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Game of Coins

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Abstract

We formalize the current practice of strategic mining in multi-cryptocurrency markets as a game, and prove that any better-response learning in such games converges to equilibrium. We then offer a reward design scheme that moves the system configuration from any initial equilibrium to a desired one for any better-response learning of the miners. Our work introduces the first multi-coin strategic attack for adaptive and learning miners, as well as the study of reward design in a multi-agent system of learning agents.

1 Introduction

Cryptocurrencies are an arms race. Hundreds of digital coins have crept into the worldwide market in the last decade [6], including more than a dozen with over a billion dollar Market Cap, e.g., [11, 1, 8, 3, 9]. The vast majority of cryptocurrencies are based on the notion of proof of work (PoW) [25]. As a result, the major strategic players in the context of cryptocurrencies are *miners* who devote their power to solving computational puzzles to find PoWs [25, 11].

The miners for a particular coin usually gain rewards that are proportional to the power they invest in the coin out of the total invested power (in the coin) by all miners. Each coin, therefore, can be viewed as having some *weight* that reflects the reward it divides among its miners. In practice, a coin's weight (or reward) depends on its transaction rate, transaction fees, and its fiat exchange rate.

While the above description is not complete, it does capture the fundamental decision faced by the miner: where should I mine? One indication for reward-based coin switching can be found online in websites like www.whattomine.com [10], where miners enter their mining parameters (technology, power, cost, et cetra) and get a list of coins they can mine for, ordered by their profitability. Another interesting example happened on November 12 (2017) [5], when a dramatic change in the Bitcoin to Bitcoin Cash [1] (a spin-off from Bitcoin) exchange rate led to a major inrush of miners from Bitcoin to Bitcoin Cash (see Figure 1).

All in all, the structure of the cryptocurrency market suggests that we face here a game among miners, where each miner wishes to mine coins of heavy weights while avoiding competition with other miners. In this paper we introduce for the first time the study of the cryptocurrency market as a game, consisting of a set of strategic players (miners) with possibly different mining powers and a set of coins with possibly different rewards (weights). The miners are free to choose to mine for any coin from the set, and we consider general better-response learning of the miners. That is, whenever

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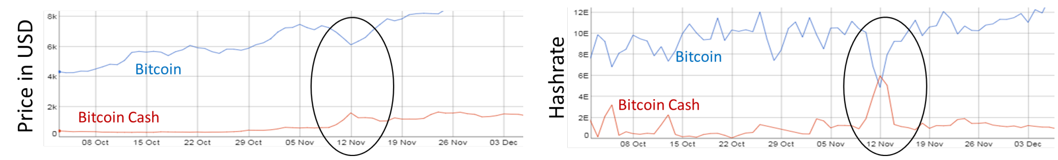


(a) Bitcoin and Bitcoin Cash exchange rates over time. (b) Hashrate corresponds to the number of miners.

Figure 1: Miners move from Bitcoin to Bitcoin Cash.

any miner may benefit from deviating (i.e., changing the coin it mines for), some miner will take a step that improves his payoff; we allow an arbitrary sequence of such individual improvement steps (sometimes called improving path [24]). In our first major result we prove that any such better response learning converges to a (pure) equilibrium regardless of miner powers and coin rewards! This result is obtained by showing an *ordinal potential*, which according to [24], implies that arbitrary better response learning converges to equilibrium.

Having at hand the above fundamental result, we move to a discussion of strategic manipulation [27]. While many efforts have been invested in the study of crypto-related manipulations [15, 29, 26, 14], we introduce for the first time the manipulation of the miners' learning and optimization process. Given that a shift in the weight of a coin may influence miner behavior [10], in the cryptocurrency setting, it is quite possible for an interested party to affect this weight, either by creating additional transactions with high fees (sometimes called whale transactions [22]) or by manipulating the coin exchange rate [16, 7, 4, 2]. This way, a miner (or another interested party) can attempt to change the system equilibrium to a better one for them. We show that under broad circumstances, for every equilibrium of such a game, there exists a miner and another equilibrium in which the miner's payoff is higher. The question is therefore: can one design rewards (i.e., temporarily increase coin weights) in a way that will lead the system from a given equilibrium to a desired one, so that the system will remain in the desired equilibrium after reverting to the original weights? Note that such reward design allows the manipulator to pay a finite cost while gaining an advantage indefinitely.

The above reward design problem is challenging since miners might take *any* better response step, and may make their moves in any order. Given the (modified) weights, we can use our previous major result to claim that any better response learning will converge to an equilibrium. Notice that the latter may not be the desired one, but now we can modify the rewards again. In the second major result of this paper we show that such desired reward design for learning agents is feasible! Namely, we provide a (multi-step) algorithm for assigning rewards in equilibrium states that moves learning agents from any initial equilibrium to a desired one.

In summary, our contributions are as follows:

1. We formalize strategic mining in multi-cryptocurrency markets as a game (Section 2).
2. We prove that any better-response learning in such games, starting from an arbitrary configuration, converges to equilibrium (Section 3).
3. We show that, in many cases, for every equilibrium there is a miner and another equilibrium in which the miner's payoff is higher (Section 4).
4. We offer a reward design scheme that moves the system configuration from any initial equilibrium to a desired one for any better-response learning of the miners (Section 5).

For space limitations, the proofs of some of the claims we state here are deferred to Appendices C - F.

1.1 Related work

Results on better response learning convergence to pure equilibrium are rare and are typically restricted to games with exact potential [18, 24], which coincide with congestion games. We show that our game does not have an exact potential (Section 3), and in fact our game belongs to the larger class of ID congestion games, where the payoff of a player depends on the player and the identity of other players who choose a similar resource, rather than on their number only. While there exist extensions of congestion games in which better-response learning converges to equilibrium (e.g., a restricted form of player-specific congestion games [23], which does not include our game), such results are extremely rare in the context of ID congestion games.

Unlike works on learning in games that emphasize adapting specific machine learning algorithms to minimize regret [18, 28, 34, 13], we assume minimal rationality on behalf of the players, i.e., that they follow an arbitrary better response step improving their individual payoffs.

Our work also expands literature on reward design [31, 17, 32], and to the best of our knowledge, is the first to introduce reward design for learning agents in a multi-agent setting. While seminal works in reward design assign/modify state rewards in a reinforcement learning context [33], we design rewards for equilibrium states for *any* better response learning.

Though several previous works presented game theoretical analyses for cryptocurrencies [22, 15, 29, 26, 12, 21, 20, 14, 30], the vast majority of them deal (in one way or another) with miners' incentives to follow the coins' mining protocols. Our work is the first to extend the study to a multi-coin setting and establish fundamental game theoretical results therein.

2 Model

A system in our model is a tuple $\langle \Pi, C \rangle$, where Π is a finite set of n miners (players) and C is a finite set of coins (resources). A miner $p \in \Pi$ has mining power $m_p \in \mathbb{R}_+$, which it can invest in one of the coins , i.e., the set of possible actions of p is C. We denote the *set of configurations* of a system $Q = \langle \Pi, C \rangle$ as $S_Q \triangleq C^n$ and denote by $s.p$ the action of player $p \in \Pi$ in configuration $s \in S_Q$. When clear from the context, we omit the subscript indicating the system and simply write S. Given $s \in S$ and $c \in C$, we denote by $P_c(s) \subseteq \Pi$ the set of miners who mine for c in s, i.e., $P_c(s) \triangleq \{p \in \Pi \mid s.p = c\}$, and by $M_c(s)$ their total mining power, i.e., $M_c(s) \triangleq \Sigma_{p \in P_c(s)} m_p$. For $s \in S_Q,\ p \in \Pi,\ c \in C$ we denote by (s_{-p}, c) the configuration that is identical to s except that $s.p$ is replaced by c.

