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Motion Aided Sampling and Reconstruction

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Abstract – Motivated by motion compensated filtering in image processing we consider the problem of sampling and reconstruction of signals with sampling rates below the Nyquist rate. It is assumed that temporal dependence can be induced via motion. This way, the data consists of both spatial and temporal sampling and we analyze here the conditions for reconstruction for a number of typical motions.

Keywords: Sampling, super resolution, reconstruction

EDICS: 2–RECO

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1 Introduction

Motivated by the problem of motion compensated filtering in video processing and super resolution problems (see e.g. [1]), we consider the following problem: Let $I_0(x)$ be a W_x-band limited signal, namely

$$\widehat{I_0}(\omega_x) = 0 \quad \text{for} \quad |\omega_x| > \frac{W_x}{2} \tag{1}$$

where $\widehat{I_0}(\omega_x)$ denotes the Fourier transform of $I_0(x)$. Suppose that $I_0(x)$ is sampled at intervals Δx where $\Delta x > \frac{2\pi}{W_x}$. It is well known that, in this case, $I_0(x)$ cannot be reconstructed from the data $\{I_0(\ell\Delta x)\}_{\ell\in\mathbb{Z}}$. Here we assume that the signal can be 'moved' and through this motion a temporal dimension is added to the problem. Namely, we generate a *two dimensional* signal given by

$$I(x,t) = I_0(x - f(t)) \tag{2}$$

where $f(t)$ represents the motion. Initially, we consider general motion. All we require is that $f(0) = 0$. Suppose now the data we generate results from sampling both in the x direction (with the same sampling interval Δx as before) and in the temporal direction with sampling interval Δt. Hence the data we have now is $\{I(\ell\Delta x, n\Delta t)\}_{(\ell,n)\in\mathbb{Z}\times\mathbb{Z}}$.

The problem we address in this paper is, under *what conditions* can one reconstruct $I_0(x)$ from the data $\{I(\ell\Delta x, n\Delta t)\}_{(\ell,n)\in\mathbb{Z}\times\mathbb{Z}}$. We provide a general reconstruction formula which is applicable to all cases treated here. Our main thrust is on existence and not on the associated question of practical reconstruction. Note that, since we assume the knowledge of $f(t)$, once $I_0(x)$ is reconstructed, $I(x,t)$ can be reconstructed as well by using (2).

Previous work on this problem includes the special case of constant velocity motion [5] and [4]. Also, some results have been reported on constant acceleration motion in [5] and [5]. However, in the latter case, the question of whether or not reconstruction is actually possible has not been addressed. Here we give necessary and sufficient conditions for reconstruction with general global motion. Constant velocity or acceleration are special cases of our result.

2 The General Case

We establish first the result for the general case as posed in the previous section and then relate the results to some specific types of motions.

Before proceeding let us introduce some notation which we will use in the sequel (this notation is common in Number Theory, see e.g. [6]). For integers $m, n \in \mathbb{Z}$, $m|n$ means that m *divides* into n, $\gcd(m,n)$ refers to the *greatest common devisor* of m and n and $m \equiv n \pmod{Q}$, for $Q \in \mathbb{N}$, means that $Q|(m-n)$ and is called *congruence relationship.* Given $Q \in \mathbb{N}$, $\overline{n}$ denotes the set of all integers congruent to n and the set $\{0, 1, ..., Q-1\}$ is called the *set of least (nonnegative) residues modulo* Q and the set $\left\{\overline{0}, \overline{1}, ..., \overline{Q-1}\right\}$ is the *least residue system modulo* Q.

Let us also denote

$$N = \left\lceil \frac{W_x \Delta x}{2\pi} \right\rceil \tag{3}$$

$$c = \frac{2\pi}{\Delta x} \tag{4}$$

and the mapping $F : Z \to [0, 1)$ defined by

$$F[n] = \left\lceil \frac{f(n\Delta t)}{\Delta x} \right\rceil - \frac{f(n\Delta t)}{\Delta x} \tag{5}$$

where $\lfloor a \rfloor$ denotes the largest integer smaller than a and $\lceil a \rceil$ denotes the smallest integer larger than a. Then, clearly

$$0 \leq \Delta x F[n] < \Delta x \quad \text{for all} \quad n \in \mathbb{Z} \tag{6}$$

We next introduce:

Definition *Denote by* $\overline{n_F}$ *the set of integers which result in the same value under the mapping* F *(namely,* $F(n) = F(n_1)$ *for all* $n, n_1 \in \overline{n_F}$*) and let* $\mathcal{N}_F$ *denote the set* $\mathcal{N}_F = \{n_m\}_{m=1}^{M}$ *such that* $\overline{(n_m)_F}$ *are all disjoint and* $\bigcup_{m=1}^{M} \overline{(n_m)_F} = \mathbb{Z}$*. Then, the Resolution Gain Factor (RGF)* M_F *is the number of elements in* $\mathcal{N}_F$*.*

We note that by the definition of $f(t)$ we can assume that always $0 \in \mathcal{N}_F$ and arbitrarily choose $n_1 = 0$.

