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On Sampling Invariant Signal Representation

Or: On the effect of sampling on algebraic relations

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Abstract

Most signal processing systems are based on discrete-time signals although the origin of many sources of information is analog. In this work we consider the task of signal representation by a set of basis functions. Presently, without prior knowledge of the signal beyond its samples, no bound on the potential representation error is available. The question raised in this paper is to what extent the sampling process keeps algebraic relations, such as inner product, intact. By interpreting the sampling process as a linear bounded operator, an upper bound on the representation error is derived and demonstrated. Based on our theorems, one can then determine the maximum representation error induced by the sampling process. We further propose a new approximation scheme for the calculation of the inner product, which is optimal in the sense of minimizing the maximum representation error. Our results are applicable to signal processing systems where analog signals are represented by their sampled versions.

I. Introduction

Signal processing applications are concerned mainly with digital data, although the origin of many sources of information is analog. Such signals include for example speech and audio, optical signals, radar and sonar as well as mechanical and biomedical signals. Representing a time-continuous (analog) signal by its samples has been widely used since Nyquist formulated the sampling theorem. It is also well known that applying this representation scheme to non bandlimited signals introduces approximation errors: the signal does not belong to $Span\{sinc(\cdot)\}$ and the samples do not correspond to its orthogonal projection. Indeed, according to the sampling theorem, these errors become smaller as the sampling interval shortens ([15]). However, there are cases in which achieving a low approximation error requires high, unrealizable, sampling rates. For this reason, mainly, alternative basis functions such as Gabor functions, wavelets, Hermite functions, Legendre functions, Laguerre functions and the like are often used instead ([2] [5] [6] [7] [10] [11] [13]).

Finding representation coefficients for these alternative bases involves inner-product calculations within the analog domain, rather than simply consider the sampled version of the signal itself as in the bandlimited case. When the signal is not given analytically, this in turn is somewhat difficult to implement. It is even impossible to perform in cases where the signal is already given by its samples only.

To overcome this difficulty, it is acceptable to approximate the original inner product by the discrete-time approximation:

$$\langle f, \varphi \rangle \cong T \cdot \sum_{n} f(nT) \cdot \varphi(nT),$$

where $f(t)$ is the original signal and $\varphi(t)$ is a known (basis) function. Here too, relying on the sampling theorem, the error of this approximation scheme becomes smaller as the sampling interval shortens. However, having no prior knowledge of the original signal $f(t)$, except for its samples, no bounds on the resulted approximation error are presently available.

With this regard of extracting the representation coefficients of a signal from its samples, several previous works are of relevance; In [14] the authors address the problem of representing a discrete time signal by introducing the discrete Gabor representation. Their derivation is focused, however, on properly representing the (sampled) discrete time signal rather than properly representing the original (analog) signal itself. In [13] it is shown that within the context of MRA (multi-resolution analysis), there are situations in which one can extract the representation coefficients of a signal while having its sampled version alone. In [9] the authors address the problem of initialization of a MRA representation scheme, while having only the signal's sampled version, as well. Nevertheless, having utilized the sampled-data control theory, their results assume certain a-priory knowledge on the frequency characteristics of the sampled signals. There are also numerous works addressing the issue of reconstructing a signal from its samples ([12] and references therein), most of them assume a shift-invariant approximation model.

Keeping the basic approximation scheme, abovementioned, the question raised and considered in this work is whether the sampling process keeps algebraic relations, shared within the analog domain, intact. We consider the operation widely used in vector representation, the inner product, and propose a new discrete approximation scheme for this calculation, allowing an optimal approach to this widely used approximation.

II. The Problem

We address the following problem: given a function $\varphi(t)\in L_2$, how can one optimally approximate the inner product of $\langle f,\varphi\rangle$, by having only the samples of $f(t)$? Furthermore, if calculated this way, what is the approximation error?

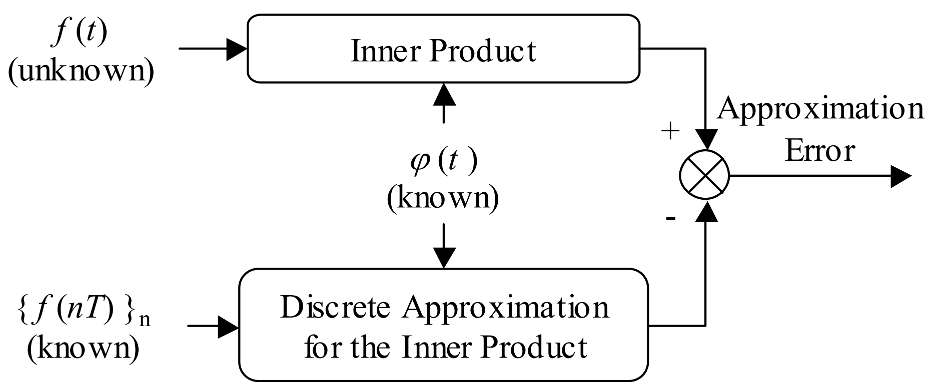

Figure 1: Statement of the problem - given a function $\varphi(t)$, how can one optimally approximate the inner product of $\langle f,\varphi\rangle$, by having only the samples of $f(t)$?

III. Sampling as a Linear Operator

We consider finite energy analog sources, i.e., $\int |f(t)|^2 dt < \infty$. For practical reasons, it is most convenient to adopt the common inner product of the Hilbert space L_2 given by,

$$\langle f, g \rangle_{L_2} = \int f(t) \cdot g(t) dt ,$$

where for simplicity, only real functions are considered. Approximating both f and φ by piecewise constant functions can be interpreted as approximating the L_2 inner product by an l_2 inner product of two sequences:

$$\langle f, g \rangle_{L_2} \cong \langle f(nT), T \cdot g(nT) \rangle_{l_2} .$$

This interpretation, however, should be done with much prudence (Figure 2); Sampling a function of L_2 does not necessarily yield a sequence of l_2. Furthermore, functions, which are considered identical in L_2, might give rise to distinct sequences in l_2. Those two drawbacks prevent one from interpreting the uniform sampling process as a bounded linear operator acting on L_2 functions to yield l_2 sequences.

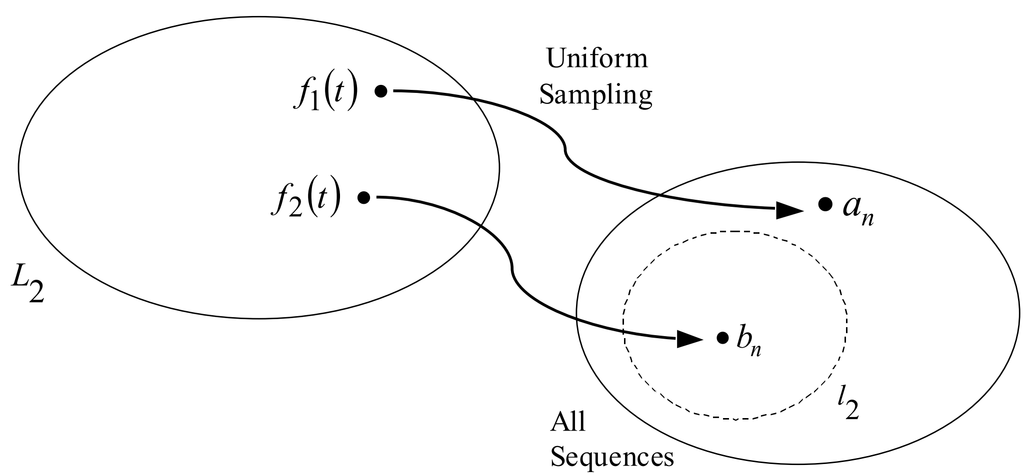

Figure 2: When uniformly sampling a function of L_2, it is not guaranteed that the resulted sequence is also of finite energy.

The following theorem addresses this issue (Fig. 3).

Theorem 1: Let $f(t)$ be a Sobolev function of order one, i.e. $f(t), f'(t) \in L_2$. Uniformly sampling it by an interval T results in a sequence of finite energy, $a[n]$, upper bounded by

$$\|a\|_{l_2} \leq \frac{1}{\sqrt{T}}\|f(t)\|_{L_2} + \frac{1}{4}\sqrt{2T}\|f'(t)\|_{L_2} < \infty$$

Proof: See Appendix.

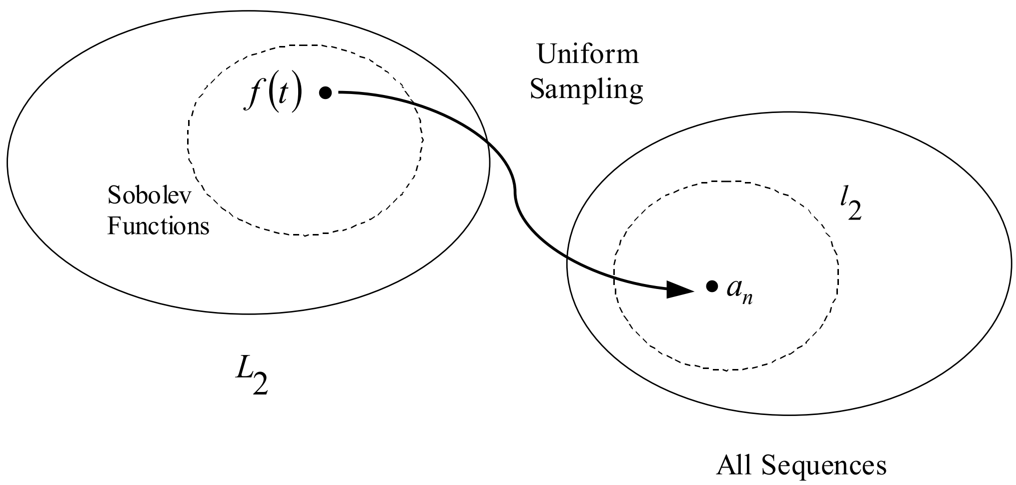

Figure 3: Uniformly sampling a Sobolev function of order one, i.e. $f(t), f'(t) \in L_2$, yields a sequence of finite energy.