A *reward function* $F : C \to \mathbb{R}_+$ maps coins to rewards. A game $G_{\Pi,C,F}$ consists of a system $\langle \Pi, C \rangle$ and a reward function F. Every coin in a game $G_{\Pi,C,F}$ divides its reward among all the players that mine for it, and the miners' payoffs are defined as follows: For $s \in S$, the *revenue per unit (RPU)* of coin c in s is $RPU_c(G_{\Pi,C,F})(s) \triangleq \frac{F(c)}{M_c(s)}$. When clear from the context, we omit the the parameter indicating the game. The payoff function of a miner $p \in \Pi$ is $u_p(s) \triangleq m_p \frac{F(s.p)}{M_{s.p}(s)} = m_p \cdot RPU_{s.p}(s)$.

Given a game $G_{\Pi,C,F}$, a configuration $s \in S$, a miner $p \in \Pi$, and a coin $c \in C$, we say that *p moves* from $s.p$ to c in s if it changes its action from $s.p$ to c. A move from $s.p$ to c is a *better response step* for p if $u_p(s) < u_p((s_{-p}, c))$. We say that a miner $p \in \Pi$ is *stable* in a configuration s in game $G_{\Pi,C,F}$ if p has no better response steps in s. A configuration s is *stable* or a (pure) *equilibrium* if every miner $p \in \Pi$ is stable in s. A *better response learning* from s in $G_{\Pi,C,F}$ is a sequence of configurations resulting from a sequence of better response steps starting from s, which is either infinite or ends with a stable configuration. In case it is finite, we say that it *converges* to its final configuration.

A function $f : S \to \mathbb{R}$ is an *ordinal potential* for a game $G_{\Pi,C,F}$ if for any two configurations $s, s' \in S$ s.t. some better response step of a miner $p \in \Pi$ leads from s to s', it holds that $f(s) < f(s')$. If, in addition, $f(s') - f(s) = u_p(s') - u_p(s)$, then f is an *exact potential*. By [24], if a game $G_{\Pi,C,F}$ has an ordinal potential, then every better response learning converges.

3 Better response learning convergance

In this section we prove that although a game $G_{\Pi,C,F}$ has no exact potential, every better-response learning of the miners in game $G_{\Pi,C,F}$ converges to a stable configuration (pure equilibrium) regardless of the sets Π and C and the reward function F. To gain intuition, the reader is referred to our Appendix A and B, where we show how to construct a particular equilibrium in a game $G_{\Pi,C,F}$ for any Π, C and F, and give a simple ordinal potential function for the symmetric case in which F is a constant function i.e., $\forall c, c' \in C$, $F(c) = F(c')$, respectively.

No exact potential. We start by showing that our game does not have an exact potential.

Proposition 1. *The game $G_{\Pi,C,F}$ does not always have an exact potential.*

Proof. Let $G_{\Pi,C,F}$ be a game where $\Pi = \{p_1, p_2\}$, $m_{p_1} = 2, m_{p_2} = 1$, $C = \{c_1, c_2\}$, and $F(c_1) = F(c_2) = 1$. Assume by way of contradiction that $G_{\Pi,C,F}$ has an exact potential function H, and consider the following four configurations:

- $s^1 = \langle c_1, c_1 \rangle$. Payoffs: $u_{p_1}(s^1) = \frac{F(c_1) \cdot m_{p_1}}{m_{p_1} + m_{p_2}} = 2/3$, $u_{p_2}(s^1) = \frac{F(c_1) \cdot m_{p_2}}{m_{p_1} + m_{p_2}} = 1/3$.
- $s^2 = \langle c_1, c_2 \rangle$. Payoffs: $u_{p_1}(s^2) = \frac{F(c_1) \cdot m_{p_1}}{m_{p_1}} = 1$, $u_{p_2}(s^2) = \frac{F(c_2) \cdot m_{p_2}}{m_{p_2}} = 1$.
- $s^3 = \langle c_2, c_2 \rangle$. Payoffs: $u_{p_1}(s^3) = \frac{F(c_2) \cdot m_{p_1}}{m_{p_1} + m_{p_2}} = 2/3$, $u_{p_2}(s^3) = \frac{F(c_2) \cdot m_{p_2}}{m_{p_1} + m_{p_2}} = 1/3$.
- $s^4 = \langle c_2, c_1 \rangle$. Payoffs: $u_{p_1}(s^4) = \frac{F(c_2) \cdot m_{p_1}}{m_{p_1}} = 1$, $u_{p_2}(s^4) = \frac{F(c_1) \cdot m_{p_2}}{m_{p_2}} = 1$.

Note that $H(s_2) - H(s_1) + H(s_3) - H(s_2) + H(s_4) - H(s_3) + H(s_1) - H(s_4) = 0$. However, by definition of exact potential, we get that $(H(s_2) - H(s_1)) + (H(s_3) - H(s_2)) + (H(s_4) - H(s_3)) + (H(s_1) - H(s_4)) = (2/3) + (-1/3) + (2/3) + (-1/3) = 2/3 \neq 0$. A contradiction.

□

Ordinal potential. To show an ordinal potential, we use the following definitions:

For a configuration $s \in S$ in a game $G_{\Pi,C,F}$, we define $list(s)$ to be the sequence of pairs in $\{\langle RPU_c(s), c \rangle \mid c \in C\}$ ordered lexicographically from smallest to largest. Denote by $v_i(s)$ the coin (second element of the pair) in the i^{th} entry in $list(s)$. Consider the ordered set $\langle L, \prec_L \rangle$, where $L \triangleq \{list(s) \mid s \in S\}$ is the set of all possible lists in $G_{\Pi,C,F}$, and $\prec_L$ is the lexicographical order. The *rank* of a list $list(s) \in L$, $rank(list(s))$, is the rank of $list(s)$ in $\prec_L$ from smallest to largest.

Note that since Π and C are finite, we know that S and L are finite. The following two observations establish a connection between better response steps and the $RPUs$ of the associated coins.

Observation 1. *Consider a game $G_{\Pi,C,F}$, $s \in S$, $v_i(s) \in C$, and $p \in \Pi$ s.t. $s.p = v_i(s)$. Then in every better response step of p that changes $s.p$ to a coin $v_j(s)$, it holds that $j > i$.*

Observation 2. *Consider a game $G_{\Pi,C,F}$. If some better response step from configuration s to configuration s' of a miner p changes $s.p = c$ to $s'.p = c'$, then $RPU_c(s) < min(RPU_c(s'), RPU_{c'}(s'))$.*

We are now ready to prove that any game $G_{\Pi,C,F}$ has an ordinal potential function.

Theorem 1. *For any finite sets Π and C of miners and coins and reward function F, $H(s) \triangleq rank(list(s))$ is an ordinal potential in the game $G_{\Pi,C,F}$.*

Proof. Consider two configurations $s, s' \in S$ s.t. some better response step of a miner $p \in \Pi$ leads from configuration s to configuration s', and let $v_i(s) = s.p$ and $v_j(s) = s'.p$. We need to show that $H(s) < H(s')$. Since only the RPUs of $v_i(s)$ and $v_j(s)$ are affected we get that

$$\forall k \neq i, j \; RPU_{v_k(s)}(s) = RPU_{v_k(s)}(s'). \tag{1}$$

By Observation 1, we get that $j > i$, and thus $\forall k,\ 1 \leq k < i$, $RPU_{v_k(s)}(s) = RPU_{v_k(s)}(s')$. By Observation 2, we get that $min(RPU_{v_i(s)}(s'), RPU_{v_j(s)}(s')) > RPU_{v_i(s)}(s)$, and thus, together with the definition of v_i and Equation 1, we get that $\forall k,\ i \leq k \leq |C|$, $RPU_{v_k(s)}(s') \geq RPU_{v_i(s)}(s)$. Therefore, none of them "move down" to a position before i in $list(s')$ and so

$$\forall k,\ 1 \leq k < i, \langle RPU_{v_k(s)}(s), v_k(s) \rangle = \langle RPU_{v_k(s')}(s'), v_k(s') \rangle. \tag{2}$$

That is, the first $i-1$ elements of $list(s)$ are equal to the first $i-1$ elements of $list(s')$. Hence, it suffices to show that the i^{th} element of $list(s')$ is lexicographically larger than the i^{th} element of $list(s)$. Let $v_l(s) = v_i(s')$. From Equation 2, we know that $l \geq i$, so there are two possible cases:

- First, $l \in \{i, j\}$. The theorem follows from Observation 2.