Let

$$x_m = \Delta x F\left[n_i\right] \text{ for every } n_m \in \mathcal{N}_F \tag{7}$$

Then we can state our main result:

Theorem *$I_0(x)$ can be reconstructed from the data $\{I(\ell\Delta x, n\Delta t)\}_{(\ell,n)\in\mathbb{Z}\times\mathbb{Z}}$ if and only if*

$$M_F \geq N \tag{8}$$

The reconstruction formula is:

$$I_0(x) = \sum_{k=-\infty}^{\infty}\sum_{m=1}^{N} I\left(\left(k + \left\lceil \frac{f(n_m\Delta t)}{\Delta x}\right\rceil\right)\Delta x, n_m\Delta t\right)\varphi_m(x - k\Delta x) \tag{9}$$

where $x_m = \Delta x F[n_m]$, $n_m \in \mathcal{N}_F$,

$$\varphi_m(x) = \frac{1}{c}\int_{-\frac{cN}{2}}^{-\frac{c(N-2)}{2}} \Phi_m(\omega_x, x)\, e^{jx\omega_x} d\omega_x \tag{10}$$

and $\{\Phi_m(\omega_x, x)\}_{m=1}^{N}$ are the solutions of the following set of linear equations

$$\sum_{m=1}^{N} e^{j(\omega_x + rc)x_m}\Phi_m(\omega_x, x) = e^{jrcx} \quad \textit{for} \quad r = 1, ..., N \tag{11}$$

in which x is arbitrary and $\omega_x \in \left(-\frac{cN}{2}, -\frac{c(N-2)}{2}\right)$.

Proof: First we note that (8) ensures that (11) can indeed be written with distinct x_m's. Furthermore, the matrix of coefficients of the equations in (11) has the form

$$\begin{bmatrix} 1 & e^{j\omega_x x_2} & e^{j\omega_x x_3} & \cdots & e^{j\omega_x x_{N-}} \\ 1 & e^{j(\omega_x + c)x_2} & e^{j(\omega_x + c)x_3} & \cdots & e^{j(\omega_x + c)x_N} \\ 1 & e^{j(\omega_x + 2c)x_2} & e^{j(\omega_x + 2c)x_3} & \cdots & e^{j(\omega_x + 2c)x_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(\omega_x + Nc)x_2} & e^{j(\omega_x + Nc)x_3} & \cdots & e^{j(\omega_x + Nc)x_N} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{jcx_2} & e^{jcx_3} & \cdots & e^{jcx_N} \\ 1 & e^{j2cx_2} & e^{j2cx_3} & \cdots & e^{j2cx_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{jNcx_2} & e^{jNcx_3} & \cdots & e^{jNcx_N} \end{bmatrix} \cdot diag\left\{1, e^{j\omega_x x_2}, ..., e^{j\omega_x x_N}\right\}$$

Since $0 \leq x_m \neq x_r < \Delta x$, from (4) we have $e^{jcx_m} \neq e^{jcx_r}$. Then, recognizing that the first matrix is a Vandermonde matrix, this implies that it is nonsingular and so is the whole coefficient matrix. Hence, (8) ensures the existence of $\{\Phi_m(\omega_x, x)\}_{m=1}^{N}$.

Once this is established we have converted the problem to a special case of a result due to Papoulis (see e.g. [3] or [2]) and (9) follows.

An immediate observation from (9) is that one does not need the whole data set in order to reconstruct the signal. Furthermore, when we look at the data available as samples of $I_0(x)$ we note that in fact we have generated a periodic (or recurrent) sampling pattern with M_F irregularly spaced (in general) samples in each period. As the above theorem states, in order to be able to reconstruct $I_0(x)$ we need to make sure that $f(t)$ and Δt are such that (8) is satisfied. From an implementation point of view, if $M_F > N$, one would want to choose the subset of N values $\{x_m\}$ which are closest to being uniformly spaced in the interval $[0, \Delta x)$. This will result in the best conditioned matrix of coefficients in (11).

Remark *The reconstruction functions $\varphi_m(x - k\Delta x)$ in (9) can be viewed as impulse responses of reconstruction filters. This approach has been described in [8] where these same functions have been derived in a somewhat different way. However, the conditions for their existence, and hence, the conditions for reconstruction, are the same. This is our main interest in this paper.*

Next, we investigate some special cases of interest.

3 Special Cases

In this section we look at motions with constant velocity or constant acceleration which have been considered in the literature. We also look at periodic motions, which we feel are of practical interest. For each motion we determine the conditions on the motion parameters in relation to the sampling rates $\Delta x, \Delta t$ so that (8) is satisfied.