The importance of this theorem resides in the fact that the sampling process can now be thought of as a linear bounded operator, which can be further investigated within the framework of functional analysis. It so happens, however, that the set of Sobolev functions is an incomplete vector space considering the regular L_2 norm. Nevertheless, introducing an alternative inner product,

$$\langle f, g \rangle_{W_2} = \langle f, g \rangle_{L_2} + \langle f', g' \rangle_{L_2},$$

allows for this vector space to be complete, thus a Hilbert space. Denoting this Hilbert space by W_2, it is the Sobolev space of order one ([1]), and the following lemmas can be stated.

Lemma 1: Given $T > 0$, the uniform sampling operator,

$$S_T f : W_2 \to l_2$$
$$S_T f = \{f(nT)\}_n$$

is a bounded linear operator (Figure 4).

Lemma 2: The sampling operator S_T is given by,

$$S_T : W_2 \rightarrow l_2$$
$$S_T f = \sum_n \langle f(t), u(t - nT) \rangle_{W_2} \cdot e_n$$

where $u(t) = e^{-|t|}/2$, and $\{e_n\}$ is the standard basis of l_2.

Proof: The functional $s_{t0}(f) = f(t_0)$ is a bounded linear functional. By Riesz representation theorem ([8]), it is also given by

$$s_{t_0}(f) = \langle f(t), u(t - t_0) \rangle_{W_2},$$

where $u(t) \in W_2$ is unique and the equality holds for any $f(t) \in W_2$. Rewriting the latter in the frequency domain,

$$\frac{1}{2\pi} \int F(w) \cdot \overline{U(w) e^{-j\omega t_0}} \left(1 + \omega^2\right) d\omega = \frac{1}{2\pi} \int F(\omega) \cdot \overline{e^{-j\omega t_0}} d\omega,$$

leads to $U(\omega) = 1 / (1+\omega^2)$, thus $u(t) = e^{-|t|}/2$ (Figure 5). □

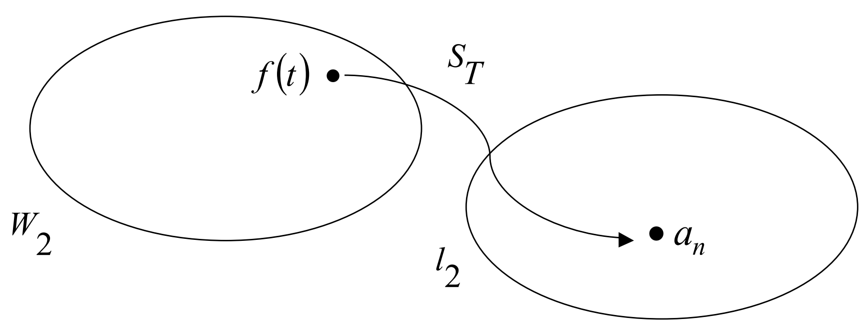

Figure 4: The uniform sampling operator is a linearly bounded operator acting on the Sobolev space of order one to yield a sequence of l_2.

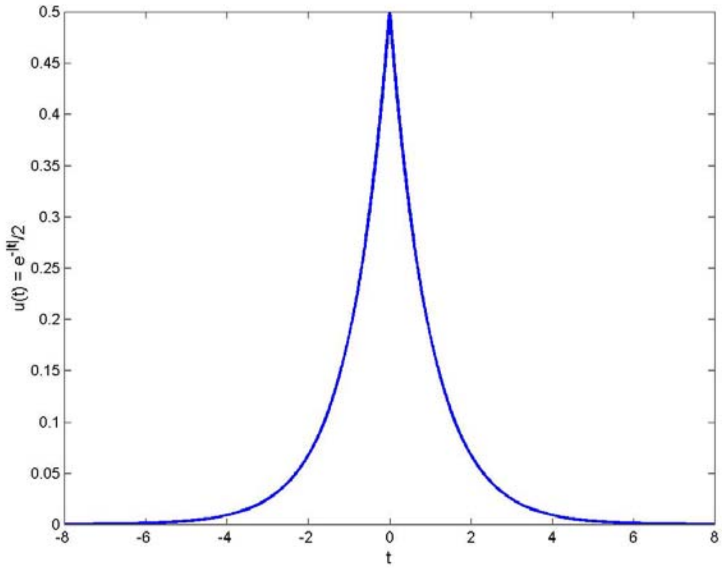

Figure 5: $u(t)$, shown here, describes the sampling functional in terms of a Sobolev inner product, i.e. $f(t_0)=\langle f, u(t-t_0)\rangle_{W2}$.

IV. Intertwining Relations of L_2, l_2 and W_2 Inner Products

Lemma 3: Let $b[n]\in l_2$ be a known sequence. Then (Figure 6),

$$\forall f \in W_2, \quad \langle S_T f, b\rangle_{l_2} = \langle f, S_T^* b\rangle_{W_2}$$

where,

$$\left(S_T^* b\right)(t) = \sum_n b[n]\cdot u(t-nT)$$

and $u(t)$ is given in Lemma 2.

Proof: S_T, which is a bounded linear operator of two Hilbert spaces, has an adjoint operator ([8]), S_T^*: $l_2 \to W_2$. That is, for every $b[n]\in l_2$, there is a unique function $(S_T^* b)(t) \in W_2$ such that,

$$\forall f \in W_2, \ \langle S_T f, b\rangle_{l_2} = \langle f, S_T^* b\rangle_{W_2}.$$

The adjoint operator can be found by,

$$\begin{aligned}
\langle S_T f, b\rangle_{l_2} &= \left\langle \sum_n \langle f, u(t-nT)\rangle_{W_2} \cdot e_n, b \right\rangle_{l_2} \\
&= \sum_n \langle f, u(t-nT)\rangle_{W_2} \cdot \langle e_n, b\rangle_{l_2} \\
&= \sum_n \left\langle f, \langle e_n, b\rangle_{l_2} \cdot u(t-nT) \right\rangle_{W_2} \\
&= \left\langle f, \sum_n \langle e_n, b\rangle_{l_2} \cdot u(t-nT) \right\rangle_{W_2} = \left\langle f, S_T^* b \right\rangle_{W_2} .
\end{aligned}$$

□

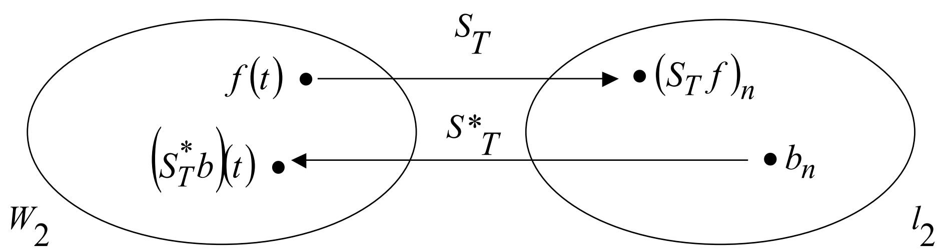

Figure 6: Intertwining relations of the inner product of W_2 and l_2. Considering the uniform sampling operator S_T and its adjoint, $\langle S_T f , b\rangle_{l_2} = \langle f , S_T^* b\rangle_{W_2}$. S_T and S_T^* are given in Lemma 2 and Lemma 3 respectively.