- Second, $l > i,\ l \neq j$. In this case,

$$\begin{aligned}\langle RPU_{v_i(s)}(s), v_i(s)\rangle &< \langle RPU_{v_l(s)}(s), v_l(s)\rangle && \text{(lexicographical order of } list(s))\\ &= \langle RPU_{v_l(s)}(s'), v_l(s)\rangle && \text{(Equation 1)}\\ &= \langle RPU_{v_i(s')}(s'), v_i(s')\rangle && \text{(Definition of } v_l(s)),\end{aligned}$$

as needed.

□

4 There is often a better equilibrium

Before moving to our second major result in which we describe a manipulation through dynamic reward design that transitions the system between equilibria, in this section we show that under broad circumstances, in every stable configuration there is at least one miner who has higher payoff in another stable configuration. This means that such a miner will gain from moving the system there. Specifically, we prove this for games that satisfy the following assumptions (note that we use these assumptions only in this section):

Assumption 1 (Never alone). *For a configuration $s \in S$ in a game $G_{\Pi,C,F}$, if there is a coin $c \in C$ s.t. $|P_c(s)| \leq 1$, then there is a miner $p \in \Pi$ s.t. changing $s.p$ to c is a better response step for p.*

Although this assumption cannot hold when $|\Pi| < 2|C|$, it often holds in practice since the number of miners must be much larger than the number of coins for the cryptocurrency to be secure (truly decentralized).

Assumption 2 (Generic game). *For any two coins $c \neq c' \in C$ and two sets of players $P, P' \subseteq \Pi$ in a game $G_{\Pi,C,F}$, $\frac{F(c)}{\Sigma_{p\in P} m_p} \neq \frac{F(c')}{\Sigma_{p\in P'} m_p}$.*

This assumption is common in game theory [19], and it makes sense in our game since mining power in practice is measured in billions of operations per hour and coin rewards are coupled with coin fiat exchange rates, so exact equality is unlikely.

The following observation follows from Assumption 1 and the fact that coins that are chosen by at least one miner always divide their entire reward. It stipulates that in every stable configuration, the sum of the payoffs the miners get is equal to the sum of the coins' rewards.

Observation 3 (All stable configurations are globally optimal). *For every stable configuration $s \in S$ in a game $G_{\Pi,C,F}$ under Assumption 1, it holds that $\sum_{p\in\Pi} u_p(s) = \sum_{c\in C} F(c)$.*

From Observation 3 and Assumption 2 it is easy to show the following claim:

Claim 4. *Consider a game $G_{\Pi,C,F}$ under Assumptions 1 and 2. If the game has more than one stable configuration, then for every stable configuration s there exist a miner p and a stable configuration s' s.t. $u_p(s') > u_p(s)$.*

It remains to show that $G_{\Pi,C,F}$ has more than one stable configuration. Consider $\Pi = \{p_1, \ldots, p_n\}$ s.t. $m_{p_1} \geq m_{p_2} \geq \ldots \geq m_{p_n}$, and $\forall i, 1 \leq i \leq n$, let $\Pi_i = \{p_1, \ldots, p_i\}$. We first show that the game $G_{\Pi_2,C,F}$ has two different configurations in which miners p_1, p_2 do not share a coin and at most one of them is unstable. Then, we inductively construct two configurations in $G_{\Pi_i,C,F}$, $\forall i, 3 \leq i \leq n$, based on the two configurations in $G_{\Pi_{i-1},C,F}$, in which all miners in Π_{i-1} keep their locations and all miners except maybe the one that was unstable in $G_{\Pi_1,C,F}$ are stable. The construction step is captured by Claim 5, where $p_{new} = p_i$ in the i^{th} step.

Claim 5. *Let F be a reward function. Consider a system $Q = \langle \Pi, C\rangle$, and a configuration $s \in S_Q$. Now consider another system $Q' = \langle \Pi', C\rangle$ s.t. $\Pi' = \Pi \cup \{p_{new}\}$, $p_{new} \notin \Pi$, and $m_{p_{new}} \leq min\{m_p | p \in \Pi\}$. Let $c = \underset{c' \in C}{\operatorname{argmax}}\ F(c') \frac{m_{p_{new}}}{M_{c'}(s) + m_{p_{new}}}$ and consider a configuration $s' \in S_{Q'}$ s.t. for all $p \in \Pi$ $s'.p = s.p$ and $s'.p_{new} = c$. Then p_{new} is stable in s' in game $G_{\Pi',C,F}$, and every player $p \in \Pi$ that is stable in s in $G_{\Pi,C,F}$ is also stable in s' in $G_{\Pi',C,F}$.*

Finally, we show that the two configurations we construct in $G_{\Pi,C,F}$ are stable: Let p_{ns} be the (possibly) unstable miner. By Assumption 1 (note that the assumption refers only to game $G_{\Pi,C,F}$),

p_{ns} cannot be alone in a coin (otherwise there must be another unstable miner), and thus it shares the coin with a smaller stable miner, which we show implies that p_{ns} is stable.

Our results are captured by the following proposition, which follows from Claim 4 and and the inductive construction using Claim 5.

Proposition 2. *Consider a game $G_{\Pi,C,F}$ under Assumptions 1 and 2. Then for every stable configuration s in $G_{\Pi,C,F}$ there exist a miner p and a stable configuration $s' \neq s$ in which $u_p(s') > u_p(s)$.*

5 Reward design: moving between equilibria

In this section we consider a system $Q = \langle \Pi, C \rangle$, where $\Pi = \{p_1, \dots p_n\}$ s.t. $m_{p_1} > m_{p_2} > \dots > m_{p_n}$. For every reward function F and every two stable configurations $s_0, s_f \in S_Q$ in game $G_{\Pi,C,F}$ we describe a mechanism to move the system from s_0 to the desired configuration s_f by temporarily increasing coin rewards. Note that once we lead the system to s_f, we can return to the original rewards (i.e., stop manipulating coin weights) because s_f is stable in $G_{\Pi,C,F}$. Therefore, a manipulator who gains from moving to a desired stable configuration can do it with a bounded cost.

We first define a reward design function that maps system configurations to reward functions.

Definition 1 (reward design function)**.** *Consider a system Q. A reward design function F for system Q is a function mapping every configuration $s \in S_Q$ to a reward function, i.e., $F(s) : C \to \mathbb{R}_+$.*

Dynamic reward design. Consider a system $\langle \Pi, C \rangle$ and a reward function F. A dynamic reward design mechanism for game $G_{\Pi,C,F}$ is an algorithm that for any two stable configurations s_0, s_f in $G_{\Pi,C,F}$ moves the system from s_0 to s_f by following the protocol in Algorithm 1.

Algorithm 1 protocol to move a system $\langle \Pi, C \rangle$ with reward function F from s_0 to s_f.

