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Metamaterial Slabs Coupling

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Abstract

Metamaterial slab coupling to another metamaterial slab or to a dielectric slab has been studied, taking into consideration regular trigonometric modes as well as hyperbolic modes. The first TM even mode has been given special attention since it exhibits interesting filtering properties. The coupling coefficient frequency variation of the structures has been investigated. It has been found that, for a given frequency, the thinner the width of the slabs, or the lower their permittivity, the higher the coupling between them.

1. Introduction

Veselago [1] has shown that the Poynting vector of a plane wave is anti-parallel to its phase-velocity vector in materials whose permittivity and permeability are both negative. These materials have been consequently termed metamaterials, backward-wave materials, left-handed materials, etc. Lindell et al. [2] and Lakhtakia et al. [3] have reviewed these materials. Pendry [4] has shown that a metamaterial slab could focus evanescent modes and resolve objects only a few nanometers wide in the optical domain. Ziolkowsky et al. [5] have studied metamaterials both analytically and numerically. Engheta [6] has made a theoretical analysis on thin subwavelength cavity resonators containing metamaterials. Alu et al. [7] have studied the radiation from a traveling-wave current sheet at the interface between a conventional material and a metamaterial. Cory et al. [8] have studied the longitudinal propagation coefficient dependence on frequency of regular modes having a real transverse wave-number. Wu et al. [9] have shown that additional modes having an imaginary transverse wave-number coexist with the regular modes. In this work we have taken into consideration the contribution of the two kinds of modes to the coupling processes.

2. The propagation coefficients

We have studied the coupling between two slabs of widths $2t_1$ and $2t_2$ respectively separated by a distance $2d$ as shown in Fig. 1. These slabs could be both metamaterial with permittivities $-\kappa_1\varepsilon_0$ and $-\kappa_2\varepsilon_0$ respectively (κ_1 and κ_2 are positive numbers), and permeability $-\mu_0$, or one of them could be dielectric with permittivity $\kappa_{1,2}\varepsilon_0$ and permeability μ_0. The surrounding medium is air. We have assumed throughout that the field components of the electromagnetic waves have a time-dependence $e^{j\omega t}$ and that they propagate in the z-direction with an $e^{-j\beta z}$ variation.

The transverse propagation coefficient k_x of regular trigonometric modes ($k_x \triangleq \ell$ inside the slab and $k_x \triangleq -jh$ outside it) for a single metamaterial slab [8]–[9] could be found from the following equation:

$$(\ell_i t_i)^2 + (h_i t_i)^2 = (k_0 t_i)^2(\kappa_i - 1) \qquad (i = 1, 2) \tag{1}$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ and κ_i is the relative permittivity of slab i, in conjunction with one of the following equations:

$$(\ell_i t_i) \cdot \tan(\ell_i t_i) = -\kappa_i (h_i t_i) \qquad \text{(TM ODD)} \qquad (i = 1, 2) \tag{2a}$$

$$(\ell_i t_i) \cdot \cot(\ell_i t_i) = \kappa_i (h_i t_i) \qquad \text{(TM EVEN)} \qquad (i = 1, 2) \tag{2b}$$

$$(\ell_i t_i) \cdot \tan(\ell_i t_i) = -(h_i t_i) \qquad \text{(TE ODD)} \qquad (i = 1, 2) \tag{3a}$$

$$(\ell_i t_i) \cdot \cot(\ell_i t_i) = (h_i t_i) \qquad \text{(TE EVEN)} \qquad (i = 1, 2)\,. \tag{3b}$$

The longitudinal propagation coefficient β_i for the single metamaterial slab could be found subsequently, once the transverse propagation coefficients have been determined, according to the following equation:

$$\beta_i^2 = \kappa_i k_o^2 - \ell_i^2 = k_0^2 + h_i^2 \qquad (i = 1, 2)\,. \tag{4}$$

Explicit expressions for the field components of these modes can be found in [8]–[9].

The transverse propagation coefficient k_x of additional hyperbolic modes [9] ($k_x \triangleq \pm j\ell$ inside the slab and $k_x \triangleq -jh$ outside it) for a single metamaterial slab, could be found from the following equation:

$$-(\ell_i t_i)^2 + (h_i t_i)^2 = (k_0 t_i)^2 (\kappa_i - 1) \qquad (i = 1, 2) \tag{5}$$

in conjunction with one of the following equations:

$$(\ell_i t_i) \cdot \tanh(\ell_i t_i) = \kappa_i (h_i t_i) \qquad \text{(TM ODD)} \qquad (i = 1, 2) \tag{6a}$$

$$(\ell_i t_i) \cdot \coth(\ell_i t_i) = \kappa_i (h_i t_i) \qquad \text{(TM EVEN)} \qquad (i = 1, 2) \tag{6b}$$

$$(\ell_i t_i) \cdot \tanh(\ell_i t_i) = (h_i t_i) \qquad \text{(TE ODD)} \qquad (i = 1, 2) \tag{7a}$$

$$(\ell_i t_i) \cdot \coth(\ell_i t_i) = (h_i t_i) \qquad \text{(TE EVEN)} \qquad (i = 1, 2)\,. \tag{7b}$$