3.1 Motion with constant velocity

Let us consider the case

$$f(t) = Vt \tag{12}$$

Then, clearly, if $F[n_1] = F[n_2]$ for some $n_1 < n_2$ we must have

$$\frac{V\Delta t}{\Delta x} = \frac{m}{n_2 - n_1}$$

for some integer m. Hence, if $\frac{V\Delta t}{\Delta x}$ is an irrational number $F[n_1] \neq F[n_2]$ for any $n_1 \neq n_2$, which means that $\mathcal{N}_F = \mathbb{Z}$. Hence, $M_F = \infty$ and reconstruction (at least theoretically) is possible for any bandwidth signal.

Let us assume now that $\frac{V\Delta t}{\Delta x}$ is a rational number. Then we make the claim:

Claim 1 *Let* $\frac{V\Delta t}{\Delta x} = \frac{R}{Q}$ *such that* $\gcd(R, Q) = 1$. *Then*

$$M_F = Q \tag{13}$$

Proof: As we already observed, $F[n_2] = F[n_1]$ iff

$$\frac{m}{n_2 - n_1} = \frac{V\Delta t}{\Delta x} = \frac{R}{Q} \tag{14}$$

Since R and Q are coprime integers this will hold iff $Q|(n_2 - n_1)$ (namely, Q divides $n_2 - n_1$). Or

$$F[n_2] = F[n_1] \Leftrightarrow n_2 \equiv n_1 \,(\operatorname{mod} Q) \tag{15}$$

This means that $\mathcal{N}_F$ is the *set of least residues* (as mentioned earlier), namely, is equal to $\{0, 1, ..., Q-1\}$ which completes the proof.

Remark *This type of motion has been extensively considered in the literature and the reconstruction methods presented, typically use filters which are referred to as 'Motion Compensated Filters' (see e.g. [5]). The corresponding reconstruction formulae make use of the whole data set hence, they are not unique. Indeed, there is some freedom in the choice of these reconstruction filters. It can be shown that there exists a choice* *of filter which results in exactly the formula given by (9). However, in most of the existing literature, while recognizing that there are some 'critical velocities' for which reconstruction is not possible, no conditions have been presented. As far as we know, these type of reconstruction conditions appeared first in [4] for the constant velocity case only.*

Using our terminology here, the condition given in [4] is

$$\min_{0<n\leq\left\lfloor\frac{W_x\Delta x}{2\pi}\right\rfloor} F[n] > 0 \tag{16}$$

and can readily be shown to be equivalent to (13).

3.2 Motion with constant acceleration.

The study of this case is motivated, again, by related work in video processing (see e.g. [5] and [1]). In these references the authors show that frequency domain insights are not helpful here and propose to use short time Fourier transforms for the reconstruction. However, the question whether reconstruction is possible, at all, has not been addressed.

We have here

$$f(t) = at^2 \tag{17}$$

where a is constant and, without loss of generality, we assume it is positive. Then,

$$F[n] = \left\lceil \frac{a(\Delta t)^2}{\Delta x} n^2 \right\rceil - \frac{a(\Delta t)^2}{\Delta x} n^2 \tag{18}$$

and we observe again, that if $\frac{a(\Delta t)^2}{\Delta x}$ is irrational $F[n_1] \neq F[n_2]$ whenever $n_1 \neq n_2$, which in turn means that $M_F = \infty$ (recall that M_F is the count of elements in $\mathcal{N}_F$). Let us then consider the case when $\frac{a(\Delta t)^2}{\Delta x} = \frac{R}{Q}$ is a rational number ($\gcd(Q\ ,R) = 1$). We can readily show that $F[n_1] = F[n_2]$ for any integer value R if and only if it is true for $R = 1$, so from here on, without loss of generality, we will assume $\frac{a(\Delta t)^2}{\Delta x} = \frac{1}{Q}$.

We make the following claim:

Claim 2 *Let* $\frac{a(\Delta t)^2}{\Delta x} = \frac{1}{Q}$, $Q > 0$ *be a given integer. Its unique factorization (see e.g. [6])*

is given by

$$Q = 2^{m_o} \prod_{i=1}^{I} p_i^{m_i} \tag{19}$$

where $p_i > 2$ *are distinct prime numbers* $m_o \geq 0$ *and* $m_i > 0$, . *Then*

$$M_F = M_0 \prod_{i=1}^{I} \left(\left(\frac{p_i - 1}{2} \right) \left(p^{m_i - 1} + \left\lfloor \frac{m_i - 1}{2} \right\rfloor \right) + 1 \right) \tag{20}$$

where

$$M_0 = \begin{cases} 1 & \text{for } m_o = 0 \\ \frac{4 + 2^{m_o - 1}}{3} & \text{for } m_o \text{ even} \\ \frac{5 + 2^{m_o - 1}}{3} & \text{for } m_o \text{ odd} \end{cases} \tag{21}$$

Proof: (See Appendix A).