$s \leftarrow s_0$
repeat
  choose a reward design function H s.t. for all $c \in C$, $H(s)(c) \geq F(c)$
  allow better-response learning in $G_{\Pi,C,H(s)}$, starting from s, to converge to some stable configuration s' ▷ convergence is due to Theorem 1
  $s \leftarrow s'$
until $s = s_f$

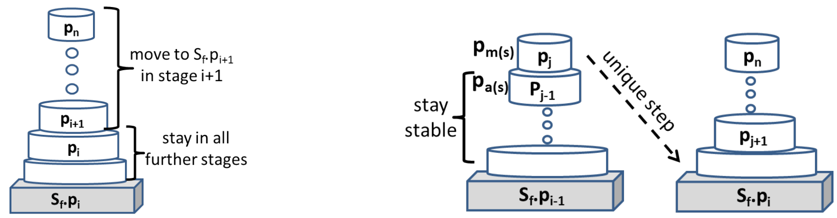
5.1 Reward design algorithm

To describe a dynamic reward design algorithm we need to specify the reward design function for every loop iteration in Algorithm 1. Intuitively, we observe that miners with less mining power are easily moved between coins, meaning that we can increase a coin reward so that a *small miner* with little mining power will benefit from moving there, but *bigger miners* with more mining power prefer to stay in their current locations. Therefore, the idea is to evolve the current configuration to $s_f \in S$ in $n = |\Pi|$ stages, where in stage i, we move the $n - i + 1$ miners with the smallest mining powers to the location (coin) of miner p_i in the final configuration s_f (i.e., $s_f.p_i$) while keeping the remaining miners in their (final) places. To this end, we define n intermediate configurations. For i, $1 \leq i \leq n$, we define s_i as:

$$s_i.p_k = \begin{cases} s_f.p_k & \forall 1 \leq k \leq i \\ s_f.p_i & \forall i < k \leq n \end{cases} \tag{3}$$

That is, in s_i, miners $p_1, \dots, p_i$ are in their final locations and miners $p_i, \dots, p_n$ are in the final location of miner p_i. Note that $s_n = s_f$. Figure 2a illustrates the stage transitions in the algorithm.

Notice that since we allow arbitrary better response learning (in every iteration), choosing a reward design function is a subtle task; miners can move according to any better response step, and we cannot control the order in which miners move. One may attempt to design a reward function so that in the resulting game there is exactly one unstable miner with exactly one better response step in the current configuration. However, even given such a function, after that miner takes its step, other miners might become unstable, which can in turn lead to a learning process that depends on the order in which miners move and on the choices they make (in case they have more than one better response



(a) Configuration s_i.

(b) Iteration moving $p_{m_i(s)}$; $p_{a_i(s)}$ is the anchor.

Figure 2: Reward design algorithm: (a) stages; and (b) iteration within stage i. Boxes represent coins, discs represent miners. The unlabeled bottom discs represent possible bigger miners who are in their final locations.

step). Hence, the main challenge is to be able to restrict the set of the possible stable configurations reached by learning phase in each iteration.

In every loop iteration of stage $i > 1$ we pick a miner p_k that we want to move from $s_f.p_{i-1}$ to $s_f.p_i$ (as explained shortly) and choose the reward function carefully so that (1) p_k's only better response step is $s_f.p_i$, (2) all other miners are stable, and (3) in every stable configuration reached by better response learning after p_k's step, p_k is in $s_f.p_i$, all miners $p_{k+1}, \ldots, p_n$ are in either $s_f.p_{i-1}$ or $s_f.p_i$, and all the other (bigger) miners remain in their (final) locations.

Moreover, our proof shows by induction that our reward design function of stage i (defined below) guarantees that the set of possible configurationas reached by learning in stage $i > 1$ is

$$T_i \triangleq \{s \in S \mid (\forall k,\ 1 \le k \le i-1 :\ s.p_k = s_f.p_k) \wedge (\forall k, i \le k \le n :\ s.p_k \in \{s_f.p_i, s_f.p_{i-1})\}\}.$$

Notice that the stage starts at $s_{i-1} \in T_i$. We now explain how we choose the reward design function for stage i. First, for every configuration $s \in T_i \setminus \{s_i\}$, the index of the miner we want to move from $s_f.p_{i-1}$ to $s_f.p_i$ (called *mover*) is

$$m_i(s) = min\{j \mid \forall l,\ j < l \le n : s.p_l = s_f.p_i\}.$$

Note that for every $s \in T_i$, $i \le m_i(s) \le n$. Moreover, $p_{m_i(s)} \notin P_{s_f.p_i}(s)$ and $m_i(s_{i-1}) = n$. Let $a_i(s) = m_i(s) - 1$. Intuitively, we use $p_{a_i(s)}$ as an *anchor* in configuration s; we choose a reward function that increases the reward of coin $s_f.p_i$ as high as possible without making the anchor unstable. As a result, all the miners in $P_{s_f.p_{i-1}}(s)$ (who are bigger than or equal to the anchor) remain stable, and miner $p_{m_i(s)}$ has a unique better response step to move to $s_f.p_i$. Figure 2b illustrates $m_i(s)$ and $a_i(s)$ for some configuration $s \in T_i$.

In order to make sure that miners not in $P_{s_f.p_{i-1}}(s) \cup P_{s_f.p_i}(s)$ also remain stable, and in order to guarantee that that every better response learning after $p_{m_i(s)}$'s step converges to a configuration in T_i, we choose a reward function that evens out the RPUs of all coins other than $s_f.p_i$. For $s \in S$, let $R(s) = max\{RPU_c(s) \mid c \in C\}\ \forall s \in S$. The reward design function H_i for stage $i > 1$ is:

$$\forall i > 1\ \forall s \in T_i\ \forall c \in C,\ H_i(s)(c) = \begin{cases} R(s) \cdot (M_c(s) + m_{p_{a_i(s)}}) & \text{for } c = s_f.p_i \\ R(s) \cdot M_c(s) & \text{otherwise} \end{cases} \tag{4}$$

Note that the RPUs of all coins except $s_f.p_i$ in the game $G_{\Pi,C,H_i(s)}$ are equal to $R(s)$. In addition, note that if a miner bigger than or equal to $p_{a_i(s)}$ moves to $s_f.p_i$, then $s_f.p_i$'s RPU becomes no bigger than $R(s)$. However, since $m_{p_{m_i(s)}} < m_{p_{a_i(s)}}$, $p_{m_i(s)}$ has a unique better response step to move to $s_f.p_i$. Therefore, we get that our reward design function allows us to control the first step of the learning process. In the next section we give more intuition on how it also restricts the stable configuration at the end of any learning process at stage i to the set T_i.

As for the fist stage, note that we need to move all miners to coin $s_f.p_1$, so intuitively we only need to increase its reward high enough. We therefore choose:

$$\forall s \in S\ \forall c \in C, H_1(s)(c) = \begin{cases} max\{F(c') \mid c' \in C\} \cdot \Sigma_{p \in \Pi} m_p & \text{for } c = s_f.p_1 \\ F(c) & \text{otherwise} \end{cases} \tag{5}$$

In Algorithm 2 we present our reward design algorithm, and in the next section we outline the proof that every stage eventually completes.

Algorithm 2 Dynamic reward design algorithm.

$s \leftarrow s_0$
for i=1 … n **do**
  repeat ▷ H_i is defined in Equations 4 and 5
    allow better-response learning in $G_{\Pi,C,H_i(s)}$, starting from s, to converge to some stable configuration s'
    $s \leftarrow s'$
  until $s = s_i$ ▷ s_i is defined in Equations 3

5.2 Proof outline

The proof for stage 1 is straightforward so we skip it. Consider stage $i > 1$. We prove in the appendix the following technical lemma about stable configurations in the stage:

Lemma 1. *Consider a configuration $s \in T_i \setminus \{s_i\}$. Then every better response learning in the game $G_{\Pi,C,H_i(s)}$ that starts at s converges to a configuration $s' \in T_i$ such that:*

1. $\forall k,\ 1 \leq k < m_i(s)$, $s'.p_k = s.p_k$.
2. $s'.p_{m_i(s)} = s_f.p_i$.