The longitudinal propagation coefficient β_i for the single metamaterial slab could be found subsequently, once the transverse propagation coefficients have been determined, according to the following equation:

$$\beta_i^2 = \kappa_i k_0^2 + \ell_i^2 = k_0^2 + h_i^2 \qquad (i = 1, 2)\,. \tag{8}$$

Explicit expressions for the field components of these modes can be found in [9].

We will be principally concerned with the first TM even propagation mode since its dispersion diagram has shown interesting filtering properties [8]. The hyperbolic first TM even mode has no cut-off [9]. When $k_0 t = 0$, $\beta t = \ell t = ht = \operatorname{arc\,coth} \kappa$ as can be seen from Eqs. (5), (6b) and (8). This mode ends when $\ell t = 0$, from which we deduce that $ht = 1/\kappa$, $k_0 t = 1/\kappa\sqrt{\kappa - 1}$ and $\beta t = 1/\sqrt{\kappa(\kappa - 1)}$, according to the same equations. Precisely at this point begins the trigonometric first TM even mode as can be seen from Eqs. (1), (2b) and (4). This mode ends when $ht = 0$, from which we deduce that $\ell t = \pi/2$ and $k_0 t = \beta t = \pi/2\sqrt{\kappa - 1}$, according to the same equations.

3. The coupling coefficient

It can be shown [10] that for weak coupling, the total electric field is given as follows:

$$\mathbf{E} = A(z)\mathbf{e}_1(x, y)e^{-j\beta_1 z} + B(z)\mathbf{e}_2(x, y)e^{-j\beta_2 z} \tag{9}$$

where $\mathbf{e}_1(x, y)$ and $\mathbf{e}_2(x, y)$ describe the fields variation with the transverse coordinates (x, y) for guides 1 and 2, while β_1 and β_2 are the longitudinal propagation coefficients for these guides when they are separated. It is assumed that only one mode propagates along the guides. The differential equation of the (z-dependent) amplitude $A(z)$ is given as follows [10]:

$$\frac{d^2 A(z)}{dz^2} + j2\xi \frac{dA(z)}{dz} + K^2 A(z) = 0 \tag{10}$$

where $\xi = \frac{1}{2}(\beta_2 - \beta_1)$ and K is the coupling coefficient of the guides.

The coupling coefficient K which appears in Eq. (10) depends on $\mathbf{e}_1(x, y)$ and on $\mathbf{e}_2(x, y)$, and is not a function of z. Numerous methods have been devised to evaluate this quantity and the method due to Arnaud [11] has been adopted in this paper. It has been used successfully to find the transverse coupling between adjacent guides of various shapes and composition [12]. It is given by:

$$K^2 = \frac{\left[\oint_C (\mathbf{E}_1 \times \mathbf{H}_2^+ - \mathbf{E}_2^+ \times \mathbf{H}_1) \cdot \mathbf{d}L\right]^2}{4\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathbf{E}_1 \times \mathbf{H}_1) \cdot \hat{\mathbf{z}}\, dx dy\right]\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathbf{E}_2 \times \mathbf{H}_2) \cdot \hat{\mathbf{z}}\, dx dy\right]} = \frac{\mathcal{C}^2}{\mathcal{P}_1 \mathcal{P}_2}\,. \tag{11}$$

The indices 1 and 2 refer to the appropriate guides, the adjoint fields $(\mathbf{E}^+, \mathbf{H}^+)$ are given by $\mathbf{E}^+ = (E_x, E_y, -E_z)$, $\mathbf{H}^+ = (-H_x, -H_y, H_z)$, and the contour C runs along the y-axis and closes at infinity. Only the y-axis contribution is to be taken into consideration [11] so that $\mathbf{d}L = -\hat{\mathbf{x}}dy$. The terms $\mathcal{C}, \mathcal{P}_1$ and $\mathcal{P}_2$ are given by:

$$\mathcal{C} = \frac{1}{2}\int_{-\infty}^{\infty} (E_{2z}H_{1y} + E_{2y}H_{1z} - E_{1z}H_{2y} - E_{1y}H_{2z})dy \tag{12}$$

$$\mathcal{P}_{1,2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathbf{E}_{1,2} \times \mathbf{H}_{1,2}) \cdot \hat{\mathbf{z}}dxdy\,. \tag{13}$$

