol{z} - \boldsymbol{x}). \tag{15}$$

Our approach to the control problem is to guess a solution for the PDE, and verify that this is indeed the unique solution, thus obtaining the limiting value of the control problem through Theorem 1. While there is no general method to assist in guessing a solution, one may obtain some intuition from the related differential game. Indeed, the structure of this game [2] suggests piecewise linear solutions.

We conclude this section by observing a simplification in the structure of the Hamiltonian, that will be used in the sequel. Recall that the Hamiltonian is given by

$$H(\boldsymbol{p}) = \sup_u \inf_m H(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{m}),$$

where by the definitions

$$H(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{m}) = c + \sum_{i=1}^{J} \left[\bar{\lambda}_i p_i + \lambda_i l\left(\frac{\bar{\lambda}_i}{\lambda_i}\right) \right] + \sum_{i=1}^{J} u_i \left[-\bar{\mu}_i \boldsymbol{\gamma}_i \cdot \boldsymbol{p} + \mu_i l\left(\frac{\bar{\mu}_i}{\mu_i}\right) \right].$$

Using convexity and the fact that the slope of l at 0^+ is $-\infty$, the minimum over $\boldsymbol{m}$ is attained at $\bar{\lambda}_i = \lambda_i e^{-p_i}$, $\bar{\mu}_i = \mu_i e^{\boldsymbol{\gamma}_i \cdot \boldsymbol{p}}$. A straightforward calculation then shows that

$$\begin{aligned} H(\boldsymbol{p}, \boldsymbol{u}) \doteq \inf_m H(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{m}) &= c + \sum_{i=1}^{J} [(\lambda_i - \bar{\lambda}_i) + u_i(\mu_i - \bar{\mu}_i)] \\ &= c + \sum_{i=1}^{J} [\lambda_i(1 - e^{-p_i}) + u_i \mu_i (1 - e^{\boldsymbol{\gamma}_i \cdot \boldsymbol{p}})]. \end{aligned} \tag{16}$$

3 Remarks and discussion

3.1 Interpretation

The explicit solutions of the PDE have interpretations in the original queueing systems. In this subsection we do not attempt to prove statements rigorously, but only to discuss informally these

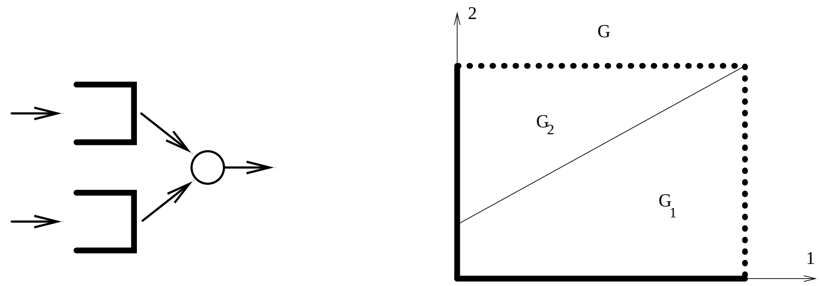

Figure 2: Priority to class i when the state is in G_i, $i = 1, 2$.

interpretations. These statements and their proof require among other things the notion of differential games which we do not include in this paper.

Consider first Theorem 2. Let $\boldsymbol{x}$ be a point in the interior of G where the gradient of V is well defined. Then the PDE is satisfied in the classical sense at this point. Thus, $H(DV(\boldsymbol{x})) = H(-\alpha_j \boldsymbol{e}_j) = 0$, where $j = j(\boldsymbol{x})$ depends on $\boldsymbol{x}$. In this case U is given as

$$\left\{ u : \sum_{i=1}^{J} u_i \leq 1, u_i \geq 0, i = 1, \ldots, J \right\}.$$

Using (16), the equation $H(-\alpha_i \boldsymbol{e}_j) = 0$ is therefore written as

$$\sup_{u \in U} \left[c + \lambda_j (1 - e^{\alpha_j}) + \sum_{i=1}^{J} u_i \mu_i (1 - e^{-\alpha_j}) 1_{i=j} \right] = 0$$

where the sum equals simply $u_j \mu_j (1 - e^{-\alpha_j})$. Clearly the supremum is attained at $u = 1_j$ (short for $u_i = 1_{i=j}$). Roughly speaking, this means that it is optimal to serve class $j(\boldsymbol{x})$ at the state $\boldsymbol{x}$. In the totally symmetric case, where $\mu_i = \mu$, $\lambda_i = \lambda$, $z_i = z$ for all i, the solution takes the form $V(\boldsymbol{x}) = \alpha \min_i (z - x_i)$. In this case, the optimal service discipline can be interpreted as "serve the longest queue." An asymmetric two dimensional example is given in Figure 2, where the domain G is divided into two subdomains G_1 and G_2 in accordance with the structure (13), and the optimal service discipline corresponds to giving priority to class i when the state is within G_i, $i = 1, 2$. This discipline gives priority *to the queue with the (weighted) shortest free buffer space.*