As part of the proof, we show that within stage i, all the reached configurations (both stable and unstable) are in T_i. Let $c = s_f.p_{i-1}$ and $c' = s_f.p_i$. After $p_{m_i(s)}$ moves to c' according to its only better response step, in the resulting configuration s', the RPUs of all coins not in $\{c, c'\}$ remain $R(s)$. Moreover, $RPU_c(s') = \frac{H_i(c)}{M_c(s')} = \frac{R(s)\cdot M_c(s)}{M_c(s)-m_{p_{m_i(s)}}}$, and $RPU_{c'}(s') = \frac{H_i(c')}{M_{c'}(s')} = \frac{R(s)\cdot(M_c(s)+m_{p_{a_i(s)}})}{M_c(s)+m_{p_{m_i(s)}}}$. Therefore, although $RPU_c(s') > RPU_c(s)$, it is still not high enough to drive miners not in $P_{c'}(s')$ (by definition, bigger than $p_{m_i(s)}$) to move to it. So the only miners that possibly have better response steps at s' are miners in $P_{c'}(s')$ who wish to move to c. Moreover, the total mining power of the miners who actually move to c is smaller than $p_{m_i(s)}$, otherwise, c's RPU will go below $R(s)$. In the proof we use the above intuition to formulate an invariant that captures the lemma statement and prove it by induction on better response steps. The lemma then follows from Theorem 1 (every better response learning converges to a stable configuration).

We next use Lemma 1 to prove that every stage $i > 1$ completes in a finite number of loop iterations. To this end, we associate with every configuration $s\ T_i$ a binary vector $v(s)$ indicating, for each $j \geq i$, whether p_j is in $P_{s_f.p_i}(s)$, where it needs to be at the end of the stage. Consider the ordered set $\langle V, \prec_v\rangle$, where $V \triangleq \{0,1\}^{n-i+1}$ is the set of all binary vectors of length $n-i+1$, and $\prec_v$ is the lexicographical order. For a configuration $s \in T_i$, we define $vec(s)$ to be a vector in V such that:

$$\forall j,\ 1 \leq j \leq n-i+1,\ vec(s)[j] = \begin{cases} 1 & \text{if } p_{j+i-1} \in P_{s_f.p_i}(s) \\ 0 & \text{otherwise} \end{cases}$$

and the function $\Phi_i : T_i \rightarrow \{1, .., |V|\}$ to be the rank of $vec(s)$ in V.

Theorem 2. *Every stage $i > 1$ of Algorithm 2 completes in a finite number of loop iterations.*

Proof. By definitions of stage i and set T_i, the first configuration of stage i is $s_{i-1} \in T_i$. By definition of $m_i(s)$, for all $s \in T_i \setminus \{s_i\}$, $m_i(s) \notin P_{s_f.p_i}(s)$. By inductively applying Lemma 1, we get that every loop iteration in stage i ends in a configuration in T_i. Therefore, consider a loop iteration of stage i that starts in configuration $s \neq s_i$ and ends in configuration s', we get by Lemma 1 that $m_i(s) \in P_{s_f.p_i}(s')$ and $\forall k,\ i \leq k < m_i(s)$, $s.p_k = s'.p_k$. Therefore $\Phi(s') > \Phi(s)$. Now since the set T_i is finite, we get the after a finite number of iterations we reach configuration s_i.

□

6 Discussion

Our work studies and challenges the crypocurrency market from a novel angle – the strategic selections by adaptive miners among multiple coins. There are several central followups one may consider. First, our reward design is effective for arbitrary better-response learning, but one may wonder about its speed of convergence under specific markets. In addition, we consider convergence to equilibrium, and one may consider also convergence to a bad (possibly unstable) configuration in which, for example, a particular miner will have a dominant position in a coin, killing (at least for a while) the basic guarantee of non-manipulation (security) for that coin and allowing him to get a bigger portion of the reward. One also may wonder about the asymmetric case where some coins can be mined only by a subset of the miners.

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Appendix A Existence of an equilibrium

We show here how to find a stable configuration (pure equilibrium) in the game $G_{\Pi,C,F}$ for any Π, C, and F. We do this by induction, selecting coins for miners in descending order of mining power.

Claim 6. *Consider a reward function F, a system $Q = \langle \Pi, C\rangle$, and another system $Q' = \langle \Pi', C\rangle$ s.t. $\Pi' = \Pi \cup \{p_{new}\}$, $p_{new} \notin \Pi$, and $m_{p_{new}} \leq min\{m_p | p \in \Pi\}$. Then, if the game $G_{\Pi,C,F}$ has a stable configuration, than the game $G_{\Pi',C,F}$ has a stable configuration as well.*

Proof. Let s be a stable configuration in $G_{\Pi,C,F}$. We use it to build a configuration s' in $G_{\Pi',C,F}$ in the following way: for all $p \neq p_{new}$ set $s'.p = s.p$, and set $s'.p_{new}$ to $c = \underset{c' \in C}{\operatorname{argmax}} F(c') \frac{m_{p_{new}}}{M_{c'}(s)+m_{p_{new}}}$. We now show that configuration s' is stable in $G_{\Pi',C,F}$.

First, consider p_{new}. Since we pick c to be $\underset{c' \in C}{\operatorname{argmax}} F(c') \frac{m_{p_{new}}}{M_{c'}(s)+m_{p_{new}}}$, we get that $\forall c' \in C$, $F(c)\frac{m_{p_{new}}}{M_c(s')} \geq F(c')\frac{m_{p_{new}}}{M_{c'}(s')+m_{p_{new}}}$, so p_{new} is stable in s'. Next, consider a miner p s.t. $s.p = c' \neq c$. Since s is stable, we know that $\forall c'' \in C$, $F(c')\frac{m_p}{M_{c'}(s)} \geq F(c'')\frac{m_p}{M_{c''}(s)+m_p}$. Now since $M_c(s) < M_c(s')$ and for all $c'' \neq c$ $M_{c''}(s) = M_{c''}(s')$, we get that $\forall c'' \in C, F(c')\frac{m_p}{M_{c'}(s')} \geq F(c'')\frac{m_p}{M_{c''}(s')+m_p}$. Meaning that p is stable in s'.

Finally, consider $p \neq p_{new}$ s.t. $s'.p = c$. We need to show that for all $c' \neq c$, $F(c')\frac{m_p}{M_{c'}(s')+m_p} \leq F(c)\frac{m_p}{M_c(s')}$. Again, since we pick c to be $\underset{c' \in C}{\operatorname{argmax}} F(c')\frac{m_{p_{new}}}{M_{c_j}(s)+m_{p_{new}}}$, we know that $\forall c' \neq c$, $F(c)\frac{m_{p_{new}}}{M_c(s)+m_{p_{new}}} \geq F(c')\frac{m_{p_{new}}}{M_{c'}(s)+m_{p_{new}}}$, and thus $\forall c' \neq c$, $F(c)\frac{1}{M_c(s')} \geq F(c')\frac{1}{M_{c'}(s')+m_{p_{new}}}$. Now by the claim assumption, $m_{p_{new}} \leq m_p$. Therefore, $\forall c' \neq c$, $F(c)\frac{1}{M_c(s')} \geq F(c')\frac{1}{M_{c'}(s')+m_{p_{new}}} \geq F(c')\frac{1}{M_{c'}(s')+m_p}$. Thus, $\forall c' \neq c$, $F(c)\frac{m_p}{M_c(s')} \geq F(c')\frac{m_p}{M_{c'}(s')+m_p}$. Meaning that p is stable in s'.

□

Proposition 3. *For any set of of miners $\Pi = \{p_1, \ldots, p_n\}$, set of coins C, and a reward function F, the game $G_{\Pi,C,F}$ has a stable configuration.*

Proof. Order miners by mining power so that $m_{p_1} \geq m_{p_2} \geq \ldots \geq m_{p_n}$. First we show that the game $G_{\{p_1\},C,F}$ has a stable configuration. Let $c = \underset{c' \in C}{\operatorname{argmax}} F(c')$, and define a configuration s in $G_{\{p_1\},C,F}$ as follows: $s.p_1 = c$. Since $\forall c' \in C$, $F(c)\frac{m_{p_1}}{M_c(s)} = F(c) \geq F(c') = F(c')\frac{m_{p_1}}{M_{c'}(s)+m_{p_1}}$, we get that the configuration s is stable in $G_{\{p_1\},C,F}$. The lemma follows by inductively applying Claim 6.