As an illustration, say $Q = 360 = 2^3 3^2 5$, using the above formula, the RGF is $M_F = 36$. We recall that this result means that if $\frac{a(\Delta t)^2}{\Delta x} = \frac{R}{360}$ (R any natural number) we could use the data generated to reconstruct a signal of bandwidth up to $36 \frac{2\pi}{\Delta x}$.

3.3 Periodic Motion

Perhaps the most interesting type of motion to consider for practical applications is periodic motion where we assume that there exists a $T > 0$ for which $f(t + T) = f(t)$. As is well known, sampling a periodic function does not necessarily result in a periodic sequence unless $\frac{\Delta t}{T} = \frac{R}{Q}$ - a rational number. Obviously, in this case, $f((n + Q)\Delta t) = f(n\Delta t)$ and if the resulting period Q, is less than N, reconstruction will be impossible. Hence, $\frac{\Delta t}{T}$ irrational or $Q \geq N$ is a necessary condition for reconstruction in this case. To generate necessary and sufficient conditions we need to consider more specific possible choices for periodic $f(t)$.

Case 1: $f(t) = V\left(\left\lceil \frac{t}{T} \right\rceil - \frac{t}{T}\right)$ *(See Fig. 1).*

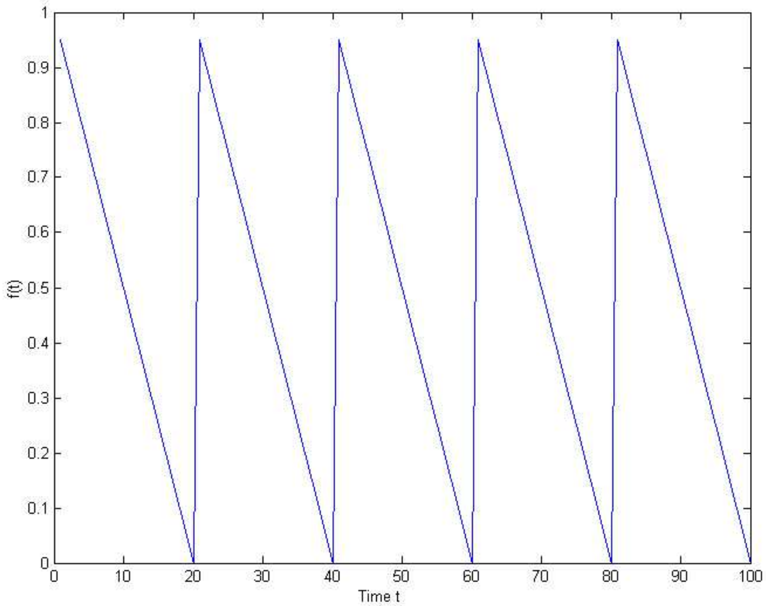

Figure 1: Periodic Motion - Case 1 ($T = 20$).

For this motion we get

$$F[n] = \left\lceil \frac{V\left(\left\lceil \frac{\Delta t}{T} n \right\rceil - \frac{\Delta t}{T} n\right)}{\Delta x} \right\rceil - \frac{V\left(\left\lceil \frac{\Delta t}{T} n \right\rceil - \frac{\Delta t}{T} n\right)}{\Delta x} \tag{22}$$

We can then prove the following claim:

Claim 3 *Let* $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ *and* $\frac{V}{\Delta x} = \frac{R_2}{Q_2}$ *with* $\gcd(Q_i, R_i) = 1$, $i = 1, 2$, *and let* $g = \gcd(Q_1, R_2)$. *Then the RGF is given by*

$$M_F = \min\left(Q_1, \frac{Q_1 Q_2}{g}\right) \tag{23}$$

Proof: Define

$$F_1[n] = \left\lceil \frac{R_1}{Q_1} n \right\rceil - \frac{R_1}{Q_1} n \tag{24}$$

then $0 \leq m_n = Q_1 F_1[n] < Q_1$ are integers and since, $F_1[n_1] = F_1[n_2] \Leftrightarrow R_1 n_1 \equiv R_1 n_2 \pmod{Q_1} \Leftrightarrow n_1 \equiv n_2 \pmod{Q_1}$, we have a one to one correspondence between the sets$\{0, 1, ..., Q_1 - 1\}$ and $\{m_0, m_1, ..., m_{Q_1 - 1}\}$. Furthermore, for any $n_1, n_2 \in \{0, 1, ..., Q_1 - 1\}$, $n_1 \equiv n_2 \pmod{Q_1} \Rightarrow n_1 = n_2$.