In the case of tandem queues, the solution to the PDE stated in Theorem 3 may be interpreted as follows. Since each queue is served by one server, the set U in this case has the form $U = [0, 1]^J$. Let again $\boldsymbol{x}$ be an interior point of G where the gradient is well defined. The form (15) implies that for some $j = j(\boldsymbol{x})$,

$$DV(\boldsymbol{x}) = -\boldsymbol{\beta}^j = -\beta_j \sum_{i=1}^{j} \boldsymbol{e}_i \,.$$

Using (16) the equation $H(-\boldsymbol{\beta}^j) = 0$ becomes

$$\sup_{u \in U} \left[c + \lambda (1 - e^{\beta_j}) + \sum_{i=1}^{J} u_i \mu_i (1 - e^{-\beta_j}) 1_{i=j} \right] = 0.$$

It is seen that the supremum is attained at any u for which $u_j = 1$ and $u_i \in [0,1]$, $i \neq j$. The interpretation in terms of the service policy is that it is important to serve class j, while optimality does not depend on the service given to other classes. *The station $j = j(x)$ can therefore be regarded as a bottleneck:* it is crucial to serve station j when at state x. This interpretation is valid at points where $V(\boldsymbol{x})$ is smooth, and in particular, does not hold on the boundary. Indeed, on the boundary service must be given to the full buffer, whether or not it is a bottleneck $(i = j)$. Since the PDE describes the limit of the stochastic control problem, the bottleneck stations have a similar property in the stochastic problem: Although it may be optimal for all servers to not idle when the system is near an interior point $\boldsymbol{x}$, it is significant for the bottleneck server to work while if the other servers idle the cost is affected only by little.

From the above discussion, j is a bottleneck at state x if $\beta_j(y_1 + \cdots + y_j) < \beta_i(y_1 + \cdots + y_i)$ for all $i \neq j$, where $\boldsymbol{y} = \boldsymbol{z} - \boldsymbol{x}$. This has the following familiar interpretation. Consider the draining time for each queue, obtained as follows. Consider buffer content as a fluid, and β_i as the draining rate of the fluid from buffer i. To obtain the draining time of buffer j, set $\beta_i = \beta_j$ for $i < j$ and turn off all arrivals. The draining time is then the time until queue j (and necessarily all preceding queues) become empty. Buffer j is then a bottleneck at state $\boldsymbol{x}$ if its draining time from state $\boldsymbol{x}$ is the largest.

This gives quite a simple partition of the space as regards to which station is a bottleneck. Some further analysis is carried out in Section 4, where it is seen that the set of all bottlenecks is contained in A' (see (18)). It should be emphasized that being a bottleneck is a function of the system's state. One interesting and perhaps counterintuitive phenomenon is that it is possible that a station is nearly empty while others are far from being empty, and still it is a bottleneck.

3.2 The perturbed rates equations

Recall that our criterion deals mainly with overflows that occur over a short interval of time. Such overflows are rare events, and for those to happen, the rates must seem to be "perturbed" by nature (or by the minimizing player—see [2]). The variables $\boldsymbol{m} \in M$ are the perturbed rates. Under the optimal policy for this player the perturbed rates satisfy some equations stated below. In the example of tandem queues we note that the perturbed rates satisfy the following two equations: for any $\boldsymbol{u}$, if $\boldsymbol{m}$ is optimal, that is, achieves the minimum in (16), then

1.
$$\bar{\lambda} \prod_{i=1}^{J} \bar{\mu}_i = \lambda \prod_{i=1}^{J} \mu_i,$$

or, equivalently,

$$\log \frac{\bar{\lambda}}{\lambda} + \sum_{i=1}^{J} \log \frac{\bar{\mu}_i}{\mu_i} = 0$$

and

2. If $\boldsymbol{u}$ is optimal then

$$\bar{\lambda} + \sum_{i=1}^{J} u_i \bar{\mu}_i = c + \lambda + \sum_{i=1}^{J} u_i \mu_i \,.$$

The first relation follows directly from (16) using the relation $\bar{\lambda} = \lambda e^{-p_1}$, $\bar{\mu}_i = \mu_i e^{\boldsymbol{\gamma}_i \cdot \boldsymbol{p}}$, as follows.

$$\begin{aligned} \bar{\lambda} \prod_{i=1}^{J} \bar{\mu}_i &= \lambda e^{-p_1} \prod_{i=1}^{J} \mu_i e^{\boldsymbol{\gamma}_i \cdot \boldsymbol{p}} \\ &= \left(\lambda \prod_{i=1}^{J} \mu_i \right) e^{-p_1 + \boldsymbol{p} \cdot \sum_{i=1}^{J} \boldsymbol{\gamma}_i} \\ &= \lambda \prod_{i=1}^{J} \mu_i \end{aligned}$$

since by definition

$$\sum_{i=1}^{J} \boldsymbol{\gamma}_i = \boldsymbol{e}_1 \,.$$

For the second relation observe that if $\boldsymbol{u}$ is optimal and the value is differentiable, then by definition of the PDE the Hamiltonian (16) equals zero, and the relation follows from the middle term in (16).