□

Appendix B Ordinal potential for the symmetric case

We consider here the symmetric case where all coin rewards are equal, and show that any game in this case has a simple potential function.

Proposition 4. *Consider a finite set of players Π, a finite set of coins C, and a reward function F s.t. $\forall c, c' \in C$ $F(c) = F(c')$. Then $H(s) \triangleq \sum_{c \in C} \frac{1}{M_c(s)}$ is a potential function in the game $G_{\Pi,C,F}$.*

Proof. Consider two configurations $s, s' \in S$ s.t. some better response step of a miner $p \in \Pi$ leads from configuration s to configuration s'. We need to show that $H(s) > H(s')$. Let $c = s.p$ and $c' = s'.p$. By definition of a better response step, $F(c')\frac{m_p}{M_{c'}(s)+m_p} > F(c)\frac{m_p}{M_c(s)}$. Since $F(c) = F(c')$, we get that $\frac{1}{M_{c'}(s)+m_p} > \frac{1}{M_c(s)}$, and thus

$$M_c(s) > M_{c'}(s) + m_p. \tag{6}$$

Now since for every $c'' \notin \{c', c\}$, $M_{c''}(s) = M_{c''}(s')$, we get that

$$H(s') - H(s) = \frac{1}{M_{c'}(s')} + \frac{1}{M_c(s')} - (\frac{1}{M_{c'}(s)} + \frac{1}{M_c(s)}).$$

Moreover, since p moves from c to c' in s, we get that

$$H(s') - H(s) = \frac{1}{M_{c'}(s) + m_p} + \frac{1}{M_c(s) - m_p} - (\frac{1}{M_{c'}(s)} + \frac{1}{M_c(s)}).$$

Thus, in order to show that $H(s) > H(s')$, we need to show that

$$\frac{1}{M_{c'}(s) + m_p} + \frac{1}{M_c(s) - m_p} < \frac{1}{M_{c'}(s)} + \frac{1}{M_c(s)}.$$

Equivalently, that

$$\frac{M_c(s) - m_p + m_p + M_{c'}(s)}{(M_{c'}(s) + m_p)(M_c(s) - m_p)} < \frac{M_c(s) + M_{c'}(s)}{M_c(s) M_{c'}(s)},$$

or that

$$M_c(s) M_{c'}(s) < (M_{c'}(s) + m_p)(M_c(s) - m_p),$$

or finally that

$$M_c(s)(M_{c'}(s) + m_p) - M_c(s) m_p < (M_{c'}(s) + m_p) M_c(s) - (M_{c'}(s) + m_p) m_p).$$

The proposition follows from Equation 6

□

Appendix C Observations' proofs for the ordinal potential

Observation 1 (restated). *Consider a game $G_{\Pi,C,F}$, $s \in S$, $v_i(s) \in C$, and $p \in \Pi$ s.t. $s.p = v_i(s)$. Then in every better response step of p that changes $s.p$ to a coin $v_j(s)$, it that $j > i$.*

Proof. By the definition of a better response step, $\frac{F(v_j(s))}{M_{v_j(s)}(s) + m_p} > \frac{F(v_i(s))}{M_{v_i(s)}(s)}$, and thus $RPU_{v_j(s)}(s) = \frac{F(v_j(s))}{M_{v_j(s)}(s)} > \frac{F(v_i(s))}{M_{v_i(s)}(s)} = RPU_{v_i(s)}(s)$. By definition of $v(s)$, we get that $j > i$.

□

Observation 2 (restated). *Consider a game $G_{\Pi,C,F}$. If some better response step from configuration s to configuration s' of a miner p changes $s.p = c$ to $s'.p = c'$, then $RPU_c(s) < min(RPU_c(s'), RPU_{c'}(s'))$.*

Proof. By definition of a better response step, $RPU_{c'}(s') > RPU_c(s)$. In addition, since $M_c(s') = M_c(s) - m_p$, we get that $RPU_c(s') = \frac{F(c)}{M_c(s')} = \frac{F(c)}{M_c(s) - m_p} > \frac{F(c)}{M_c(s)} = RPU_c(s)$.

□

Appendix D There is often a better Eeuilibrium: proofs

We prove here the Claims from Section 4

Claim 4 (restated). *Consider a game $G_{\Pi,C,F}$ under Assumption 2. If the game has more than one stable configuration, then for every stable configuration s there exist a miner p and a stable configuration s' s.t. $u_p(s') > u_p(s)$.*

Proof. Consider a stable configuration s. By assumption, there exists another stable configuration $s' \neq s$. Therefore, there is a player p and coins $c \neq c'$ s.t. $p \in P_c(s)$ and $p \in P_{c'}(s')$. By Assumption 2, $\frac{F(c)}{M_c(s)} \neq \frac{F(c')}{M_{c'}(s')}$. Thus, $u_p(s') \neq u_p(s)$. If $u_p(s') > u_p(s)$, then we are done. Otherwise, by Observation 3, there is another player $p' \neq p$ s.t. $u_{p'}(s') > u_{p'}(s)$.

□

Claim 7. *Consider a game $G_{\Pi,C,F}$, a configuration $s \in S$, a coin $c \in C$, and two miners $p, p' \in P_c(s)$ s.t. $m_p \leq m_{p'}$. If p is stable in s, then p' is stable in s as well.*

Proof. Since p is stable in s, we get that $F(c)\frac{m_p}{M_c(s)} \geq F(c')\frac{m_p}{M_{c'}(s)+m_p}$ for every $c' \in C$. Thus, $F(c)\frac{1}{M_c(s)} \geq F(c')\frac{1}{M_{c'}(s)+m_p}$ for every $c' \in C$. Since, $m_p \leq m_{p'}$, it follows that $F(c)\frac{1}{M_c(s)} \geq F(c')\frac{1}{M_{c'}(s)+m_{p'}}$ for every $c' \in C$. Thus, $F(c)\frac{m_{p'}}{M_c(s)} \geq F(c')\frac{m_{p'}}{M_{c'}(s)+m_{p'}}$ for every $c' \in C$, and p' is stable in s.

□

Claim 5 (restated). *Let F be a reward function. Consider a system $Q = \langle \Pi, C \rangle$, and a configuration $s \in S_Q$. Now consider another system $Q' = \langle \Pi', C \rangle$ s.t. $\Pi' = \Pi \cup \{p_{new}\}$, $p_{new} \notin \Pi$, and $m_{p_{new}} \leq min\{m_p | p \in \Pi\}$. Let $c = \underset{c' \in C}{\operatorname{argmax}}\ F(c')\frac{m_{p_{new}}}{M_{c'}(s)+m_{p_{new}}}$ and consider a configuration $s' \in S_{Q'}$ s.t. for all $p \in \Pi$ $s'.p = s.p$ and $s'.p_{new} = c$. Then p_{new} is stable in s' in game $G_{\Pi',C,F}$, and every player $p \in \Pi$ that is stable in s in $G_{\Pi,C,F}$ is also stable in s' in $G_{\Pi',C,F}$.*

Proof. By construction of configuration s', $M_c(s') = M_c(s) + m_{p_{new}}$ and $\forall c' \neq c$ $M_{c'}(s') = M_{c'}(s)$. Therefore, by the way we pick c, we get that $F(c)\frac{m_{p_{new}}}{M_c(s')} \geq F(c')\frac{m_{p_{new}}}{M_{c'}(s')}$ for all $c' \in C$, and thus p_{new} is stable in s'. Now consider a player $p \in \Pi$ that is stable in s, we show that p is stable also in s'. Consider two cases:

- First, $p \in P_{c'}(s')$ s.t. $c \neq c'$. By construction, $p \in P_{c'}(s)$, and since p is stable in s we know that $F(c')\frac{m_p}{M_{c'}(s)} \geq F(c'')\frac{m_p}{M_{c''}(s)+m_p}$ for every $c'' \in C$. Now since $M_c(s') > M_c(s)$ and for all $c'' \neq c$, $M_{c''}(s') = M_{c''}(s)$, we get $F(c')\frac{m_p}{M_{c'}(s')} \geq F(c'')\frac{m_p}{M_{c''}(s')+m_p}$ for every $c'' \in C$. Meaning that p is stable in s'.
- Second, $p \in P_c(s')$. Since p_{new} is stable in s' and $m_p \geq m_{p_{new}}$, we get by Claim 7 that p is stable in s'.