Example 1: Let $f(t)$ be a normalized Gaussian,

$$f(t) = \frac{1}{\sqrt[4]{\pi}} e^{-t^2/2} ,$$

and let $b[n]$ be the finite sequence,

$$b[n] = \begin{cases} n & n = 1,2,3 \\ 0 & else \end{cases} .$$

Sampling $f(t)$ at $T = 0.7$ yields the following (Figure 7):

$$\langle S_T f, b\rangle_{l_2} = 1 \cdot f(0.7) + 2 \cdot f(1.4) + 3 \cdot f(2.1) = 1.4002 ,$$

where the same result is obtained by

$$\left\langle f, S_T^* b \right\rangle_{W_2} = \left\langle f \ , \ 1 \cdot e^{-|t-0.7|} + 2 \cdot e^{-|t-1.4|} + 3 \cdot{}^{-|t-2.1|} \right\rangle_{W_2} = 1.4002 .$$

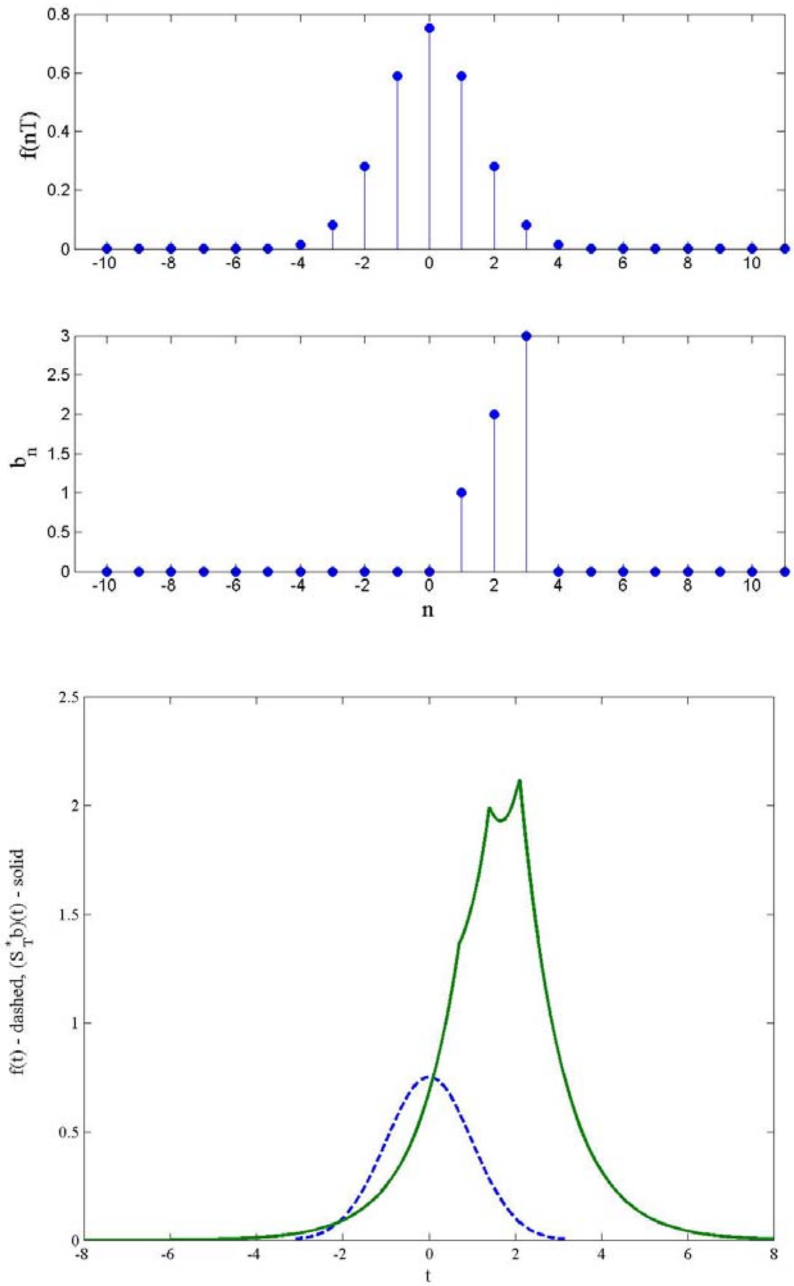

Figure 7: An example of the intertwining relations of W_2- and l_2- inner products. $f(t)$ is a Gaussian function and $b[n]$ is a finite sequence. Performing $\langle S_T f\ ,\ b\rangle_{l2}$ upper and middle) is equivalent to performing $\langle f\ ,\ S_T^*\ b\rangle_{W2}$ (bottom). The sampling interval is $T = 0.7$.

Lemma 4: Let $\varphi(t) \in L_2$ be a known function. Then (Figure 8),

$$\forall f \in W_2 : \ \langle f, \varphi \rangle_{L_2} = \langle f, \varphi^* \rangle_{W_2} ,$$

where $\varphi^*(t) \in W_2$ is given by the convolution $\varphi^*(t) = \varphi(t) * u(t)$, and $u(t)$ is given in Lemma 2.

Proof: Consider the identity operator $I f = f$, where I: $W_2 \to L_2$. This operator is bounded since an L_2-norm is always smaller than a W_2 norm. Therefore, it possesses an adjoint counterpart, I^*, guarantying that for any $\varphi(t) \in L_2$ there exists $\varphi^*(t) = I^* \varphi$, such that,

$$\forall f(t) \in W_2 , \ \langle If, \varphi \rangle_{L_2} = \langle f, I^* \varphi \rangle_{W_2} .$$

Writing this relation in the frequency domain yields the equality $\Phi^*(\omega) = \Phi(\omega) / 1 + \omega^2$. □

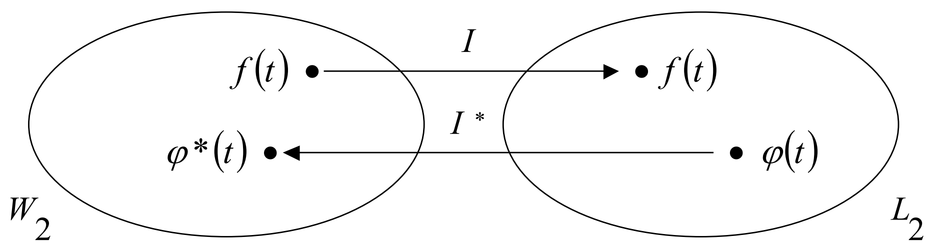

Figure 8: Intertwining relations of the inner products of W_2 and L_2. Performing $\langle f, \varphi \rangle_{L2}$ is equivalent to performing $\langle f, \varphi^* \rangle_{W2}$. φ^* is as given in Lemma 4.

Example 2: Let f and φ be normalized Gaussian functions as in Example 1. Their L_2 - inner product equals 1. The same result is obtained by performing the inner product in W_2 (Figure 9), i.e.,

$$\langle f, \varphi^*(t) \rangle_{W_2} = 1 .$$

Lemma 5: Let $\varphi(t) \in L_2$ be a known function, and let $b[n] \in l_2$ be a known sequence. Then (Figure 10),

$$\forall f \in W_2 : \ \langle f, \varphi \rangle_{L_2} - \langle S_T f, b \rangle_{l_2} = \langle f, \varphi^* - S_T^* b \rangle_{W_2} ,$$

where S_T is the uniform sampling operator given in Lemma 2, S_T^* its adjoint as given in Lemma 3: and $\varphi^*(t)$ is dependant on $\varphi(t)$ as given in Lemma 4.

Proof: It has been shown that, $\langle f,\varphi \rangle_{L2} = \langle f,\varphi^* \rangle_{W2}$. It has also been shown that $\langle S_T f , b\rangle_{l2} = \langle f , S_T^* b\rangle_{W2}$. Therefore,

$$\langle f,\varphi\rangle_{L_2} - \langle S_T f, b\rangle_{l_2} = \langle f,\varphi^*\rangle_{W_2} - \left\langle f, S_T^* b\right\rangle_{W_2} = \left\langle f,\varphi^* - S_T^* b\right\rangle_{W_2}.$$

□

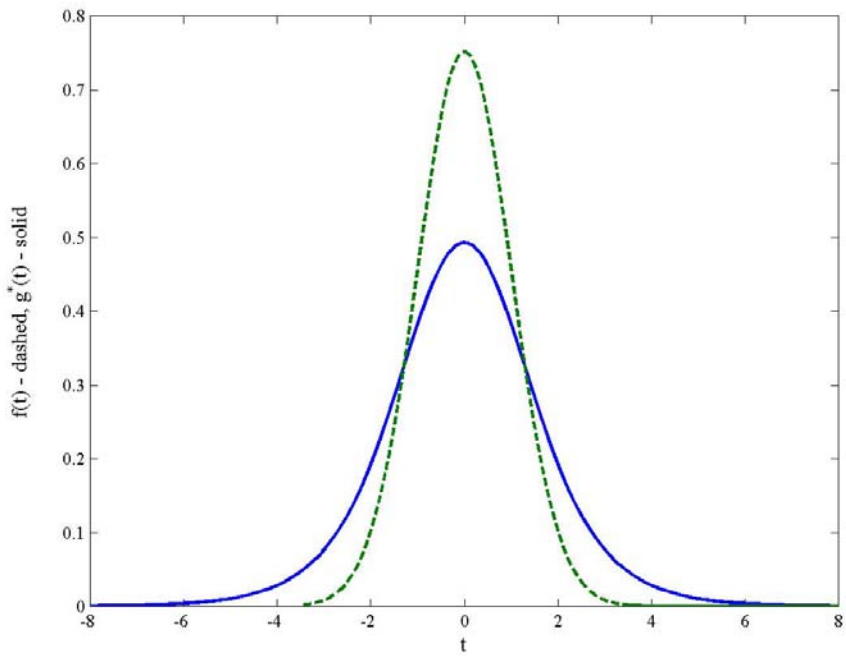

Figure 9: An example of the intertwining relations of W_2- and L_2- inner products. Performing $\langle f,\varphi \rangle_{L2}$ is equivalent to performing $\langle f,\varphi^* \rangle_{W2}$. f and φ are normalized Gaussian functions (dashed), and φ^* is as given in Lemma 4 (solid).

Lemma 6: Let $f,\varphi \in H$, where H is a Hilbert space. Then,

$$\langle f,\varphi\rangle = \|f\| \cdot \|\varphi\|$$

iff $f(t) = \alpha \cdot \varphi(t)$ for any $\alpha \in C$.