A generalization of these equations to a more general network is possible and will appear elsewhere, together with a formal proof.

The first relation appears to be a new observation. The second relation was noted by Avram [3], and can be regarded as an analogue of a well-known fact for networks without control, in the context of large deviations for exit problems. This appears as a consequence of time-reversal arguments in [11]. For example, the way an M/M/1 queue overflows is by reversing the arrival rate and service rate, that is

$$\bar{\lambda} = \mu \qquad \text{and } \bar{\mu} = \lambda.$$

This is obtained from our relations if we set $c = 0$. In the more general controlled case, our results hold under the optimal control.

Remark: Onno Boxma pointed out that the roots α_i and β_i are related to the busy cycle period of queues. For example, if B denotes the busy cycle period for an M/M/1 queue under the stationary distribution, then $Ee^{cB} = e^{\beta}$, where

$$c + \lambda(1 - e^{\beta}) + \mu(1 - e^{-\beta}) = 0$$

(compare with (14)). This interesting relation is yet to be explored.

4 Proof of the main result

For integers $i \leq j$, let $[i,j] \doteq \{1,\dots,j\}$. Recall that $I(\boldsymbol{x}) = \{i \in [1,J] : x_i = 0\}$, denote also $B(\boldsymbol{x}) = \{i \in [1,J] : x_i = z_i\}$, and note that $I(\boldsymbol{x})$ and $B(\boldsymbol{x})$ do not intersect. Since points $\boldsymbol{x}$ for which $x_1 = z_1$ are not in G, we have $1 \notin B(\boldsymbol{x})$ for all $\boldsymbol{x} \in G$. Writing the explicit form (12) of the solution to (14) as

$$e^{\beta_i} = \frac{(c+\lambda+\mu_i) + \sqrt{c^2+\lambda^2+\mu_i^2+2c(\lambda+\mu_i)}}{2\lambda},$$

it follows that the positive solutions β_i to (14) are monotone in μ_i, in the sense that

$$\mu_i < \mu_j \quad \Longleftrightarrow \quad \beta_i < \beta_j. \tag{17}$$

Next we show that the minimum over $i = 1,\dots,J$ in (15) can equivalently be performed over a smaller (state-dependent) set.

Lemma 1 *The minimum in* (15) *is obtained over the indices in the set* $A(\boldsymbol{x})$*, defined by* (18) *and* (23)*. That is,*

$$V(\boldsymbol{x}) = \min_{i=1,\dots,J} \boldsymbol{\beta}^i \cdot (\boldsymbol{z}-\boldsymbol{x}) = \min_{i \in A(\boldsymbol{x})} \boldsymbol{\beta}^i \cdot (\boldsymbol{z}-\boldsymbol{x}).$$

Proof: Fix $\boldsymbol{x} \in G$ and recall that $z_1 - x_1 > 0$. Let

$$A' = \{i \in [1,J] : \mu_i \leq \mu_j, \quad \text{for all } j < i\}. \tag{18}$$

By convention, $1 \in A'$. If

$$\boldsymbol{\beta}^i \cdot (\boldsymbol{z}-\boldsymbol{x}) \leq \min_{j \neq i} \boldsymbol{\beta}^j \cdot (\boldsymbol{z}-\boldsymbol{x}), \tag{19}$$

then $\beta_i \leq \beta_j$ for $j \in [1,i]$, and it follows from (17) that $i \in A'$. Thus the minimum in (15) equals

$$V(\boldsymbol{x}) = \min_{i \in A'} \boldsymbol{\beta}^i \cdot (\boldsymbol{z}-\boldsymbol{x})$$

for all $\boldsymbol{x} \in G$ (although not every $i \in A'$ is a minimizer).
Fix now $i \in A'$, and suppose that

$$\text{there is some } j > i \text{ so that } [i+1,j] \subset B(\boldsymbol{x}) \text{ and } \mu_i \geq \mu_j. \tag{20}$$