We can now rewrite $F[n]$ as

$$F[n] = \left\lceil \frac{R_2}{Q_2} F_1[n] \right\rceil - \frac{R_2}{Q_2} F_1[n]$$

or

$$\begin{aligned} F[n] &= \left\lceil \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \right\rceil - \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \\ &= \left\lceil \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \right\rceil - \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \end{aligned}$$

where, by definition, $\gcd\left(\frac{Q_1 Q_2}{g}, \frac{R_2}{g}\right) = 1$. Hence, $F[n_1] = F[n_2] \Leftrightarrow m_{n_1} \equiv m_{n_2} \left(\mathrm{mod}\ \frac{Q_1 Q_2}{g}\right) \Rightarrow$ the set of $n \in \{0, 1, ..., Q_1 - 1\}$ for which $F[n_1] \neq F[n_2]$ (namely, the set $\mathcal{N}_F$) is given by $\{0, 1, ..., Q_1 - 1\} \cap \left\{n : 0 \leq m_n < \frac{Q_1 Q_2}{g} - 1\right\} \Rightarrow M_F = \min\left(Q_1, \frac{Q_1 Q_2}{g}\right)$ as claimed.

It can be observed from this claim that, if $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ while $\frac{V}{\Delta x}$ is irrational $M_F = Q_1$. On the other hand, if $\frac{\Delta t}{T}$ is irrational $M_F = \infty$ no matter what $\frac{V}{\Delta x}$ is.

Case 2: $f(t) = V\left|\left\lceil \frac{t}{T} - \frac{1}{2} \right\rceil - \frac{t}{T}\right|$ *(See Fig. 2).*

Here

$$F[n] = \left\lceil \frac{V\left|\left\lceil \frac{\Delta t}{T} n - \frac{1}{2} \right\rceil - \frac{\Delta t}{T} n\right|}{\Delta x} \right\rceil - \frac{V\left|\left\lceil \frac{\Delta t}{T} n - \frac{1}{2} \right\rceil - \frac{\Delta t}{T} n\right|}{\Delta x} \tag{25}$$

For this case, we prove the following claim:

Claim 4 *Let* $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ *and* $\frac{V}{\Delta x} = \frac{R_2}{Q_2}$ *with* $\gcd(Q_i, R_i) = 1$, $i = 1, 2$, *and let* $g = \gcd(Q_1, R_2)$. *Then the RGF is given by*

$$M_F = \min\left(\left\lfloor \frac{Q_1 - 1}{2} \right\rfloor + 1, \frac{Q_1 Q_2}{g}\right) \tag{26}$$

Proof: The proof is quite similar to the proof of Claim 3. Define

$$F_1[n] = \left|\left\lceil \frac{R_1}{Q_1} n - \frac{1}{2} \right\rceil - \frac{R_1}{Q_1} n\right| \tag{27}$$

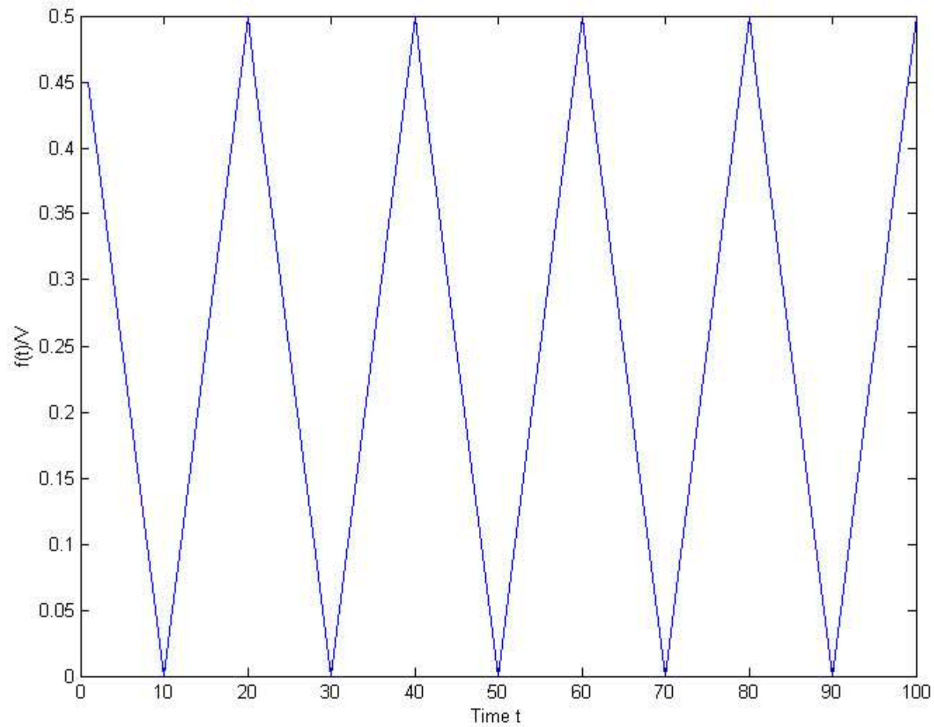

Figure 2: Periodic Motion - Case 2 ($T = 20$).