The general pattern of the frequency dependence of the coupling coefficient K could be broadly predicted as follows, taking into consideration both hyperbolic and trigonometric solutions. We note that Poynting's vector in the surrounding medium, $\mathbf{P} = \hat{\mathbf{z}}\frac{1}{2}Re(E_x H_y^*) \propto \beta h^2 e^{-2hx}/\omega$. Then, in this region, assuming the slab has a given t and a given κ, at the higher end of the frequency range ($k_0 t \sim \pi/2\sqrt{\kappa - 1}$), ht is very low ($ht \sim 0$), so that the coupling is very low (due to the h^2 term), because the energy density of each slab is very low in the vicinity of the neighbouring slab. Afterwards, the coupling grows as the frequency diminishes, and at the lower end of the frequency range ($k_0 t \sim 0$), ht reaches its highest value ($ht \sim \mathrm{arc\ coth}\,\kappa$). Between the two limits of the frequency range, the coupling could grow steadily, or it could reach a maximum, and decline to zero if for very low frequencies ht is so high that most of the energy density of each slab concentrates near its own surface and scarcely reaches the surface of the neighbouring slab (due to the e^{-2hx} term). One sees then that the value of the coupling in the frequency range depends on the interplay between the various frequency-dependent terms composing the energy density. For a given frequency, h grows as t diminishes or κ diminishes, and the coupling dependence on t or κ follows its dependence on h as described above. The same pattern of coupling variation with frequency has been generally observed in [12].

4. Numerical results

4.1. The propagation coefficients

The normalized transverse propagation coefficient in the surrounding medium (ht) is given as a function of the normalized transverse propagation coefficient in the metamaterial slab (ℓt) for the hyperbolic and for the trigonometric modes with κ as parameter ($\kappa = 1.5, 2, 3$) in Fig. 2. The longitudinal propagation coefficient β is given as a function of the frequency f for the hyperbolic and for the trigonometric modes with κ as parameter ($\kappa = 1.5, 2, 3$) in Fig. 3. The width of the slab $2t = 7.5\,\mathrm{mm}$ in these two figures. We see that the hyperbolic solution, which begins at $k_0 t = 0$ and ends at $k_0 t = 1/\kappa\sqrt{\kappa - 1}$, is followed by the trigonometric solution, which begins where the hyperbolic solution ends, and ends at $k_0 t = \pi/2\sqrt{\kappa - 1}$.

4.2. Coupling between two metamaterial slabs

We first study the coupling between two similar metamaterial slabs whose separation is $2d = 1\,\mathrm{mm}$, of width $2t = 7.5\,\mathrm{mm}$, and permittivities $\varepsilon = -1.5\varepsilon_0$, $-2\varepsilon_0$, $-3\varepsilon_0$, as described in section 4.1. The coupling coefficient K frequency variation is shown in Fig. 4. We note that for a given frequency, the lower the refractive index, the higher the coupling.

The dispersion diagrams for three metamaterial slabs having the same permittivity ($\varepsilon = -2\varepsilon_0$) and various widths ($2t = 5$, 7.5, $10\,\mathrm{mm}$) are given in Fig. 5. The coupling coefficient K frequency variation for two such identical metamaterial waveguides separated by a distance $2d = 1\,\mathrm{mm}$ is given in Fig. 6. We note that for a given frequency, the thinner the slab, the higher the coupling.

4.3. Coupling between a metamaterial slab and a dielectric slab

The coupling between a dielectric slab and a metamaterial or another dielectric slab has also been studied. Three cases have been considered: $2t_1 = 5\,\mathrm{mm}$, $\varepsilon_1 = -2\varepsilon_0$ and $2t_2 = 10\,\mathrm{mm}$, $\varepsilon_2 = 3\varepsilon_0$; $2t_1 = 5\,\mathrm{mm}$, $\varepsilon_1 = -2\varepsilon_0$ and $2t_2 = 15\,\mathrm{mm}$, $\varepsilon_2 = 3\varepsilon_0$; $2t_1 = 10\,\mathrm{mm}$, $\varepsilon_1 = 3\varepsilon_0$ and $2t_2 = 15\,\mathrm{mm}$, $\varepsilon_2 = 3\varepsilon_0$. The distance between the slabs is always $2d = 1\,\mathrm{mm}$. The frequency variation of the longitudinal propagation coefficient β is shown in Fig. 7 and the frequency variation of the coupling coefficient K is shown in Fig. 8 for these three cases. The graphs of the coupling show very interesting features. The metamaterial-dielectric structures possess a pass-band and show a maximum. This maximum occurs at the frequency for which the β's of the two slabs are roughly equal. In this case the surface waves propagating along the two slabs decay almost identically away from their surface.