Since $i \in A'$, if $i+1 \notin A'$ then $\mu_{i+1} > \mu_i$. But then, if in addition $i+2 \notin A'$ we have $\mu_{i+2} > \mu_i$. So, if $k \notin A'$ for all $i+1 \leq k \leq j-1$ then $\mu_k > \mu_i$, and together with $\mu_j \leq \mu_i$ we have that $j \in A'$. We conclude that under (20),

$$\text{there is } k \in [i+1,j] \text{ such that } k \in A'. \tag{21}$$

By (20), $\beta_i \geq \beta_j$ and so

$$\boldsymbol{\beta}^i \cdot (\boldsymbol{z} - \boldsymbol{x}) \geq \boldsymbol{\beta}^j \cdot (\boldsymbol{z} - \boldsymbol{x}). \tag{22}$$

Therefore, for k as in (21) and under the assumptions in (20),

$$\boldsymbol{\beta}^i \cdot (\boldsymbol{z} - \boldsymbol{x}) \geq \boldsymbol{\beta}^k \cdot (\boldsymbol{z} - \boldsymbol{x}),$$

and since $k \in A'$, we need not consider i in the minimum. As a result, (15) is equivalently given as

$$V(\boldsymbol{x}) = \min_{i \in A(\boldsymbol{x})} \boldsymbol{\beta}^i \cdot (\boldsymbol{z} - \boldsymbol{x}), \quad \boldsymbol{x} \in G,$$

where $A(\boldsymbol{x})$ is obtained from $A'(\boldsymbol{x})$ by deleting those $i \in A'(\boldsymbol{x})$ which are followed by empty queues, $[i+1, j] \subset B(\boldsymbol{x})$, and such that $\mu_i \geq \mu_j$. This leaves in $A(\boldsymbol{x})$ an index i from $A'(\boldsymbol{x})$ only if either $z_{i+1} > x_{i+1}$, or $z_k = x_k$ for $i+1 < k \leq j$ implies $\mu_i < \mu_j$. More formally,

$$\begin{aligned} A(\boldsymbol{x}) &= A' \setminus \{i \in A' : \exists\, j > i,\ [i+1, j] \subset B(\boldsymbol{x}),\ \mu_i \geq \mu_j\} \\ &= \{i \in A' : \text{ either } i+1 \notin B(\boldsymbol{x}); \text{ or } j > i \text{ and } [i+1, j] \subset B(\boldsymbol{x}) \text{ imply } \mu_i < \mu_j\}. \end{aligned} \tag{23}$$

■

The form of the proposed solution is the minimum of smooth functions. The set of superdifferentials at a point where the function is not smooth can be seen (using the definition (7)) to consist of the convex hull of the gradients of the smooth functions defining it, at that point. Also, at the boundary, the fact that there are less constraints introduced by (7) on $p \in D^+V$ than there are when x is in the interior, has an effect of enlarging the set further. Thus, for example, the set of superdifferentials of the zero function from $\mathbb{R}_+$ to $\mathbb{R}$ at zero is $\mathbb{R}_+$. Using these considerations and the rectangular structure of the domain, one finds the general form for the superdifferential as follows. Let $\boldsymbol{x} \in G$ be fixed. Set $A = A(\boldsymbol{x})$, $I = I(\boldsymbol{x})$, $B = B(\boldsymbol{x})$, and $O = \{1, \ldots, J\} \setminus (I \cup B)$. Then any element $\boldsymbol{p} \in D^+V(\boldsymbol{x})$ is given as

$$\boldsymbol{p} = -\sum_{i \in A} \nu_i \boldsymbol{\beta}^i + \boldsymbol{\delta},$$

for some $\{\nu_i\}$, where

$$\delta_i \geq 0,\ i \in I, \quad \delta_i \leq 0,\ i \in B, \quad \delta_i = 0,\ i \in O, \tag{24}$$

and

$$\nu_i \geq 0,\ i \in A, \quad \nu_i = 0,\ i \notin A, \quad \sum_{i \in A} \nu_i = 1. \tag{25}$$

If we denote

$$\delta_{J+1} = 0, \tag{26}$$

then, using (16), the Hamiltonian is expressed as

$$H(\boldsymbol{p}) = c + \lambda(1 - e^{-p_1}) + \sum_{i=1}^{J} 0 \vee \mu_i(1 - e^{\boldsymbol{p}\cdot\boldsymbol{\gamma}_i})$$
$$= c + \lambda(1 - e^{\sum_{i\in A}\nu_i\beta_i - \delta_1}) + \sum_{i=1}^{J} 0 \vee \mu_i(1 - e^{-\nu_i\beta_i+\delta_i-\delta_{i+1}}). \tag{27}$$

Proof of Theorem 3:
Verifying the PDE for superdifferentials. To verify (10), it suffices to show that $H(\boldsymbol{p}) \geq 0$ whenever $\boldsymbol{p} \in DV^+(\boldsymbol{x})$ and $\boldsymbol{p}\cdot\boldsymbol{\gamma}_i < 0$ for all $i \in I$.