Proof: If there is $\alpha \in C$ in which $f(t) = \alpha \cdot \varphi(t)$, then the equality immediately follows. On the other hand, suppose the equality holds, but there is no $\alpha \in C$ for which $f(t) = \alpha \cdot \varphi(t)$. Now, denoting f_1 to be the orthogonal projection of f onto φ, and $f_2 = f - f_1$, yields $\langle f,\varphi \rangle = \langle f_1+f_2,\varphi \rangle = \langle f_1,\varphi \rangle$. Since the latter expression is equivalent to $\|f\| \cdot \|\varphi\|$, as well as to $\|f_1\| \cdot \|\varphi\|$ it results in $f = \alpha \cdot \varphi$, a contradiction, which proves the lemma. □

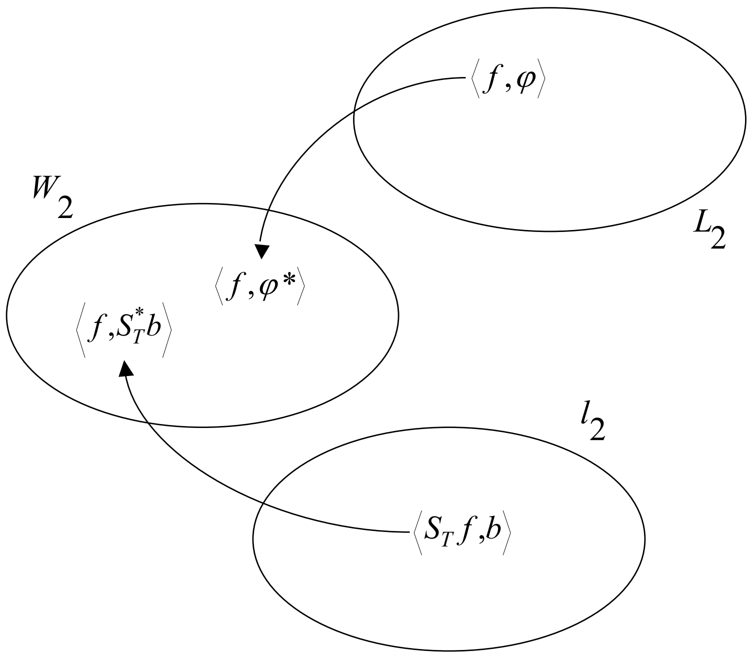

Figure 10: Intertwining relations of L_2, l_2 and W_2 Inner Products. φ is a known function of L_2 and b is a known sequence of l_2. f is an arbitrary function of W_2 to be uniformly sampled. The inner product of L_2 has a corresponding representation in W_2, and the same holds for the l_2 inner product.

Lemma 7: Let $\varphi(t) \in W_2$ be a known function. Then,

$$\forall f \in W_2, \quad \langle f, \varphi \rangle_{W_2} \leq M \cdot \|f\|_{L_2},$$

where $M = \|\varphi\|_{W_2}^2 \big/ \|\varphi\|_{L_2}$.

Proof: It has been shown that $\langle f, \varphi \rangle_{W_2} \leq \|f\|_{W_2} \|\varphi\|_{W_2}$, where equality is achieved iff $f(t) = \alpha \cdot \varphi(t)$ for any $a \in C$. a thus corresponds to the ratio $\|f\|_{W_2} \big/ \|\varphi\|_{W_2}$, as well as to the ratio $\|f\|_{L_2} \big/ \|\varphi\|_{L_2}$, which proves the lemma. □

V. Sampling Effects on L_2 Inner Product

Theorem 2: Let $\varphi(t) \in W_2$ be a known function. Given a sampling interval T, the following relation holds:

$$\forall f \in W_2, \quad \left| \langle f, \varphi \rangle_{L_2} - T \cdot \langle S_T f, S_T \varphi \rangle_{l_2} \right| \le B \cdot \|f\|_{L_2},$$

where S_T is the uniform sampling operator with an interval T. B is then given by

$$B = \frac{\left\| \varphi(t) * u(t) - T \cdot \sum_n \varphi(nT) \cdot u(t - nT) \right\|_{W_2}^2}{\left\| \varphi(t) * u(t) - T \cdot \sum_n \varphi(nT) \cdot u(t - nT) \right\|_{L_2}},$$

where $u(t) = e^{-|t|}/2$.

Proof: Set $b[n] = T \cdot \varphi(nT)$ in Lemma 5 and apply Lemma 7. □

Denoting $\varphi^*(t) = \varphi(t) * u(t)$ and $\varphi_1(t) = T \cdot \sum \varphi(nT) \cdot u(t-nt)$ allows one to interpret the vector relations shared within W_2 when approximation $\langle f,\varphi \rangle_{L2}$ by their samples, as shown in Figure 11. It should be noted that φ_1 always resides within $Span\{u(t-nT\}$, where φ^* always does not. Furthermore, there are functions $f(t) \in W_2$ that achieve the upper bound of Theorem 2. One simply sets $f(t) = \varphi^*(t) - \varphi_1(t)$.

The admissible functions - $f(t)$ - considered so far are Sobolev functions of order $n = 1$. It can be argued, however, that such functions are less common in practical applications, and it would be worth considering smoother functions, giving rise to yet more applicable upper bounds. Such functions can be regarded as members of a Sobolev space of a higher order, having the following inner product:

$$\langle f, g \rangle_{W_2^n} = \frac{1}{2\pi} \int F(\omega) \cdot \overline{G(\omega)} \cdot \left(1 + \omega^2 + \omega^4 + \cdots \omega^{2n}\right) d\omega.$$

Therefore, considering admissible Sobolev functions - $f(t)$ - of order n, deriving upper bounds on the approximation of $\langle f,\varphi \rangle_{L2}$ by their sampled versions would follow the analysis presented earlier. The key point for such an analysis would be to apply Theorem 2, where $u(t)$ is the inverse Fourier transform of

$$U(\omega) = \frac{1}{1 + \omega^2 + \omega^4 + \cdots + \omega^{2n}}.$$

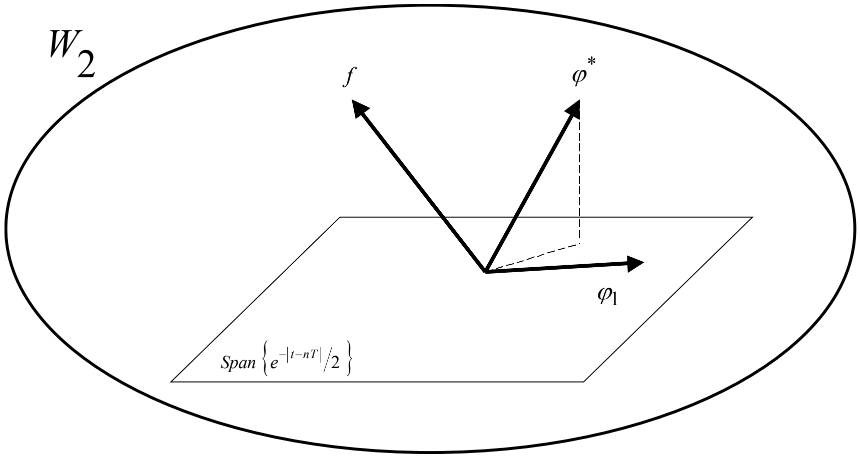

Figure 11: Vector relations for approximating $\langle f,\varphi \rangle_{L2}$ using their corresponding samples. φ^* is such that $\langle f,\varphi \rangle_{L2} = \langle f,\varphi^* \rangle_{W2}$ and φ_1 is such that $\langle S_T f, S_T\varphi \rangle_{l2} = \langle f, \varphi_1\rangle_{W2}$. S_T is the sampling operator. $\varphi^*(t) = \varphi(t) * u(t)$ and $\varphi_1(t) = T\cdot\sum\varphi(nT)\cdot u(t\text{-}nt)$, where $u(t)$ is as given in Theorem 2.