□

Lemma 2. *Any game $G_{\Pi,C,F}$ under Assumptions 1 and 2 has at least two different stable configurations.*

Proof. Let $p_1, \ldots, p_n$ be the miners in Π sorted in decreasing mining power, i.e., $m_{p_1} \geq m_{p_2} \geq \ldots \geq m_{p_n}$ and let $c_1, \ldots, c_l$ be the coins in C sorted in decreasing coin rewards, i.e., $F(c_1) \geq F(c_2) \geq \ldots \geq F(c_l)$. Note that through the proof we construct several games, but we assume Assumptions 1 and 2 only in the game $G_{\Pi,C,F}$. Let $\Pi_1, \ldots, \Pi_n$ be a sequence of sets of miners s.t. $\Pi_k = \{p_1, \ldots, p_k\}$, $1 \leq k \leq n$. Next consider two configurations s_1^2, s_2^2 in game $G_{\Pi_2,C,F}$: $s_1^2 = \langle c_1, c_2 \rangle$ (i.e., $s_1^2.p_1 = c_1$ and $s_1^2.p_2 = c_2$) and $s_2^2 = \langle c_2, c_1 \rangle$. Note that $s_1^2 \neq s_2^2$, and since $F(c_1) \geq F(c_2)$, p_1 is stable in s_1^2 and p_2 is stable in s_2^2.

We now use s_1^2, s_2^2 to inductively construct a sequence of tuples of configurations $\langle s_1^2, s_2^2 \rangle, \langle s_1^3, s_2^3 \rangle, \ldots, \langle s_1^n, s_2^n \rangle$, where $\forall\ 2 \leq k \leq n$, s_1^k, s_2^k are two configurations in Game $G_{\Pi_k,C,F}$. For every $3 \leq k \leq n$ let $x_1 = \underset{c \in C}{\operatorname{argmax}}\ F(c)\frac{m_{p_k}}{M_c(s_1^{k-1})+m_{p_k}}$ and $x_2 = \underset{c \in C}{\operatorname{argmax}}\ F(c)\frac{m_{p_k}}{M_c(s_2^{k-1})+m_{p_k}}$, we construct $s_1^k = s_1^{k-1} \times x_1$ (i.e., $\forall 1 \leq i < k$, $s_1^k.p_i = s_1^{k-1}.p_i$ and $s_1^k.p_k = x_1$) and $s_2^k = s_2^{k-1} \times x_2$.

Note that since $s_1^2 \neq s_2^2$, we get by construction that $s_1^n \neq s_2^n$. It remains to show that configurations s_1^n and s_2^n are stable. Assume by way of contradiction that it is not the case, and assume w.l.o.g. that s_1^n is not stable. By inductively applying Claim 5, and since p_1 is stable in s_1^2 we get that players p_1 and $p_3, \ldots, p_n$ are stable in s_1^n. Thus p_2 is not stable. Recall that $s_1^2.p_2 = c_2$, and thus by construction $s_1^n.p_2 = c_2$. Recall also that we assume Assumptions 1 and 2, and consider two cases:

- First, $|P_{c_2}(s_1^n)| = 1$. By Assumption 1, there is a miner $p \in \Pi$ s.t. changing $s_1^n.p$ to c_2 is a better response step for p. Thus, p is not stable in s_1^n. In addition, by definition of better response step, we know that $p_i \neq p_2$. A contradiction to p_2 being the only not stable miner in s_1^n.

- Second, $|P_{c_2}(s_1^n)| > 1$. Since $p_1 \in P_{c_1}(s_1^2)$, we get that there is a stable miner $p \in P_{c_2}(s_1^2)$ s.t. $p \in \{p_3, \ldots, p_n\}$, and thus, $m_2 \geq m_p$. Therefore, by Claim 7, we get that p_2 is stable in s_1^n. A contradiction.

$\square$

Appendix E Reward design: proof of Lemma 1

Lemma 1 (restated). *Consider a configuration $s \in T_i \setminus \{s_i\}$. Then every better response learning in the game $G_{\Pi,C,H_i(s)}$ that starts at s converges to a configuration $s' \in T_i$ such that:*

1. $\forall k,\ 1 \leq k < m_i(s)$, $s'.p_k = s.p_k$.
2. $s'.p_{m_i(s)} = s_f.p_i$.

Proof. Let $c = s_f.p_{i-1}$ and $c' = s_f.p_i$. By definition of $m_i(s)$, and since $s \neq s_i$ we know that $p_{m_i(s)} \in P_c(s)$. We first show that the only better response step in configuration s in game $G_{\Pi,C,H_i(s)}$ is that $p_{m_i(s)}$ moves to c'. Since $\forall c'' \neq c'$, $RPU_{c''}(s) = R(s)$ and $RPU_{c'}(s) > R(s)$, we get that no miner has a better response step to move to any coin $c'' \neq c'$. Since $m_{p_{m_i(s)}} < m_{p_{a_i(s)}}$, we get that $RPU_{c'}((s_{-p_{m_i(s)}}, c')) = \frac{R(s)\cdot(M_{c'}(s)+m_{p_{a_i(s)}})}{M_{c'}(s)+m_{p_{m_i(s)}}} > R(s) = RPU_c(s)$, and thus moving to c' is a better response step for $p_{m_i(s)}$ in s. Now consider a miner $p_k \notin P_{c'(s)}$ s.t. $k \neq m_i(s)$. By definition of $m_i(s)$, $k < m_i(s)$, thus $m_{p_k} > m_{p_{m_i(s)}}$, and thus, $\frac{R(s)\cdot(M_{c'}(s)+m_{p_{a_i(s)}})}{M_{c'}(s)+m_{p_k}} \leq R(s)$. All in all, we get that the only better response step in configuration s is that $p_{m_i(s)}$ moves to c'.

Let $s^0 = (s_{-p_{m_i(s)}}, c')$ be the configuration reached after $p_{m_i(s)}$ takes its step. We next prove by induction that every configuration s' that is reached by better response steps staring from s^0 satisfies the following properties:

Ψ_1. $\forall k,\ 1 \leq k < m_i(s)$: $s'.p_k = s.p_k$.

Ψ_2. $s'.p_{m_i(s)} = c'$.

Ψ_3. $\forall k,\ m_i(s) < k \leq n$: $s'.p_k \in \{c, c'\}$.

Ψ_4. $M_c(s^0) \leq M_c(s') \leq M_c(s)$.

Ψ_5. $M_{c'}(s) \leq M_{c'}(s') \leq M_{c'}(s^0)$.

Note that since $s \in T_i$, $\Psi_1 - \Psi_3$ imply that $s' \in T_i$.

Base. We show below that the properties $\Psi_1 - \Psi_5$ are satisfied for configuration s^0.

$\Psi_1 - \Psi_3$. Follow by definitions of $m_i(s)$ and T_i, and since $s^0 = (s_{-p_{m_i(s)}}, c')$.

$\Psi_4 - \Psi_5$. Trivially follows.

Induction step. Consider two configurations $s^1, s^2 \in S$ s.t. a better response step of some miner leads from s^1 to s^2, and s^1 satisfies properties $\Psi_1 - \Psi_5$. We show that s_2 satisfies properties $\Psi_1 - \Psi_5$ as well.