Step 1. We show first that $H(\boldsymbol{p}) \geq 0$ whenever $\boldsymbol{p} \in DV^+(\boldsymbol{x})$ and $\boldsymbol{p}\cdot\boldsymbol{\gamma}_i \leq 0$ for all $i = 1,\dots,J$. The forms of $\boldsymbol{\beta}^i$ and $\boldsymbol{\gamma}_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ imply the last inequality can be rewritten

$$-\nu_i\beta_i + \delta_i - \delta_{i+1} \leq 0, \; i = 1,\dots,J. \tag{28}$$

In this case, (27) becomes

$$H(\boldsymbol{p}) = h(\boldsymbol{\nu},\boldsymbol{\delta}) \doteq c + \lambda(1 - e^{\sum_{i\in A}\nu_i\beta_i - \delta_1}) + \sum_{i=1}^{J} \mu_i(1 - e^{-\nu_i\beta_i+\delta_i-\delta_{i+1}}). \tag{29}$$

The constraints we have put on $(\boldsymbol{\nu},\boldsymbol{\delta})$ define a convex set S, namely, the set of $(\boldsymbol{\nu},\boldsymbol{\delta}) \in \mathbb{R}^J \times \mathbb{R}^{J+1}$ satisfying (24), (25), (26) and (28). The set S is bounded for the following reasons. First, since $\boldsymbol{\nu}$ is a probability vector (see (25)) we have $\nu_i \in [0,1]$. Next, by (24) and $1 \notin B$, $\delta_1 \geq 0$. Finally, it follows from (28) (using the convention $\sum_s^t = 0$ if $t < s$) that

$$\delta_1 - \sum_{j=1}^{i-1}\nu_j\beta_j \leq \delta_i \leq \sum_{j=i}^{J}\nu_j\beta_j, \quad i = 1,\dots,J.$$

This shows the boundedness.

The function $(\boldsymbol{\nu},\boldsymbol{\delta}) \mapsto h(\boldsymbol{\nu},\boldsymbol{\delta})$ is concave, and therefore to prove $h \geq 0$ on S it suffices to check the inequality on the set E of extreme points of S, which contains a finite number of points since the constraints are linear.

Lemma 2 *All points $(\boldsymbol{\nu},\boldsymbol{\delta}) \in E$ have the following form:*

$$\nu_k = 1 \quad \textit{for some } k, \textit{ and}$$
$$\delta_i^{(r,k)} = \beta_k(1_{i\geq r+1} - 1_{i\geq k+1}), \quad i = 1,\dots,J, \; r = s,\dots,t-1,$$

where t, s are defined below.

Proof: We will first obtain the general form of $\boldsymbol{\nu}$ and then that of $\boldsymbol{\delta}$.

We claim that for any $(\boldsymbol{\nu}, \boldsymbol{\delta}) \in E$, $\boldsymbol{\nu}$ is of the form $\boldsymbol{\nu} = 1_k$ (short for $\nu_i = 1_{i=k}$, $i = 1, \ldots, J$), for some $k \in A$. Assume this is false. Then there are $l, m \in A$, $l < m$, for which $\nu_l, \nu_m \in (0,1)$. We will show that there is a vector $\boldsymbol{\Delta}_m$, such that replacing $(\boldsymbol{\nu}, \boldsymbol{\delta})$ by $(\boldsymbol{\nu}', \boldsymbol{\delta}') = (\boldsymbol{\nu} - \epsilon \boldsymbol{e}_m, \boldsymbol{\delta} + \epsilon \boldsymbol{\Delta}_m)$ maintains the relations (24), (26) and (28) for both $\epsilon > 0$ and $\epsilon < 0$ (provided $|\epsilon|$ is small). A similar statement will hold also for $(\boldsymbol{\nu}'', \boldsymbol{\delta}'') = (\boldsymbol{\nu} + \epsilon \boldsymbol{e}_l - \epsilon \boldsymbol{e}_m, \boldsymbol{\delta} - \epsilon \boldsymbol{\Delta}_l + \epsilon \boldsymbol{\Delta}_m)$, and as a result all of (24), (25), (26) and (28) will hold for both $\epsilon > 0$ and $\epsilon < 0$, a contradiction to $(\boldsymbol{\nu}, \boldsymbol{\delta})$ being an extreme point of S. The construction of $\boldsymbol{\Delta}_m$ based on $(\boldsymbol{\nu}, \boldsymbol{\delta})$ can be mimicked to construct $\boldsymbol{\Delta}_l$ based on $(\boldsymbol{\nu}', \boldsymbol{\delta}')$ (in particular, no use is made of the fact that ν_i sum to one, but only that some of its components are within $(0,1)$), and therefore the latter construction is omitted.