With regard to $U(\omega)$, it so happens that the polynomial in the denominator has complex roots residing on the unit circle. Thus, $u(t)$ would be comprised of a linear combination of shifted and modulated $e^{-|t|}$ functions (Figure 12). However, when considering the case where n reaches infinity,

$$\lim_{n\to\infty} U(\omega) = \begin{cases} 1-\omega^2 & |\omega|<1 \\ 0 & |\omega|\geq 1 \end{cases},$$

leading to (Figure 12),

$$\lim_{n\to\infty} u(t) = \begin{cases} \dfrac{2\sin(t)}{\pi t^3} - \dfrac{2\cos(t)}{\pi t^2} & t\neq 0 \\ \dfrac{2}{3}\cdot\dfrac{1}{\pi} & t=0 \end{cases}.$$

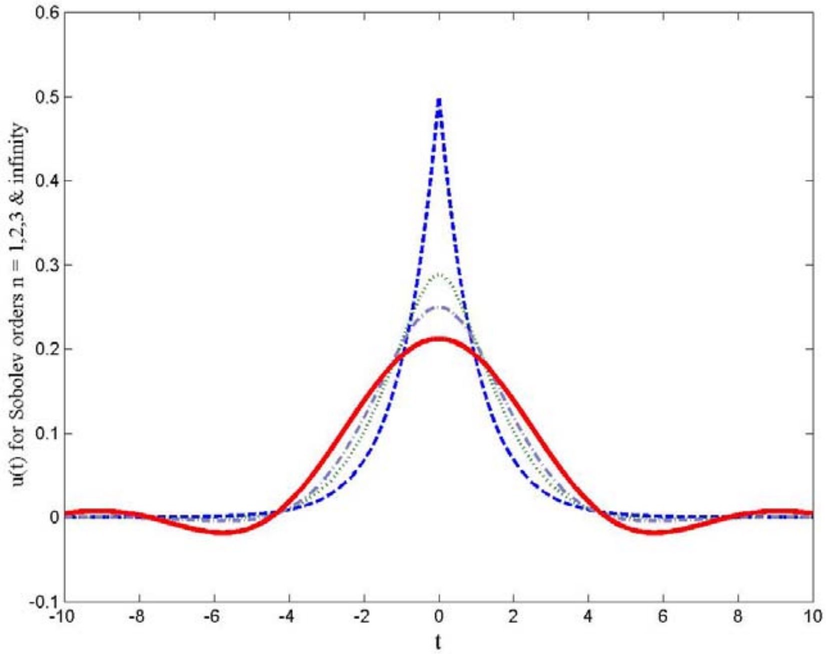

Figure 12: Deriving upper bounds on the approximation error of $\langle f,\varphi \rangle_{L2}$ by their sampled versions, where *f* is a Sobolev function of order *n* is dependant of *u*(*t*) (Theorem 2). *u*(*t*), in turn, is the inverse Fourier transform of $U(\omega) = 1/(1+\omega^2+\omega^4+\ldots+\omega^{2n})$. Shown are *u*(*t*) for *n* = 1 (dash), 2 (dot), 3 (dashed-dot) and infinity (solid).

An alternative expression for *B* of Theorem 2 can be given within the frequency domain, allowing more convenient calculations when considering practical situations. The numerator within the equation of *B* is,

$$\left\| \varphi(t) * u(t) - T \cdot \sum_n \varphi(nT) \cdot u(t-nT) \right\|_{W_2}^2 .$$

It can be expressed, though, within the frequency domain:

$$\begin{aligned}\left\| \varphi(t) * u(t) - T \cdot \sum_n \varphi(nT) \cdot u(t-nT) \right\|_{W_2}^2 &= \frac{1}{2\pi} \int \left| \Phi \cdot U - T \cdot \sum_n \varphi(nT) \cdot U \cdot e^{-j\omega nT} \right|^2 \cdot U^{-1} d\omega \\ &= \frac{1}{2\pi} \int \left| \Phi - T \cdot \sum_n \varphi(nT) \cdot e^{-j\omega nT} \right|^2 \cdot U \, d\omega \\ &= \frac{1}{2\pi} \int_{-1}^{1} \left| \Phi - T \cdot \sum_n \varphi(nT) \cdot e^{-j\omega nT} \right|^2 \cdot \left(1-\omega^2\right) d\omega \ .\end{aligned}$$

A Similar expression holds for the denominator:

$$\left\| \varphi(t) * u(t) - T \cdot \sum_n \varphi(nT) \cdot u(t - nT) \right\|_{L_2} = \sqrt{\frac{1}{2\pi} \int \left| \Phi \cdot U - T \cdot \sum_n \varphi(nT) \cdot U \cdot e^{-j\omega nT} \right|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi} \int \left| \Phi - T \cdot \sum_n \varphi(nT) \cdot e^{-j\omega nT} \right|^2 \cdot U^2 \, d\omega}$$

$$= \sqrt{\frac{1}{2\pi} \int_{-1}^{1} \left| \Phi - T \cdot \sum_n \varphi(nT) \cdot e^{-j\omega nT} \right|^2 \cdot \left(1 - \omega^2\right)^2 d\omega} \ .$$

As for a general remark, Sobolev functions of an arbitrary order are dense in L_2. Thus, restricting the admissible functions, $f(t)$, to belong to such spaces still maintains the generality of the results.

Example 3: Sampling Hermite functions. Those functions are defined by,

$$h_k(t) = \frac{H_k(t) e^{-t^2/2}}{\pi^{1/4} 2^{n/2} (k!)^2} \ ,$$

where $H_k(t)$ are the Hermite polynomials are recursively defined by,

$$\begin{cases} t \cdot H_k(t) = \frac{1}{2} H_{k+1}(t) + k \cdot H_{k-1}(t) \\ H_0(t) = 1 \ , \ H_1(t) = -2t \ , \ H_2(t) = 4t^2 - 2 \ , \ H_3(t) = -8t^3 + 12t \ , \ H_4(t) = 16t^4 - 48t^2 + 12 \ \ldots \end{cases}$$

Those functions constitute an orthonormal basis in L_2. Therefore, the representation coefficients are then found by $a_k = \langle f, \psi_k \rangle_{L2}$, where ψ_k is the Hermite function of order k (Figure 13). Having both the sampled versions of f and ψ_k , a_k would be approximated by,

$$\hat{a}_k = T \cdot \left\langle S_T f, S_T \psi_k \right\rangle_{l_2} = T \cdot \sum_n f(nT) \cdot \psi_k(nT).$$

This in turn yields an approximation error, upper bounded by $B \cdot \|f\|_{L2}$. Some bounds are shown in Figure 15 for the first and sixth Hermite functions. For a given sampling interval, T, B corresponds to the worst-case scenario (i.e., the approximation error would be as large as possible). In such a case $f(t) = \varphi^*(t) - \varphi_1(t)$.

As evident from Figure 15, smaller sampling intervals than a certain threshold give rise to an asymptotic value of B = 1. It so happens that the basis functions are effectively bandlimited for these sampling intervals, and $f(t) = \varphi^*(t) - \varphi_1(t)$ would be then orthogonal to $\varphi(t)$ within the analog domain. For those sampling intervals that are lower than the threshold, it is the W_2 norm of $f(t)$ that increases rather than the ratio of the W_2 and the L_2 norms, which maintain a constant value, leading to the asymptotic value of B. This observation is described in Figure 16, where the approximation error is now given in terms of the W_2 norms of both $f(t)$ and $\varphi(t)$.

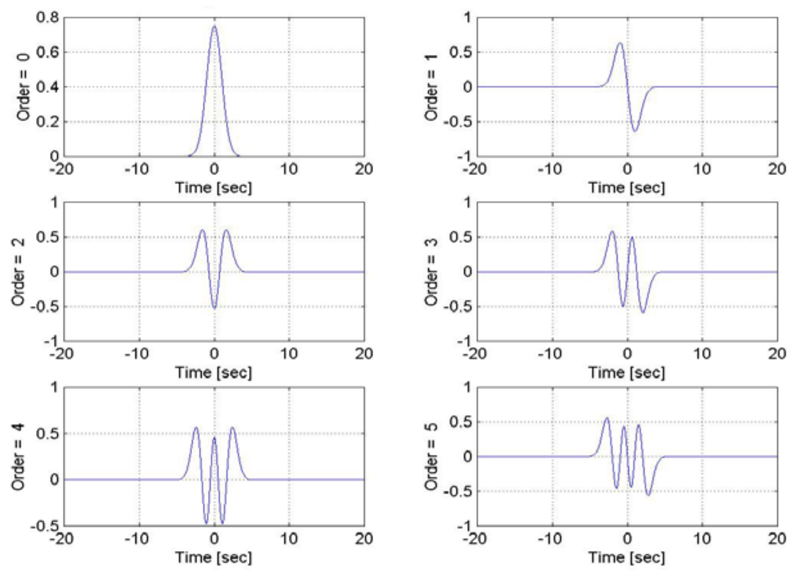

Figure 13: The Hermite functions constitute an orthonormal basis in L_2. The representation coefficients are then found by applying consecutive inner products $a_k = \langle f, \psi_k \rangle_{L2}$. Shown are $h_k(t)$ for $k = 0,1,\ldots,5$.

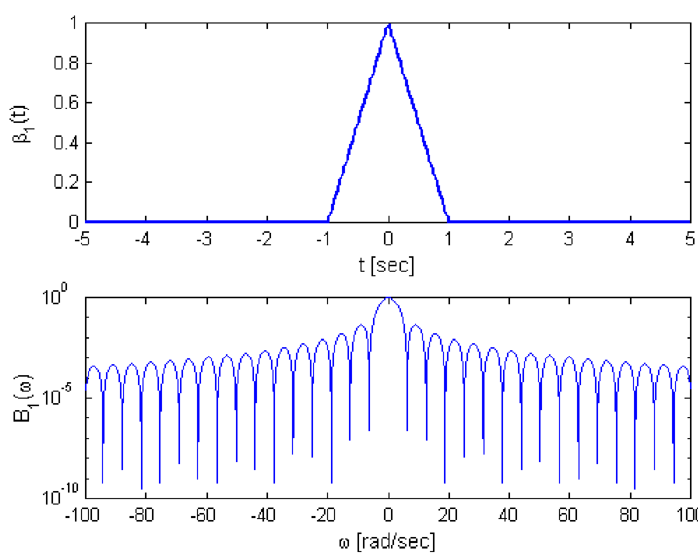

Figure 14: The B-spline of order 1, $\beta_1(t)$ in the time domain (top) and in the frequency domain (bottom).