then $0 \leq m_n = Q_1 F_1[n] \leq \left\lfloor \frac{Q_1-1}{2} \right\rfloor$ are integers and since, $F_1[n_1] = F_1[n_2] \Leftrightarrow R_1 n_1 \equiv R_1 n_2 \pmod{Q_1}$ *or* $R_1 n_1 \equiv -R_1 n_2 \pmod{Q_1}$

$\Leftrightarrow n_1 \equiv n_2 \pmod{Q_1}$ *or* $n_1 \equiv -n_2 \pmod{Q_1}$, we have a one to one correspondence between the sets $\left\{0, 1, ..., \left\lfloor \frac{Q_1-1}{2} \right\rfloor\right\}$ and $\left\{m_0, m_1, ..., m_{\left\lfloor \frac{Q_1-1}{2} \right\rfloor}\right\}$. Furthermore, for any $n_1, n_2 \in \left\{0, 1, ..., \left\lfloor \frac{Q_1-1}{2} \right\rfloor\right\}$, $n_1 \equiv n_2 \pmod{Q_1}$ *or* $n_1 \equiv -n_2 \pmod{Q_1} \Rightarrow n_1 = n_2$.

We can now rewrite $F[n]$ as

$$F[n] = \left\lceil \frac{R_2}{Q_2} F_1[n] \right\rceil - \frac{R_2}{Q_2} F_1[n]$$

or

$$F[n] = \left\lceil \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \right\rceil - \frac{R_2}{Q_1 Q_2} Q_1 F_1[n]$$
$$= \left\lceil \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \right\rceil - \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n$$

where, by definition, $\gcd\left(\frac{Q_1Q_2}{g}, \frac{R_2}{g}\right) = 1$. Hence,

$$F[n_1] = F[n_2] \Leftrightarrow m_{n_1} \equiv m_{n_2} \left(\mathrm{mod}\,\frac{Q_1Q_2}{g}\right) \Rightarrow$$

the set of $n \in \left\{0, 1, ..., \left\lfloor \frac{Q_1-1}{2} \right\rfloor\right\}$ for which $F[n_1] \neq F[n_2]$ (namely, the set $\mathcal{N}_F$) is given by $\left\{0, 1, ..., \left\lfloor \frac{Q_1-1}{2} \right\rfloor\right\} \cap \left\{n : 0 \leq m_n < \frac{Q_1Q_2}{g} - 1\right\} \Rightarrow M_F = \min\left(\left\lfloor \frac{Q_1-1}{2} \right\rfloor + 1, \frac{Q_1Q_2}{g}\right)$ as claimed.

Here too, it can be observed from the claim above that, if $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ while $\frac{V}{\Delta x}$ is irrational, $M_F = \left\lfloor \frac{Q_1-1}{2} \right\rfloor + 1$. On the other hand, if $\frac{\Delta t}{T}$ is irrational, $M_F = \infty$ no matter what $\frac{V}{\Delta x} \neq 0$ is.

4 Conclusion

We have addressed the problem of using motion as a temporal enhancement of spatial sampling rates. While a number of algorithms for reconstruction of signals from their combined spatial and temporal samples have been previously described in the literature, most current results do not address the question 'when is this reconstruction possible?'. In this paper we analyze a number of typical motions, each with its own parameters, and derive necessary and sufficient conditions which guarantee, in each case, the feasibility of signal reconstruction.

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References

[1] A. Murat Tekalp, *Digital Video Processing*, Prentice Hall PTR, 1995.

[2] A. Zayed, *Advances in Shannon's Sampling Theory*, CRC Press, London 1993.

[3] A. Papoulis, *Signal Analysis*, McGraw-Hill, New York 1977.

[4] N. Goldberg, A. Feuer and G.C. Goodwin, 'Super-Resolution Reconstruction Using Spatio-Temporal Filtering' (submitted to) Journal of Visual Communication and Image Representation, 2002.

[5] A. J. Patti, M. I. Sezan and A. M. Tekalp, 'Digital video standards conversion in the presence of accelerated motion', Signal Proc.: Image Communication, Vol. 6, pp. 213-227, June 1994.

[6] R. A. Mollin, *Fundamental Number Theory with Applications,* CRC Press, London 1998.

[7] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York 1972.

[8] J. L. Brown, 'Multi-Channel Sampling of Low-Pass Signals', IEEE Trans. on Circ. and Sys., Vol. CAS-28, No. 2, Feb. 1981.