Case 1: *Inequality* (28) *holds as a strict inequality for* $i = m$, *i.e.*, $-\nu_m \beta_m + \delta_m - \delta_{m+1} < 0$. Then $-(\nu_m - \epsilon)\beta_m + \delta_m - \delta_{m+1} \le 0$ also holds (for both ϵ positive and negative), provided that $|\epsilon|$ is small. Here we take $\boldsymbol{\Delta}_m = 0$. We see that $(\boldsymbol{\nu} - \epsilon \boldsymbol{e}_m, \boldsymbol{\delta})$ satisfies the requirements.

Case 2: *Inequality* (28) *holds with equality for* $i = m$, *i.e.*, $-\nu_m \beta_m + \delta_m - \delta_{m+1} = 0$. Since $\nu_m \beta_m > 0$, either $\delta_m > 0$ or $\delta_{m+1} < 0$. Assume $\delta_m > 0$ (the case $\delta_{m+1} < 0$ can be treated analogously, and is therefore omitted). Let m' be the smallest $j \in [1, m]$ for which (28) holds with equality for all $i \in [j, m]$. For $i \in [m', m-1]$ (the set being empty and the statement void if $m' = m$) we have $-\nu_i \beta_i + \delta_i - \delta_{i+1} = 0$, hence if $\delta_{i+1} > 0$ then $\delta_i > 0$. Together with the fact that $\delta_m > 0$, this shows that

$$\delta_i > 0, \; i \in [m', m]. \tag{30}$$

Moreover,

$$-\nu_{m'-1}\beta_{m'-1} + \delta_{m'-1} - \delta_{m'} < 0 \tag{31}$$

(the statement being void in case that $m' = 1$). Set $\boldsymbol{\Delta}_m = -\beta_m \sum_{i=m'}^{m} \boldsymbol{e}_i$, $\boldsymbol{\nu}' = \boldsymbol{\nu} + \epsilon \boldsymbol{e}_m$, and $\boldsymbol{\delta}' = \boldsymbol{\delta} + \epsilon \boldsymbol{\Delta}_m$. The perturbation $\epsilon \boldsymbol{\Delta}_m$ is chosen to cancel the change in $-\nu_m \beta_m$, and to preserve the left hand side of (28) for all $i > m' - 1$. However, by (31) the inequality is maintained for $i = m' - 1$ as well, by taking $|\epsilon|$ sufficiently small. Similarly, by (30), (24) and (26) are also maintained on taking $|\epsilon|$ small.

This completes the construction of $\boldsymbol{\Delta}_m$. As described above, this leads to a contradiction, and we conclude that any extreme point $(\boldsymbol{\nu}, \boldsymbol{\delta}) \in E$ satisfies $\boldsymbol{\nu} = 1_k$ for some $k \in A$.

When $\boldsymbol{\nu} = 1_k$ for some $k \in A$, inequalities (28) can be rewritten as

$$\delta_i \le \delta_{i+1}, \; i \ne k, \quad \delta_k \le \beta_k + \delta_{k+1}. \tag{28$'$}$$

In particular, $0 \le \delta_1 \le \cdots \le \delta_k$. Let s denote the largest $j \le k$ for which $j \in B \cup O$ (and $s = 0$ if there is no such j). The definitions of O and B imply $\delta_s = 0$, and thus

$$\delta_j = 0 \text{ for } 1 \le j \le s. \tag{32a}$$

Similarly, $\delta_{k+1} \le \cdots \le \delta_J \le 0$, and if t denotes the least $j \in [k+1, J]$ for which $j \in I \cup O$ (and $t = J + 1$ if empty), then

$$\delta_j = 0, \quad t \le j \le J. \tag{32b}$$

Hence $\boldsymbol{\delta}$ must satisfy

$$\begin{cases} 0 \le \delta_{s+1} \le \cdots \le \delta_k \le \beta_k + \delta_{k+1}, & \text{when } s \le k-1, \\ 0 \le \delta_k & \text{when } s = k. \end{cases} \tag{32c}$$

In addition,

$$\begin{cases} \delta_{k+1} \le \cdots \le \delta_{t-1} \le 0, & \text{when } t \ge k+2 \\ \delta_{k+1} \le 0 & \text{when } t = k+1. \end{cases} \tag{32d}$$

We have just shown that (24), (26) and (28') imply (32a–d). On the other hand, clearly (32a–d) implies (28'). Moreover, as follows directly from the definition of s and t,

$$i \in I \text{ for all } i \in [s+1, k] \text{ and } i \in B \text{ for all } i \in [k+1, t-1]. \tag{33}$$

This shows that (32a–d) implies (24) and (26). Thus (24), (26) and (28') are equivalent to (32a–d). Now, the set of $\boldsymbol{\delta}$ satisfying the constraints (32a–d) is easy to analyze. In particular, it is not hard to see that it has the following $t-s$ extreme points, indexed by k and $r \in [s, \ldots, t-1]$, namely

$$\delta_i^{(r,k)} = \beta_k (1_{i \ge r+1} - 1_{i \ge k+1}), \quad i = 1, \ldots, J, \; r = s, \ldots, t-1. \tag{34}$$