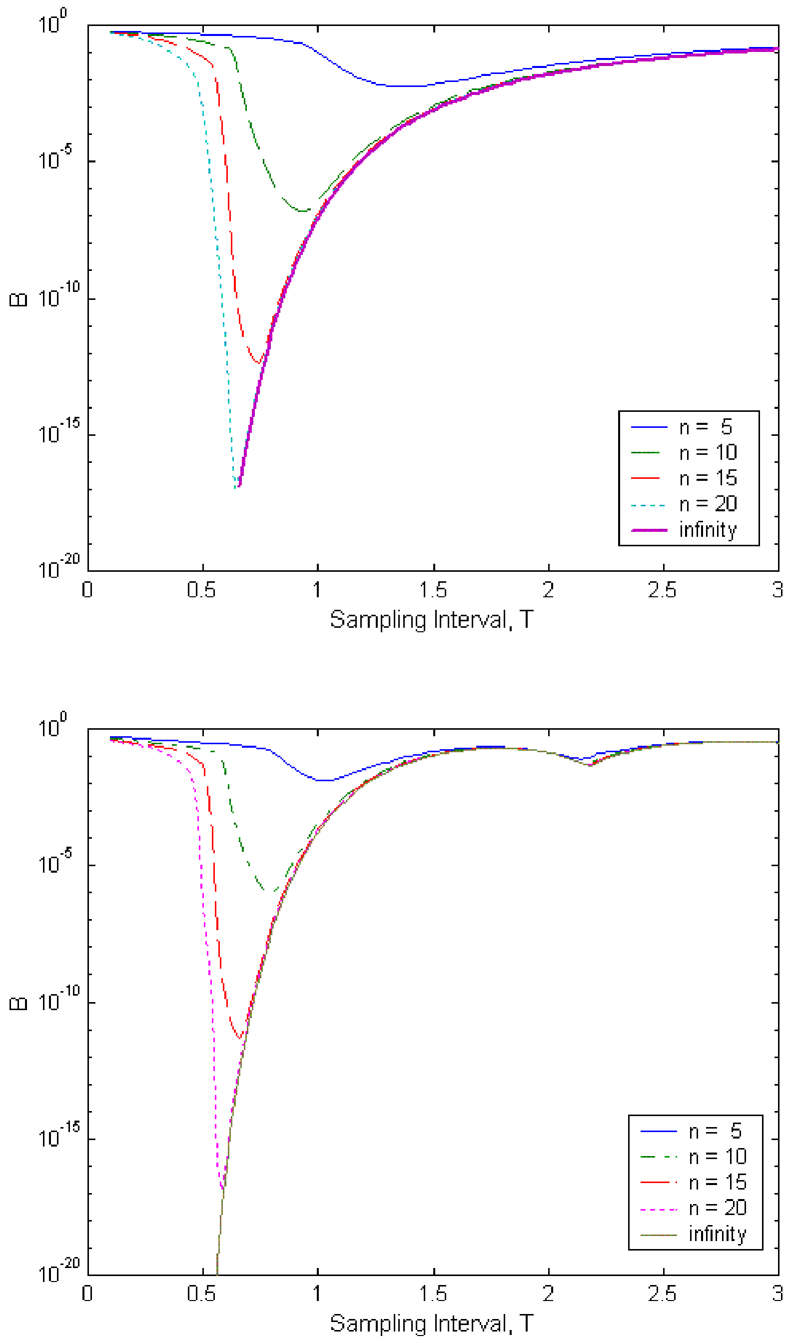

Figure 15: Upper bounds on the approximation error of $\langle f, h_k \rangle$ by their corresponding sampled versions. Here h_k is the Hermite function, where $k = 0$ (top) and $k = 5$ (bottom). The upper bound is given by $B \cdot \|f\|_{L2}$. Shown are upper bounds where the admissible functions, *f*, are Sobolev functions of several orders (n = 5,10,15,20 and infinity). For further details, refer to the text of Example 3.

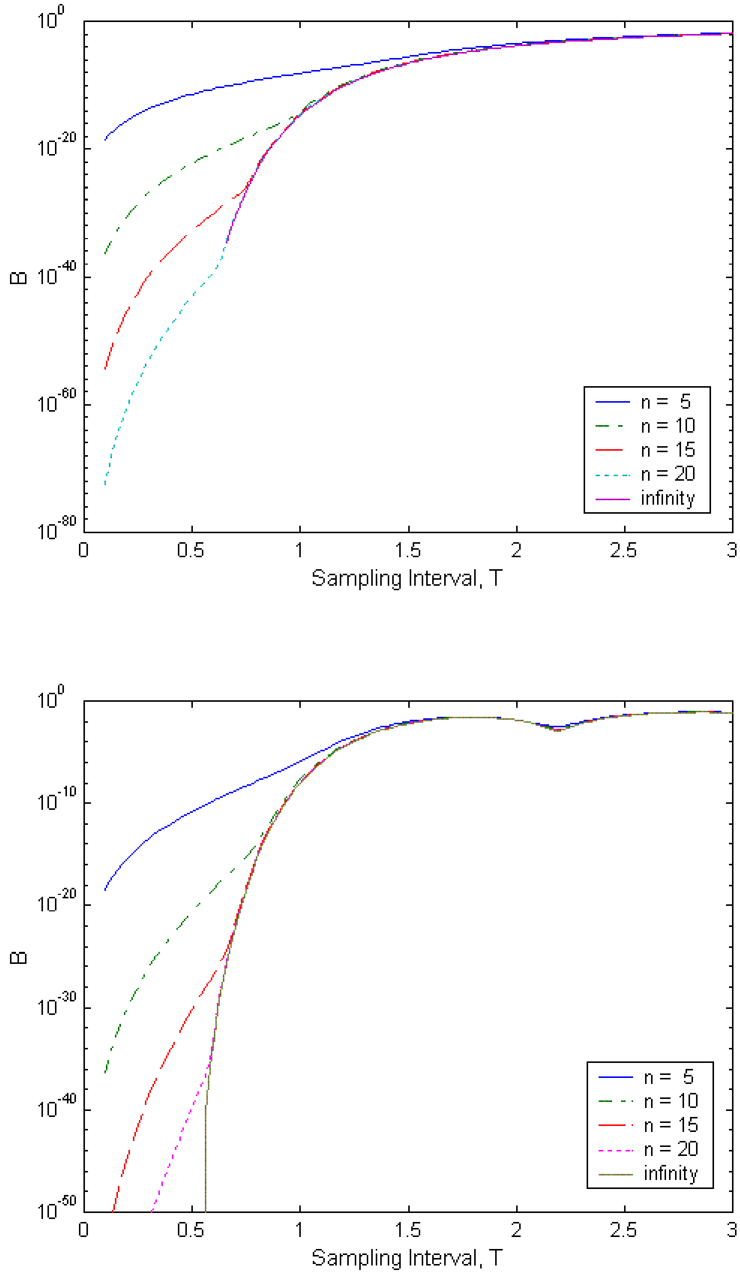

Figure 16: Upper bounds on the approximation error of $\langle f, h_k \rangle$ by their corresponding sampled versions. Here h_k is the Hermite function, where $k = 0$ (top) and $k = 5$ (bottom). The upper bound is given by $B \cdot \|f\|_{W2}$. Shown are upper bounds where the admissible functions, *f*, are Sobolev functions of several orders (n = 5,10,15,20 and infinity). For further details, refer to the text of Example 3.

Example 4: Sampling finite support functions, B-splines (Figure 14). A B-spline of order k is defined by $\beta_k(t) = \beta_0 * \beta_{k-1}$, where $\beta_0(t)$ is the box function with a support of [-½,½]. Similar to Example 3, the approximation error of a given inner product result is upper bounded by $B \cdot \|f\|_{L2}$ (Figure 17), or alternatively by $B \cdot \|f\|_{W2}$ (Figure 18). The spectral content of the B-spline function differs from that of the Hermite function. It decreases in an oscillatory manner rather than in a monotonically one. Thus, when considering relatively low sampling intervals, the spectral content of the worst case function $f(t) = \varphi^*(t)-\varphi_1(t)$ would occupy the same frequencies bands regardless of T, and this is the reason for the identical behavior evident in both Figure 17 and Figure 18.

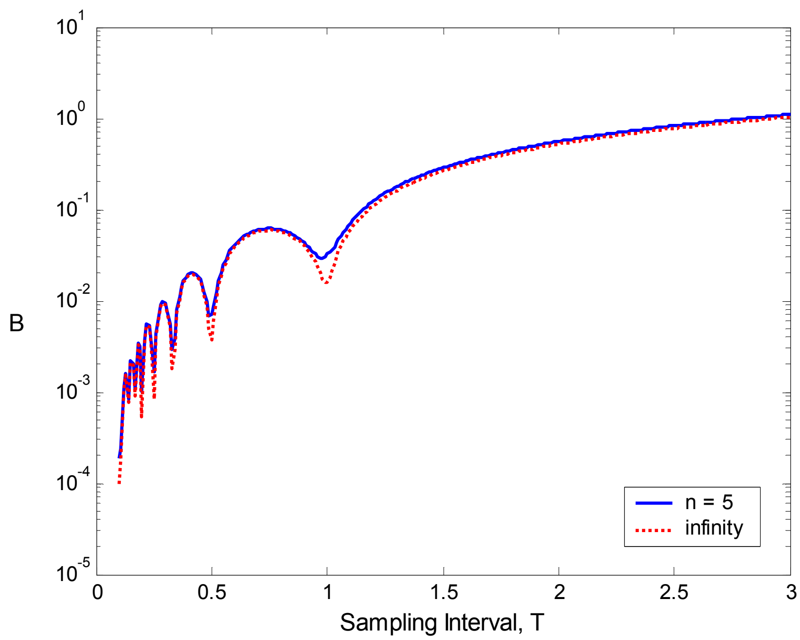

Figure 17: Upper bounds on the approximation error of $\langle f, \beta_k \rangle$ by their corresponding sampled versions. Here β_k is the B-spline function of order $k = 1$. The upper bound is given by $B \cdot \|f\|_{L2}$. Shown are upper bounds where the admissible functions, f, are Sobolev functions of orders n = 5 and infinity, respectively. For further details, refer to the text of Example 4.