The above results show that a single metamaterial slab or a pair of coupled metamaterial slabs in the first TM even mode could be used as a low-pass filter. On the other hand, one could devise a band-pass filter by coupling a metamaterial slab to a dielectric slab in the first TM even mode. The band-pass would extend from the frequency for which $h = 0$ for the dielectric slab (j), $k_{0j} = \pi/2t_j\sqrt{\kappa_j - 1}$, to the frequency for which $h = 0$ for the metamaterial slab (i), $k_{0i} = \pi/2t_i\sqrt{\kappa_i - 1}$. Evidently k_{oi} must be larger than k_{oj}.

5. Conclusion

The coupling between two metamaterial slabs, or between a metamaterial slab and a dielectric slab, has been studied, taking into consideration the regular trigonometric modes as well as the hyperbolic modes. The first TM even mode has been especially studied because it shows interesting filtering properties. A bandpass filter could be devised by coupling a metamaterial slab to a dielectric one, the limits of the passband being determined by the cutoff frequencies of the two slabs. The coupling coefficient frequency variation of these structures has been studied, with the slab width or the slab permittivity as parameter. It has been found that, for a given frequency, the thinner the width or the lower the permittivity, the higher the coupling between the slabs.

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Figure captions

Figure 1: The coupled metamaterial slabs structure.

Figure 2: The normalized transverse propagation coefficient in the slab, ℓt, as a function of the normalized transverse propagation coefficient in the surrounding medium, ht, for a slab width $2t = 7.5\,\mathrm{mm}$ and for various values of the slab relative permittivity
$\varepsilon_r = -1.5, -2, -3$:
$* * * * *$ hyperbolic solution
$\circ \circ \circ \circ \circ$ trigonometric solution.

Figure 3: The longitudinal propagation coefficient β as a function of the frequency f for a slab width $2t = 7.5\,\mathrm{mm}$ and for various values of the slab relative permittivity
$\varepsilon_r = -1.5, -2, -3$:
$* * * * *$ hyperbolic solution
$\circ \circ \circ \circ \circ$ trigonometric solution.

Figure 4: The coupling coefficient K as a function of the frequency f for a pair of slabs of width $2t = 7.5\,\mathrm{mm}$ each and various values of relative permittivity $\varepsilon_r = -1.5, -2, -3$:
$* * * * *$ hyperbolic solution
$\circ \circ \circ \circ \circ$ trigonometric solution.
The distance between the slabs is $2d = 1\,\mathrm{mm}$.

Figure 5: The longitudinal propagation coefficient β as a function of the frequency f for a slab relative permittivity $\varepsilon_r = -2$ and for various values of the slab width $2t = 5, 7.5, 10\,\mathrm{mm}$:
$* * * * *$ hyperbolic solution
$\circ \circ \circ \circ \circ$ trigonometric solution.

Figure 6: The coupling coefficient K as a function of the frequency f for a pair of slabs of relative permittivity $\varepsilon_r = -2$ each and various values of width $2t = 5, 7.5, 10\,\mathrm{mm}$:
$* * * * *$ hyperbolic solution
$\circ \circ \circ \circ \circ$ trigonometric solution.
The distance between the slabs is $2d = 1\,\mathrm{mm}$.

Figure 7: The longitudinal propagation coefficient β as a function of the frequency f for three slabs of different widths $2t$ and different relative permittivities ε_r:

1. $2t = 10\,\mathrm{mm}$, $\varepsilon_r = 3$
2. $2t = 15\,\mathrm{mm}$, $\varepsilon_r = 3$
3. $2t = 5\,\mathrm{mm}$, $\varepsilon_r = -2$.

Figure 8: The coupling coefficient K as a function of the frequency f for three pairs of slabs of different widths $2t$ and different relative permittivities ε_r:

1. $2t_1 = 5\,\mathrm{mm}$, $\varepsilon_{r1} = -2$ and $2t_2 = 10\,\mathrm{mm}$, $\varepsilon_{r2} = 3$
2. $2t_1 = 5\,\mathrm{mm}$, $\varepsilon_{r1} = -2$ and $2t_2 = 15\,\mathrm{mm}$, $\varepsilon_{r2} = 3$
3. $2t_1 = 10\,\mathrm{mm}$, $\varepsilon_{r1} = 3$ and $2t_2 = 15\,\mathrm{mm}$, $\varepsilon_{r2} = 3$.

The distance between the slabs is $2d = 1\,\mathrm{mm}$.