■

We now calculate h (cf. (29)) at each extreme point. To this end, note that if $(\boldsymbol{\nu}, \boldsymbol{\delta}) = (1_k, \boldsymbol{\delta}^{(r,k)})$ then by (34)

$$-\nu_i \beta_i + \delta_i - \delta_{i+1} = \begin{cases} -\beta_k & i = r, \\ 0 & i \ne r, \end{cases} \qquad i = 1, \ldots, J,$$

and

$$\delta_1 = \begin{cases} \beta_k & r = 0, \\ 0 & r > 0. \end{cases}$$

Substituting in (29), we have the following possibilities. If $r = 0$ then $\delta_i^{(r,k)} = \beta_k 1_{i \in [1,k]}$ thus $\sum \nu_i \beta_i - \delta_1 = 0$, and $-\nu_i \beta_i + \delta_i - \delta_{i+1} = 0$, $i = 1, \ldots, J$. Hence $h(1_k, \boldsymbol{\delta}^{(r,k)}) = c > 0$. Otherwise,

$$h(1_k, \boldsymbol{\delta}^{(r,k)}) = c + \lambda(1 - e^{\beta_k}) + \mu_r (1 - e^{-\beta_k}). \tag{35}$$

In case that $r = k$, the right hand side of (35) vanishes owing to the definition of β_k (see (14)). In case that $r < k$, recall that $k \in A$, and in particular, $k \in A'$. By (18) we therefore have $\mu_k \le \mu_r$, hence by (14), $h(1_k, \boldsymbol{\delta}^{(r,k)}) \ge 0$. Finally, consider the case where $k < r \le t-1$. By (33), $[k+1, r] \subset B$. Since $k \in A$, (23) implies that $\mu_k < \mu_r$, and we again conclude that $h(1_k, \boldsymbol{\delta}^{(r,k)}) > 0$. Having shown that $h(\boldsymbol{\nu}, \boldsymbol{\delta}) \ge 0$ for all extreme points of the set S, we conclude that the inequality holds for all $(\boldsymbol{\nu}, \boldsymbol{\delta}) \in S$.

Step 2. We now relax the condition (28). Let then $(\boldsymbol{\nu}, \boldsymbol{\delta})$ satisfy (24), (25), (26), and assume that the inequality in (28) holds for all $i \in I$ (relaxing the assumption made in Step 1, that it holds for all $1 \le i \le J$). Let $P = P(\boldsymbol{\nu}, \boldsymbol{\delta})$ denote the set of i such that

$$-\nu_i \beta_i + \delta_i - \delta_{i+1} > 0. \tag{36}$$

If P is empty then the results of Step 1 apply and $H(\boldsymbol{p}) \geq 0$. Hence assume P is not empty. Let $j = j(\boldsymbol{\nu}, \boldsymbol{\delta})$ be the least element in P.

Note that $J \notin P$. For if $J \in P$ then using (36) and (26) we find $\delta_J > 0$, and therefore $J \in I$. However, we get to assume (28) for all $i \in I$, which means that $-\nu_J\beta_J + \delta_J - \delta_{J+1} \leq 0$. This contradicts $J \in P$.

We proceed by backward induction on the value of $j(\boldsymbol{\nu}, \boldsymbol{\delta})$. Note that the sets B and I depend on x. The argument below treats simultaneously all $x \in G$ by considering all possible sets B and I.

Induction step. Assumption: For all $(I, B, \boldsymbol{\nu}, \boldsymbol{\delta})$ satisfying (24), (25), (26), such that $P(\boldsymbol{\nu}, \boldsymbol{\delta}) \cap I = \emptyset$ and $j(\boldsymbol{\nu}, \boldsymbol{\delta}) \in [i+1, i+2, \ldots, J-1]$, one has $H(\boldsymbol{p}) \geq 0$. Let $(I, B, \boldsymbol{\nu}, \boldsymbol{\delta})$ be such that $P(\boldsymbol{\nu}, \boldsymbol{\delta}) \cap I = \emptyset$ and $j(\boldsymbol{\nu}, \boldsymbol{\delta}) = i$ $(i \geq 1)$. Then $-\nu_i\beta_i + \delta_i - \delta_{i+1} > 0$. Modify δ_{i+1} by increasing it so as to get equality i.e., set $\delta'_{i+1} > \delta_{i+1}$ so that $-\nu_i\beta_i + \delta_i - \delta'_{i+1} = 0$. This modification does not change the value of $0 \vee \mu_i(1 - e^{-\nu_i\beta_i + \delta_i - \delta_{i+1}})$ (but keeps it zero), and it can only decrease (or leave unchanged) the value of $0 \vee \mu_{i+1}(1 - e^{-\nu_{i+1}\beta_{i+1} + \delta_{i+1} - \delta_{i+2}})$. Hence by (27), the value of $H(\boldsymbol{p})$ is only decreased. At the same time, $j(\boldsymbol{\nu}, \boldsymbol{\delta}') > j(\boldsymbol{\nu}, \boldsymbol{\delta})$. Note that in modifying $\boldsymbol{\delta}$, (24) need not hold for B and I. However, clearly there are other sets, B' and I' with which it holds. For example,

$$B' = \{i \in [1, J] : \delta'_i < 0\}, \quad I' = \{i \in [1, J] : \delta'_i \geq 0\}, \quad O' = \emptyset. \tag{37}$$