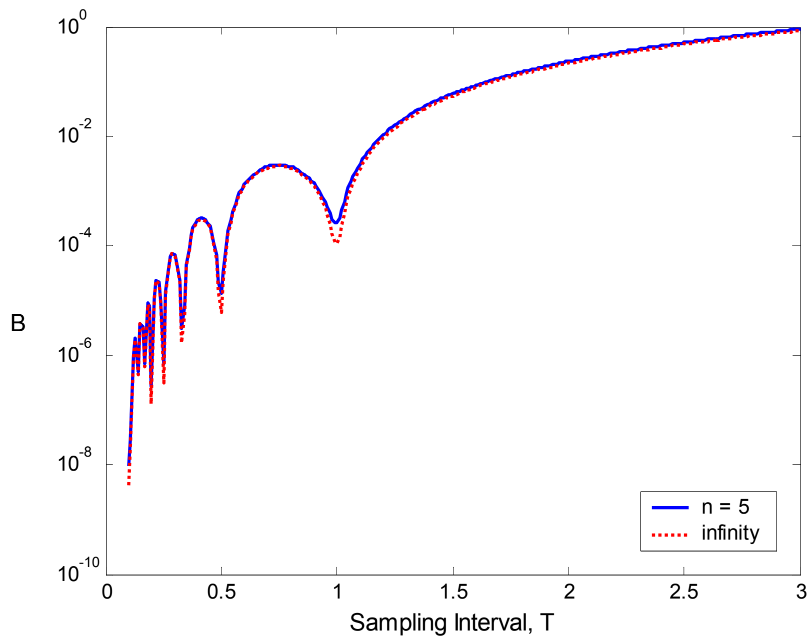

Figure 18: Upper bounds on the approximation error of $\langle f, \beta_k \rangle$ by their corresponding sampled versions. Here β_k is the B-spline function of order $k = 1$. The upper bound is given by $B \cdot \|f\|_{W2}$. Shown are upper bounds where the admissible functions, f, are Sobolev functions of orders n = 5 and infinity, respecrively. For further details, refer to the text of Example 4.

VI. Sampling Effects on L_2 Inner Product - Revisited

The standard approach to approximating $\langle f,\varphi \rangle_{L_2}$ by their corresponding samples is by no mean optimal, as shown in the next theorem.

Theorem 3: Let $\varphi(t)\in L_2$ be a known function. Given a sampling interval T, one can find an optimal $b[n]\in l_2$ that minimizes B with respect to the inequality

$$\forall f \in W_2, \quad \left| \left\langle f, \varphi \right\rangle_{L_2} - \left\langle S_T f, b \right\rangle_{l_2} \right| \le B \cdot \|f\|_{L_2}.$$

Here S_T is the sampling operator with interval T, $b[n]$ is the representation coefficients of the orthogonal projection, in the Sobolev sense, of $\varphi^* = \varphi * u$ onto $Span\{u(t\text{-}nT)\}_n$ (Figure 19) and $u(t)$ is as given in Theorem 2. B is as given in Theorem 2, replacing $\varphi(nT)$ by $b[n]$.

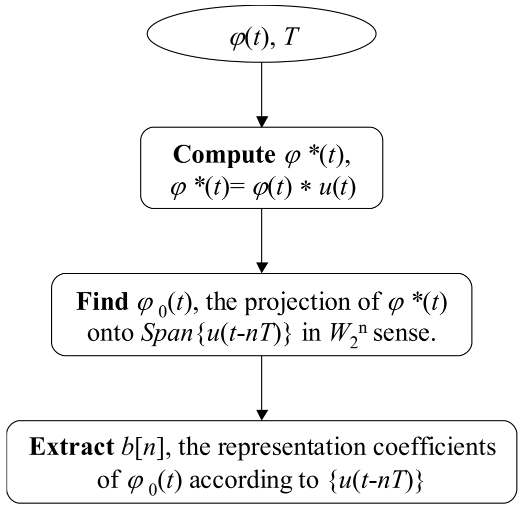

Figure 19: Optimal approximation of $\langle f, \varphi \rangle$ within the digital domain. Extracting $b[n]$ as shown here, and applying $\langle S_T f\,,\, b \rangle$ would yield the minimum possible upper bound for the approximation error.

Proof: Let φ_0 be the orthogonal projection (in W_2 sense) of φ^* onto $Span\{u(t\text{-}nT\}$. Define,

$$\varphi_2(t) = \sum_n b[n] \cdot \frac{1}{2} e^{-|t-nT|} \,,$$

$$\Delta\varphi(t) = \varphi_0(t) - \varphi_2(t) \,,$$

$$\varphi_\perp(t) = \varphi^*(t) - \varphi_0(t) \,.$$

It will be shown that $\Delta\varphi = 0$ minimizes B (Figure 20) : an explicit expression for B has been shown previously, leading to

$$B = \frac{\left\|\varphi^* - \varphi_2\right\|_{W_2}^2}{\left\|\varphi^* - \varphi_2\right\|_{L_2}} = \frac{\left\|\varphi^* - (\varphi_0 - \Delta\varphi)\right\|_{W_2}^2}{\left\|\varphi^* - (\varphi_0 - \Delta\varphi)\right\|_{L_2}} = \frac{\left\|\varphi_\perp + \Delta\varphi\right\|_{W_2}^2}{\left\|\varphi_\perp + \Delta\varphi\right\|_{L_2}}$$

$$\geq \frac{\left\|\varphi_\perp + \Delta\varphi\right\|_{W_2}^2}{\left\|\varphi_\perp + \Delta\varphi\right\|_{W_2}} = \left\|\varphi_\perp + \Delta\varphi\right\|_{W_2} = \left\|\varphi_\perp\right\|_{W_2} + \left\|\varphi g\right\|_{W_2} \quad .$$

It is evident that a minimization of B with respect to $\Delta\varphi$ yields $\Delta\varphi = 0$. □

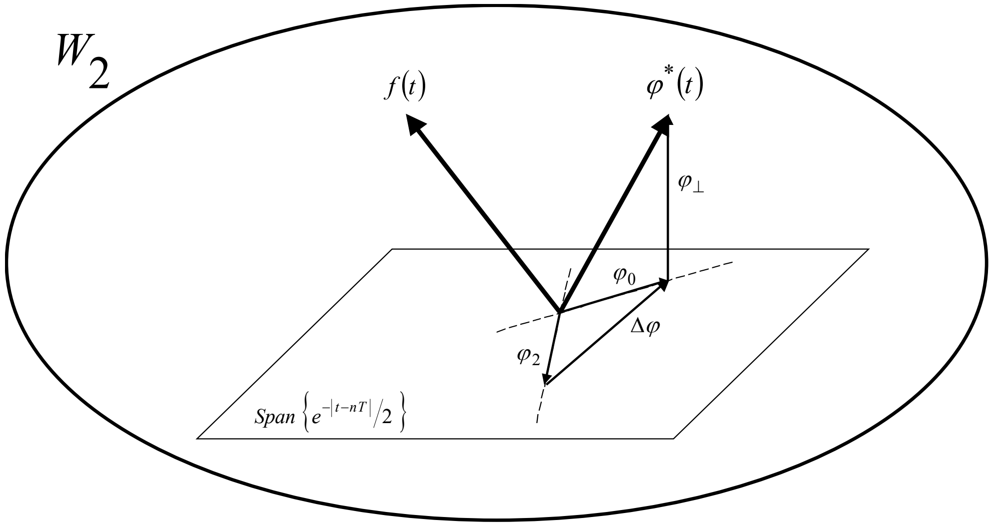

Figure 20: Vector relations for optimal approximation of $\langle f,\varphi\rangle_{L2}$. φ^* is such that $\langle f,\varphi\rangle_{L2} = \langle f,\varphi^*\rangle_{W2}$ and φ_2 is any functions within $Span\{u(t-nT)\}$. Setting φ_2 to be φ_0, namely the orthogonal projection of φ^* onto $Span\{u(t-nT)\}$ would result in the minimum possible upper bound on the approximation error.