Appendix A
The proof of Claim 2

To prove Claim 2 we first define a set of integers $\widetilde{\mathcal{N}}$

$$\widetilde{\mathcal{N}} = \left\{ 0 \le \widetilde{n}_j < Q : (n_j)^2 \equiv \widetilde{n}_j \, (\mathrm{mod}\, Q) \, , n_j \in \mathcal{N}_F \right\} \tag{A.1}$$

Note that, from Definition 2 of $\mathcal{N}_F$, we can readily see that for every $n \in \mathbb{Z}$, $(n)^2$ is congruent modulo Q to exactly one element in $\widetilde{\mathcal{N}}$. This set is commonly referred to as the *set of (least ,nonnegative) quadratic residues* (see e.g. [7]). Furthermore, since by definition, for any $n_j, n_k \in \mathcal{N}_F$, $j \neq k$, $(n_j)^2$ and $(n_k)^2$are not congruent modulo Q, we have $\widetilde{n}_j \neq \widetilde{n}_k$. Hence, there is a one-to-one correspondence between $\mathcal{N}_F$ and $\widetilde{\mathcal{N}}$ (namely, the two sets have the same number of elements, M_F). Next we need to prove the following three preliminary results:

Claim A1 *Let $Q_1, Q_2, Q \in \mathbb{N}$, $\widetilde{\mathcal{N}_1}, \widetilde{\mathcal{N}_2}, \widetilde{\mathcal{N}}$ the corresponding sets of quadratic residues and $M_{1,F}, M_{2,F}, M_F$ the respective element counts of these sets. Then, if Q_1, Q_2 are (positive) coprime and $Q = Q_1 Q_2$ we obtain*

$$M_F = M_{1,F} \cdot M_{2,F} \tag{A.2}$$

Proof: Let $\widetilde{n}_j^1 \in \widetilde{\mathcal{N}_1}$ and $\widetilde{n}_k^2 \in \widetilde{\mathcal{N}_2}$. Consider the following system of congruences:

$$\begin{aligned} (x)^2 &\equiv \widetilde{n}_j^1 \, (\mathrm{mod}\, Q_1) \\ (x)^2 &\equiv \widetilde{n}_k^2 \, (\mathrm{mod}\, Q_2) \end{aligned} \tag{A.3}$$

Then, by the Chinese Remainder Theorem (see e.g. [6] or [7]) and (A.1) we know that this system always has solutions and the squares of any two solutions differ by a multiple of Q. Define $\widetilde{n}_{j,k}$ as

$$\widetilde{n}_{j,k} = \min_{r \in \mathbb{Z}} \left\{ (x)^2 - rQ \ge 0 \right\} \tag{A.4}$$

then $0 \le \widetilde{n}_{j,k} < Q$ and, since $\widetilde{n}_j^1$ is not congruent to any other element of $\widetilde{\mathcal{N}_1}$ neither is $\widetilde{n}_k^2$ to any other element of $\widetilde{\mathcal{N}_2}$, $\widetilde{n}_{j,k} \neq \widetilde{n}_{\ell,m}$ whenever $(j,k) \neq (\ell, m)$. Hence, the set

$\{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\}$ contains exactly $M_{1,F} \cdot M_{2,F}$ elements. Next we show that

$$\widetilde{\mathcal{N}} = \{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\} \tag{A.5}$$

From (A.3) and (A.4) we have $0 \leq \widetilde{n}_{j,k} < Q$ and $(x)^2 \equiv \widetilde{n}_{j,k} \,(\mathrm{mod}\, Q)$ so $\widetilde{n}_{j,k} \in \widetilde{\mathcal{N}} \Rightarrow \widetilde{\mathcal{N}} \subseteq \{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\}$.

Let $\widetilde{n}_r \in \widetilde{\mathcal{N}}$. Hence, there exists $n \in \mathbb{Z}$ such that $(n)^2 \equiv \widetilde{n}_r \,(\mathrm{mod}\, Q)$. On the other hand, by definition of $\widetilde{\mathcal{N}_1}$ and $\widetilde{\mathcal{N}_2}$, there exist $\widetilde{n}^1_\ell \in \widetilde{\mathcal{N}_1}$ and $\widetilde{n}^2_m \in \widetilde{\mathcal{N}_2}$ such that

$$\begin{aligned} (n)^2 &\equiv \widetilde{n}^1_\ell \,(\mathrm{mod}\, Q_1) \\ &\equiv \widetilde{n}^2_m \,(\mathrm{mod}\, Q_2) \end{aligned}$$

Hence, for the corresponding $\widetilde{n}_{\ell,m}$we have $(n)^2 \equiv \widetilde{n}_{\ell,m} \,(\mathrm{mod}\, Q) \equiv \widetilde{n}_r \,(\mathrm{mod}\, Q)$ and since $0 \leq \widetilde{n}_r, \widetilde{n}_{\ell,m} < Q$ we must have $\widetilde{n}_r = \widetilde{n}_{\ell,m}$. Hence, $\widetilde{\mathcal{N}} \supseteq \{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\}$, which establishes (A.5). This completes the proof.