Moreover, $1 \notin B'$, since we had $1 \notin B$ and δ_1 was not modified. By the induction assumption we therefore obtain that $H(\boldsymbol{p}) \geq 0$.

Induction base. We show that for all $(I, B, \boldsymbol{\nu}, \boldsymbol{\delta})$ satisfying (24), (25), (26), such that $P(\boldsymbol{\nu}, \boldsymbol{\delta}) \cap I = \emptyset$ and $j(\boldsymbol{\nu}, \boldsymbol{\delta}) = J - 1$, one has $H(\boldsymbol{p}) \geq 0$. We have $-\nu_{J-1}\beta_{J-1} + \delta_{J-1} - \delta_J > 0$. Similar to before, we set $\delta'_J > \delta_J$ so that $-\nu_{J-1}\beta_{J-1} + \delta_{J-1} - \delta'_J = 0$. We claim that $P(\boldsymbol{\nu}, \boldsymbol{\delta}') = \emptyset$. Indeed, since $P(\boldsymbol{\nu}, \boldsymbol{\delta}) = \{J - 1\}$, clearly $P(\boldsymbol{\nu}, \boldsymbol{\delta}') \subset \{J\}$. However,

$$\begin{aligned} -\nu_J\beta_J + \delta'_J &= -\nu_J\beta_J - \nu_{J-1}\beta_{J-1} + \delta_{J-1} \\ &\leq 0, \end{aligned}$$

where we have used the fact that $J - 1 \in P(\boldsymbol{\nu}, \boldsymbol{\delta})$ implies $J - 1 \notin I$, and hence by (24) $\delta_{J-1} \leq 0$. This shows that $P(\boldsymbol{\nu}, \boldsymbol{\delta}') = \emptyset$. As in the previous paragraph, because of the modification of $\boldsymbol{\delta}$, (24) need not hold for the I and B we started with, but there are other sets I' and B' (defined e.g. as in (37)) with which it holds. The results of Step 1 therefore apply, and therefore $H(\boldsymbol{p}) \geq 0$.

This completes the argument by induction and establishes (10) for superdifferentials.

Verifying the PDE for subdifferentials. We are required to show that (11) holds for $\boldsymbol{p} \in D^-V(\boldsymbol{x})$, where $\boldsymbol{x} \in G \setminus \partial_c G$ (and hence $B(\boldsymbol{x}) = \emptyset$). Unless $D^-V(\boldsymbol{x})$ is empty, the general form of $\boldsymbol{p} \in D^-V(\boldsymbol{x})$, $\boldsymbol{x} \in G$ is

$$\boldsymbol{p} = -\boldsymbol{\beta}^k + \boldsymbol{\delta},$$

where $k \in A$, and

$$\delta_i \leq 0, \ i \in I, \quad \delta_i = 0, \ i \notin I. \tag{38}$$

Here we have used the fact that V is the minimum of smooth functions, and therefore away from the boundary $D^-V(\boldsymbol{x})$ can have at most one element, equal to the gradient of any minimizing function. It suffices to show that $H(\boldsymbol{p}) \le 0$ whenever $\boldsymbol{p} \cdot \boldsymbol{\gamma}_i = -1_{i=k}\beta_k + \delta_i - \delta_{i+1} > 0$ for all $i \in I$. Using (27), (38) and (14), we find

$$
\begin{aligned}
H(\boldsymbol{p}) &= c + \lambda(1 - e^{\beta_k - \delta_1}) + \sum_{i=1}^{J} 0 \vee \mu_i(1 - e^{-1_{i=k}\beta_k + \delta_i - \delta_{i+1}}) \\
&= c + \lambda(1 - e^{\beta_k - \delta_1}) + \sum_{i \notin I} \mu_i(1 - e^{-1_{i=k}\beta_k - \delta_{i+1}}) \\
&\le c + \lambda(1 - e^{\beta_k}) + \sum_{i \notin I} \mu_i(1 - e^{-1_{i=k}\beta_k}) \\
&\le c + \lambda(1 - e^{\beta_k}) + \mu_k(1 - e^{-\beta_k}) \\
&= 0.
\end{aligned}
$$

■

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