There are functions $f(t)\in W_2$ that achieve this upper bound; one simply sets $f(t) = \varphi_\perp(t)$. It is also evident that the minimum possible value of B is $B_{\min} = \|\varphi_\perp\|^2_{W2} / \|\varphi_\perp\|^2_{L2}$. In addition, Theorem 3 can be further utilized by considering smoother admissible functions , similar to the generalization of Theorem 2. However, regardless of the Sobolev order of $f(t)$, the admissible functions to be sampled, the set $\{u(t-nT)\}$ is not orthogonal within W_2^n. Thus, in order to find the orthogonal projection of φ^* onto $Span\{u(t-nT)\}$, its biorthogonal set is to be found. Denoting $\{w_n(t)\}$ to be its biorthogonal set, the following lemma is given.

Lemma 8: $\{w_n(t)\}$, the biorthogonal counterpart of $\{u(t-nT)\}$ in W_2^n sense is given by,

$$w_n(t) = w(t-nT),$$
$$w(t) = \sum_k \alpha_k \cdot u(t-kT),$$
$$\alpha_k = F^{-1}\left\{\frac{1}{F\{u(kT)\}(\theta)}\right\}(k).$$

Proof: Each $\{w_n(t)\}$ is a linear combination of $\{u(t-nT)\}$:

$$w_n(t) = \sum_l \alpha_{m,l} \cdot u(t-lT)$$

In addition, it inherits the shift-invariant characteristics of $\{u(t\text{-}nT)\}$:

$$w_n(t) = w(t - nT) = \sum_l \alpha_l u_{l+n}(t) = \sum_l \alpha_{l-n} u_l(t)$$

The biorthonormality condition can now be written by,

$$\langle u_n, w_m \rangle_{W_2} = \left\langle u_n, \sum_l \alpha_{l-m} \cdot u_l \right\rangle_{W_2^n} = \delta(n-m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases},$$

which in turn becomes:

$$\sum_l \alpha_{l-m} \cdot \langle u_n, u_l \rangle_{W_2^n} = \sum_l \alpha_{l-m} \cdot h(n-l) = \delta(n-m),$$

where $h(k) = \langle u_k, u_0 \rangle_{W2}{}^n$. Setting $k = l\text{-}m$ and $\tilde{n} = n - m$ enables one to write:

$$\sum_k \alpha_k \cdot h(n-k-m) = \delta(n-m)$$
$$\sum_k \alpha_k \cdot h(\tilde{n}-k) = \delta(\tilde{n})$$
$$\{\alpha_k * h(k)\}(\tilde{n}) = \delta(\tilde{n})$$
$$\alpha_k = F^{-1}\left\{\frac{1}{H(\theta)}\right\}(k).$$

An explicit expression for $H(\theta)$ can be found:

$$\begin{aligned} h(k) &= \langle u_k, u_0 \rangle_{W_2^n} \\ &= \frac{1}{2\pi} \int U_k(\omega) \cdot \overline{U_0(\omega)} \cdot \left(1 + \omega^2 + \cdots \omega^{2n}\right) d\omega \\ &= \frac{1}{2\pi} \int \frac{e^{-j\omega kT}}{\left(1 + \omega^2 + \cdots \omega^{2n}\right)^2} \cdot \left(1 + \omega^2 + \cdots \omega^{2n}\right) d\omega \\ &= F^{-1}\left(\frac{1}{1 + \omega^2 + \cdots \omega^{2n}}\right)(-kT) = u(-kT) = u(kT) \end{aligned}$$

$$H(\theta) = \sum_k h(k) \cdot e^{-j\theta k} = \sum_k u(kT) \cdot e^{-j\theta k}$$
$$\alpha_k = F^{-1}\left\{\frac{1}{H(\theta)}\right\}(k).$$

which concludes the proof. □

Examples utilizing this optimal approximation scheme are currently under investigation.

VII. Conclusions

Approximating the inner product of two analog signals by their corresponding sampled versions has been considered. Interpreting the sampling process as a linear bounded operator, a tight upper bound on the resulted error has been derived and demonstrated. The only constraint imposed is that the functions to be sampled would be smooth. No constraints of bandlimited are assumed.

Furthermore, a new discrete approximation scheme for the inner product of the original signals has been proposed; It reduces the approximation error's upper bound to its minimum possible value, thus optimal.

The theorems presented in this work enable one to determine the maximum potential representation error induced by the sampling process. Our results are applicable to signal processing applications, where digital signal representation of analog signals is required, based on their sampled versions.

Acknowledgments

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Appendix: Proof of Theorem 1

Denote $a[n]$ to be the ensued sequence of sampling $f(t)$ by an interval *T*. The energy of $a[n]$ is then, $\|a\|_{l_2}^2 = \sum_n a^2[n]$. Considering the Fourier transform ([4]) of $a[n]$,

$$\|a\|_{l_2}^2 = \sum_n a^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}|A(\theta)|^2\,d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{T}\sum_k F\left(\frac{1}{T}(\theta - 2\pi k)\right)\right|^2 d\theta$$
$$= \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left|\sum_k F\left(\omega - \frac{2\pi k}{T}\right)\right|^2 d\omega\,.$$

Relying on [3], an upper bound on this latter expression is to be determined; the process of sampling yields replicas of $F(\omega)$, shifted by $2\pi k/T$. Let $\{I_k\}$ be the distinct intervals defined by,

$$I_k = \left\{\omega : \left|\omega - \frac{2\pi k}{T}\right| < \frac{\pi}{T}\right\}$$

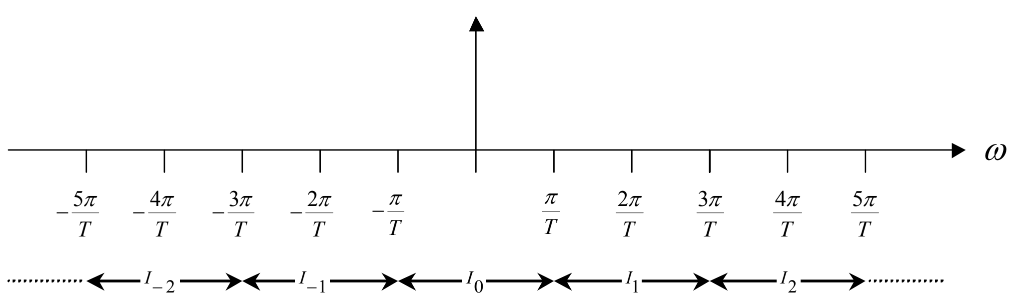

Figure 21: The sampling process yields replicas of $F(\omega)$, shifted by $2\pi k/T$. Shown are $\{I_k\}$, the replicated intervals.

Based on these intervals, let $F_k(\omega)$ to be $F(\omega)$ restricted to I_k, followed by a proper shift as to be centered around $\omega = 0$. The energy of $a[n]$ can be now written as:

$$
\begin{aligned}
\|a\|_{l_2}^2 &= \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left|\sum_k F\left(\omega-\frac{2\pi k}{T}\right)\right|^2 d\omega \\
&= \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left|\sum_k F_k(\omega)\right|^2 d\omega \\
&= \frac{1}{2\pi T}\left\|\sum_k F_k(\omega)\right\|_{L_2}^2 ,
\end{aligned}
$$

which can be further manipulated,

$$
\begin{aligned}
\|a\|_{l_2} &= \frac{1}{\sqrt{2\pi T}}\left\|\sum_k F_k(\omega)\right\|_{L_2} . \\
&\le \frac{1}{\sqrt{2\pi T}}\sum_k \|F_k(\omega)\|_{L_2}
\end{aligned}
$$

Denoting $g(t)=f'(t)$, one can write $F(\omega)=\frac{1}{j\omega}G(w)$, and for $k\neq 0$,

$$
\begin{aligned}
\|F_k(\omega)\|_{L_2} &= \sqrt{\int_{I_k}|F(\omega)|^2 d\omega} = \sqrt{\int_{I_k}\left|\frac{1}{j\omega}G(w)\right|^2 d\omega} \\
&\le \sqrt{\int_{I_k}\left(\frac{T}{(2|k|-1)\pi}\right)^2 |G(w)|^2 d\omega} \\
&= \frac{T}{(2|k|-1)\pi}\|G_k(w)\|_{L_2} .
\end{aligned}
$$

Using the Cauchy-Schawrtz inequality, this latter result enables one to derive the following expression:

$$
\begin{aligned}
\|a\|_{l_2} \le \frac{1}{\sqrt{2\pi T}}\sum_k \|F_k(\omega)\|_{L_2} &\le \frac{1}{\sqrt{2\pi T}}\|F_0(\omega)\|_{L_2} + \frac{1}{\sqrt{2\pi T}}\sum_{k\neq 0}\frac{T}{(2|k|-1)\pi}\|G_k\|_{L_2} \\
&\le \frac{1}{\sqrt{2\pi T}}\|F_0(\omega)\|_{L_2} + \frac{1}{\sqrt{2\pi T}}\left\|\sum_{k\neq 0}\frac{T}{(2|k|-1)\pi}\right\|_{l_2}\cdot\left\|\sum_{k\neq 0}\|G_k\|\right\|_{l_2} \\
&= \frac{1}{\sqrt{2\pi T}}\|F_0(\omega)\|_{L_2} + \frac{1}{4}\sqrt{\frac{T}{\pi}}\|G(\omega)\|_{L_2} \\
&= \frac{1}{\sqrt{T}}\|f(t)\|_{L_2} + \frac{1}{4}\sqrt{2T}\|f'(t)\|_{L_2} < \infty
\end{aligned}
$$

which proves the theorem. □

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