Useful equations for configuration s^1. We start by proving two useful equations on the RPUs of coins c and c' in configuration s^1 using Ψ_4 and Ψ_5:

By Ψ_4, $M_c(s^1) \leq M_c(s)$. In addition, since $H_i(s)(c) = R(s) \cdot M_c(s)$, we get that

$$RPU_c(s^1) = \frac{H_i(s)(c)}{M_c(s^1)} = \frac{R(s) \cdot M_c(s)}{M_c(s^1)} \geq \frac{R(s) \cdot M_c(s)}{M_c(s)} = R(s). \tag{7}$$

By Ψ_5, $M_{c'}(s^1) \le M_{c'}(s^0)$. In addition, since (1) $m_{p_{m_i(s)}} < m_{p_{a_i(s)}}$, (2) $M_{c'}(s) = M_{c'}(s^0) - m_{p_{m_i(s)}}$, and (3) $H_i(s)(c') = R(s) \cdot (M_{c'}(s) + m_{p_{a_i(s)}})$, we get that

$$\begin{aligned} RPU_{c'}(s^1) &= \frac{H_i(s)(c')}{M_{c'}(s^1)} = \frac{R(s) \cdot (M_{c'}(s) + m_{p_{a_i(s)}})}{M_{c'}(s^1)} \\ &= \frac{R(s) \cdot (M_{c'}(s^0) - m_{p_{m_i(s)}} + m_{p_{a_i(s)}})}{M_{c'}(s^1)} \\ &\ge \frac{R(s) \cdot M_{c'}(s^0)}{M_{c'}(s^0)} = R(s). \end{aligned} \tag{8}$$

Useful equations for configurations s^1 and s^2. We now prove two useful equations on the RPUs of coins not in $\{c, c'\}$ in configurations that satisfy $\Psi_1 - \Psi_3$.

By definition of H_i, $\forall c'' \notin \{c, c'\}$, $H_i(s)(c'') = R(s) \cdot M_{c''}(s)$. Since $s \in T_i$, $\forall k, i \le k \le n :\ s.p_k \in \{c, c'\}$, and by definition of $m_i(s)$, we know that $m_i(s) \ge i$. Therefore, by $\Psi_1 - \Psi_3$, we get that

$$\forall c'' \notin \{c, c'\},\ M_{c''}(s^1) = M_{c''}(s), \tag{9}$$

and thus

$$\forall c'' \notin \{c, c'\},\ RPU_{c''}(s^1) = \frac{H_i(s)(c'')}{M_{c''}(s^1)} = \frac{R(s) \cdot M_{c''}(s)}{M_{c''}(s)} = R(s). \tag{10}$$

Induction step proof. We are now ready to prove that s_2 satisfies properties $\Psi_1 - \Psi_5$.

$\Psi_1 - \Psi_2$. Since Ψ_1 and Ψ_2 hold in s^1, we only need to show that $p_1, \ldots, p_{m_i(s)}$ are stable in s^1. Let $k \in \{1, \ldots, m_i(s)\}$ and note that $m_{p_k} \ge m_{p_{m_i(s)}}$. By Equation 10, for all $c'' \notin \{c, c'\}$, $RPU_{c''}(s^1) = R(s)$, and by Equations 7 and 8, we know that $RPU_c(s^1) \ge R(s)$ and $RPU_{c'}(s^1) \ge R(s)$, respectively. Therefore, it remains to show that p_k does not have a better response step to c or c':

- By Ψ_4, $M_c(s^1) \ge M_c(s^0)$, and thus we get that

$$\frac{H_i(c)}{M_c(s^1) + m_{p_k}} \le \frac{H_i(c)}{M_c(s^0) + m_{p_{m_i(s)}}} = \frac{H_i(c)}{M_c(s)} = \frac{M_c(s) \cdot R(s)}{M_c(s)} = R(s).$$

  Therefore, p_k does not have a better response step to c.
- By Ψ_2, $p_{m_i(s)} \in P_{c'}(s^1)$. Therefore, if $p_k = p_{m_i(s)}$, then we are done. Otherwise, $k < m_i(s) = a_i(s) + 1$, so $m_{p_k} \ge m_{p_{a_i(s)}}$. By Ψ_5, $M_{c'}(s^1) \ge M_{c'}(s)$, and thus

$$\frac{H_i(c')}{M_{c'}(s^1) + m_{p_k}} \le \frac{H_i(c')}{M_{c'}(s) + m_{p_{a_i(s)}}} = \frac{R(s) \cdot (M_{c'}(s) + m_{p_{a_i(s)}})}{M_{c'}(s) + m_{p_{a_i(s)}}} = R(s).$$

  Therefore, p_k does not have a better response step to c'.

Ψ_3. By Equations 7 and 8, $RPU_c(s^1) \ge R(s)$ and $RPU_{c'}(s^1) \ge R(s)$. By Equation 10, for all $c'' \notin \{c, c'\}$, $RPU_{c''}(s^1) = R(s)$. Therefore, since by the inductive assumption (Ψ_3), miners $p_{m_i(s)+1}, \ldots, p_n$ are in $P_c(s^1) \cup P_{c'}(s^1)$, we get that none of them has a better response step to move to a coin $c'' \notin \{c, c'\}$, and thus Ψ_3 holds in s^2 as well.

Ψ_4. By definitions of s^0, $m_i(s)$, and T_i, we know that $P_c(s^0) \cap \{p_{m_i(s)}, \ldots, p_n\} = \emptyset$. Since Ψ_1 holds in s^0 and in s^1, we get that $\forall k,\ 1 \le k < m_i(s)$: $s^1.p_k = s^0.p_k$. Therefore, we get that $M_c(s^0) \le M_c(s^2)$. It remains to show that $M_c(s^2) \le M_c(s)$. By Equation 10, for all $c'' \notin \{c, c'\}$, $RPU_{c''}(s^1) = R(s)$. By Equations 7 and 8, $RPU_c(s^1) \ge R(s)$ and $RPU_{c'}(s^1) \ge R(s)$. Now assume by way of contradiction that $M_c(s^2) > M_c(s)$. Thus,

$$RPU_c(s^2) = \frac{H_i(c)}{M_c(s^2)} < \frac{H_i(c)}{M_c(s)} = \frac{R(s) \cdot M_c(s)}{M_c(s)} = R(s).$$

Now since $RPU_c(s^2) < R(s) \le RPU_c(s^1)$, we get that $s^2 = (s^1_{-p}, c)$ for some $p \in \Pi$. A contradiction to Observation 2.

Ψ_5. Since we already showed that s^2 satisfies $\Psi_1 - \Psi_3$, by Equation 9, we get that $\forall c'' \notin \{c, c'\},\ M_{c''}(s^2) = M_{c''}(s)$. In addition, since $s^0 = (s_{-p_{m_i(s)}}, c')$, $\forall c'' \notin \{c, c'\},\ M_{c''}(s^0) = M_{c''}(s)$. Therefore, we get that $\forall c'' \notin \{c, c'\},\ M_{c''}(s) = M_{c''}(s^0) = M_{c''}(s^2)$, and thus $M_c(s^2) + M_{c'}(s^2) = M_c(s^0) + M_{c'}(s^0) = M_c(s) + M_{c'}(s)$. Hence, Ψ_5 follows from Ψ_4.

Now together with Theorem 1, we know that every better response learning in the game $G_{\Pi,C,H_i(s)}$ that starts at s converges to some configuration s' that satisfies $\Psi_1 - \Psi_5$. Since $s \in T_i$, we get that

$$(\forall k,\ 1 \leq k \leq i-1\ : s.p_k = s_f.p_k) \wedge (\forall k, i \leq k \leq n:\ s.p_k \in \{s_f.p_i, s_f.p_{i-1}\}).$$

And since $s \neq s_i$, we get that $i \leq m_i(s)$. Thus, by Ψ_1, $\forall k,\ 1 \leq k \leq i-1 : s'.p_k = s_f.p_k$. In addition, by Ψ_3, we get that $\forall k,\ i \leq k \leq n,\ s.p_k \in \{s_f.p_i, s_f.p_{i-1}\}$. Therefore, $s' \in T_i$, and the lemma follows from Ψ_1 and Ψ_2.

$\square$

s