Claim A2 *Let* $Q = 2^{m_o}$. *Then*

$$M_o \,(= M_F) = \begin{cases} 1 & \text{for } m_o = 0 \\ \frac{4+2^{m_o-1}}{3} & \text{for } m_o \text{ even} \\ \frac{5+2^{m_o-1}}{3} & \text{for } m_o \text{ odd} \end{cases} \tag{A.6}$$

Proof: Considering the congruence relationship $x^2 \equiv \widetilde{n}_i \,(\mathrm{mod}\, Q)$ we use a result in [6] which states that for odd n_i an x satisfying the the congruence relationship exists iff $\widetilde{n}_i \equiv 1 \,(\mathrm{mod}\, g)$ where g is the greatest common devisor of $Q = 2^{m_o}$ and 8. This means that for the set $\{1, 3, ..., 2^{m_o} - 1\}$ the number of integers for which an x exists is given by

$$\begin{aligned} 1 \text{ for } m_o &= 1, 2 \\ 2^{m_o-3} \text{ for } m_o &\geq 3 \end{aligned} \tag{A.7}$$

Next we observe that we can write

$$\{0, 1, ..., 2^{m_o} - 1\} = \{0\} \bigcup_{r=1}^{m_o} 2^{m_o-r} \{1, 3, ..., 2^r - 1\} \tag{A.8}$$

and that only the sets where $m_o - r$ is even contain elements for which an x exists. Hence we consider separately odd and even m_o. For m_o odd only the sets with odd $r\,(=2r_o+1)$ are counted and we get (note that $\{0\}$ counts for 1)

$$\begin{aligned} M_o &= 2 + \sum_{r_o=1}^{\frac{m_o-1}{2}} 2^{(2r_o+1)-3} \\ &= 2 + \frac{2^{m_o-1}-1}{3} \\ &= \frac{2^{m_o-1}+5}{3} \end{aligned}$$

For m_o even only the sets with even $r\,(=2r_o)$ are counted and we get

$$\begin{aligned} M_o &= 2 + \sum_{r_o=2}^{\frac{m_o}{2}} 2^{2r_o-3} \\ &= 2 + 2\frac{2^{m_o-2}-1}{3} \\ &= \frac{2^{m_o-1}+4}{3} \end{aligned}$$

which completes the proof.

Claim A3 *Let $Q = p^m$ where $p > 2$ is prime. Then*

$$M_F = 1 + \frac{p-1}{2}\left(\left\lfloor \frac{m-1}{2} \right\rfloor + p^{m-1}\right)$$

Proof: We consider the set $\{0, 1, ..., p^m - 1\}$ and want to find for how many of its elements $\widetilde{n}_i$, the congruence $x^2 \equiv \widetilde{n}_i \pmod{Q}$ has solutions. Since we can write

$$\{0, 1, ..., p^m - 1\} = \left\{0, p, 2p, ..., p^2, ..., p^m - p\right\} \bigcup \{1, 2, ..., p-1, p+1, ..., p^m - 1\}$$

where one set contains all the elements which are devisable by p, there are p^{m-1} of them, and the second are the remaining elements, $p^{m-1}(p-1)$ of them. Or

$$\{0, 1, ..., p^m - 1\} = \{0\} \bigcup_{r=1}^{m-1} p^r \{1, 2, .., p-1\} \bigcup_{k=0}^{p^{m-1}-1} \{kp+1, ..., kp+p-1\}$$

As stated earlier, whether an element $0 \leq \widetilde{n} < p^m$ belongs to $\widetilde{\mathcal{N}}$ is equivalent to whether the congruence $x^2 \equiv \widetilde{n} \pmod{p^m}$ has a solution. From a result in [7] we have that if p does not divide into $\widetilde{n}$ the above congruence has a solution iff the congruence $x^2 \equiv \widetilde{n} \pmod{p}$ has a solution. Furthermore, in the set $\{1, 2, ..., p-1\}$ there are exactly $\frac{p-1}{2}$ elements for which the above congruence has a solution. Noting also that $x^2 \equiv (\widetilde{n}p^r) \pmod{p^m}$ can have a solution iff r is even and if the congruence $x^2 \equiv \widetilde{n} \pmod{p^{m-r}}$ has a solution and that $kp + \widetilde{n} \equiv \widetilde{n} \pmod{p}$ we can conclude that (including the element 0 for which the congruence has a trivial solution)

$$\begin{aligned} M_F &= 1 + \left\lfloor \frac{m-1}{2} \right\rfloor \frac{p-1}{2} + p^{m-1} \frac{p-1}{2} \\ &= 1 + \frac{p-1}{2} \left(\left\lfloor \frac{m-1}{2} \right\rfloor + p^{m-1} \right) \end{aligned}$$

which completes the proof.

The proof of Claim 2 then consists of recalling the factorization of a general Q as given in (19) and applying the three claims above.