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Wave propagation along a rectangular metallic wave-guide longitudinally loaded with a metamaterial slab

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Abstract

Wave propagation along a rectangular metallic wave-guide loaded with a longitudinal metamaterial slab adjacent to air, has been studied. The dispersion relations have been obtained and the $\beta - \omega$ diagrams have been drawn for various width ratios of the metamaterial to air regions. For real propagation coefficients, in the $0 < \beta < k_0$ range, the group velocity could be negative, very low or constant, depending on the frequency range and on the width ratios of the two regions. In some cases it has been found that the propagation coefficient is complex although the media are lossless: there is no net energy flow through any cross-section of the guide so that the energy present is only stored, reactive, energy and this seems to be due to strong coupling processes occurring in the device.

1. Introduction

Plane-wave propagation in materials whose permittivity and permeability are both negative has been investigated by Veselago [1] who has shown that the Poynting vector is anti-parallel to the phase-velocity vector. These materials have been variously termed as metamaterials, backward-wave materials, left-handed materials, etc. Lindell et al. [2] have reviewed isotropic and uniaxially isotropic metamaterials, and Lakhtakia et al. [3] have made an overview covering developments in the microwave range. Ziolkowsky [4] has reported the design, fabrication and testing of several metamaterials at X-band frequencies.

Many applications have been proposed. Pendry [5] has shown that a metamaterial slab could resolve objects only a few nanometers wide in the optical domain. Engheta [6] has made a theoretical analysis on thin subwavelength cavity resonators containing metamaterials. Guiding devices using metamaterials have been studied by many authors. Nefedov et al. [7] have considered theoretically the wave-guide properties of a planar two-layered wave-guide, one magnetodielectric and the other metamaterial. Topa [8] has addressed the use of metamaterials in the design of a contra-directional coupler. Eleftheriades et al. [9] have presented experimental verification of focusing using an implementation of artificial transmission line media in planar form. Alu et al. [10] have analysed wave propagation in a parallel-plate wave-guide filled with a pair of lossless slabs, one with $\varepsilon > 0, \mu < 0$ and the other with $\varepsilon < 0, \mu > 0$. Krowne et al. [11] have studied a microstrip structure containing a metamaterial. Hrabar et al. [12] have analysed a rectangular metallic wave-guide filled with metamaterial.

In this paper, the wave propagation in a rectangular metallic wave-guide loaded with a longitudinal metamaterial slab adjacent to air, has been studied. The $\beta - \omega$ diagrams have been obtained for various width ratios of the metamaterial to air regions.

2. The electromagnetic fields

The waveguide walls $(a \times b)$ are perfectly conducting. The waveguide interior is filled on its right side by a metamaterial (index 2) whose permittivity and permeability are $\varepsilon_2 = -|\kappa|\varepsilon_0$ and $\mu_2 = -\mu_0$ respectively and on its left side by air (index 1). The widths of the metamaterial and air-filled regions of the wave-guide are t and d respectively (see Fig. 1).

The electric and magnetic field components are given as follows for the LSE (TEx) modes:

$$E_y = j\beta A \sin(hx)\cos\left(\frac{m\pi}{b}y\right) \qquad (0 \leq x \leq d) \tag{1a}$$

$$E_y = j\beta B \sin\Big(\ell(a-x)\Big)\cos\left(\frac{m\pi}{b}y\right) \qquad (d \leq x \leq a) \tag{1b}$$

$$H_x = \frac{\beta^2}{j\omega\mu_0} A \sin(hx)\cos\left(\frac{m\pi}{b}y\right) \qquad (0 \leq x \leq d) \tag{2a}$$

$$H_x = \frac{-\beta^2}{j\omega\mu_0} B \sin\Big(\ell(a-x)\Big)\cos\left(\frac{m\pi}{b}y\right) \qquad (d \leq x \leq a) \tag{2b}$$

$$H_z = -\frac{\beta A}{\omega\mu_0} h \cos(hx)\cos\left(\frac{m\pi}{b}b\right) \qquad (0 \leq x \leq d) \tag{2c}$$

$$H_z = -\frac{\beta B}{\omega\mu_0} \ell \cos\Big(\ell(a-x)\Big)\cos\left(\frac{m\pi}{b}y\right) \qquad (d \leq x \leq a), \tag{2d}$$

and as follows for the LSM (TMx) modes:

$$E_x = \frac{-\beta^2}{j\omega\varepsilon_0} A \cos(hx)\sin\left(\frac{m\pi}{b}y\right) \qquad (0 \leq x \leq d) \tag{3a}$$

$$E_x = \frac{\beta^2}{j\omega|\kappa|\varepsilon_0} B \cos\Big(\ell(a-x)\Big)\sin\left(\frac{m\pi}{b}y\right) \qquad (d \leq x \leq a) \tag{3b}$$

$$E_z = \frac{A\beta}{\omega\varepsilon_0} h \sin(hx)\sin\left(\frac{m\pi}{b}y\right) \qquad (0 \leq x \leq d) \tag{3c}$$

$$E_z = \frac{B\beta}{\omega|\kappa|\varepsilon_0} \ell \sin\Big(\ell(a-x)\Big)\sin\left(\frac{m\pi}{b}y\right) \qquad (d \leq x \leq a) \tag{3d}$$

$$H_y = -j\beta A \cos(hx)\sin\left(\frac{m\pi}{b}y\right) \qquad (0 \leq x \leq d) \tag{4a}$$

$$H_y = -j\beta B \cos\Big(\ell(a-x)\Big)\sin\left(\frac{m\pi}{b}y\right) \qquad (d \leq x \leq a). \tag{4b}$$

A and B are constant amplitudes. $m = 0, 1, 2, 3 \ldots$. The following notation has been adopted in both cases: $k_{x1} = h,\ k_{x2} = \ell,\ k_y = \frac{m\pi}{b},\ k_z = \beta$. Since $k_{x_i}^2 + k_{y_i}^2 + k_{z_i}^2 = \omega^2\varepsilon_i\mu_i \triangleq k_i^2 (i = 1, 2)$, we obtain:

$$h^2 + \left(\frac{m\pi}{b}\right)^2 + \beta^2 = \omega^2\varepsilon_1\mu_1 = \omega^2\varepsilon_0\mu_0 = k_0^2 \qquad (i = 1,\ 0 \leq x \leq d) \tag{5a}$$

$$\ell^2 + \left(\frac{m\pi}{b}\right)^2 + \beta^2 = \omega^2\varepsilon_2\mu_2 = \omega^2(-|\kappa|\varepsilon_0)(-\mu_0) = \omega^2|\kappa|\varepsilon_0\mu_0 = |\kappa|k_0^2 \qquad (i = 2,\ d \leq x \leq a) \tag{5b}$$

In both cases an $e^{j(\omega t-\beta z)}$ factor has been omitted.

Subtracting Eq. (5b) from Eq. (5a) we obtain:

$$\ell^2 - h^2 = (|\kappa| - 1)k_0^2, \quad (|\kappa| > 1), \tag{6}$$

which represents an hyperbola. We could retrieve β from Eq. (5a) or Eq. (5b).

Continuity of the E_y component at $x = d$ gives for the LSE mode:

$$A\sin(hd) = B\sin(\ell t), \tag{7a}$$

while continuity of the H_z component at $x = d$ gives for the same mode:

$$Ah\cos(hd) = B\ell\cos(\ell t), \tag{7b}$$

so that the dispersion relation is given as follows by dividing Eq. (7a) by Eq. (7b):

$$\ell\tan(hd) = h\tan(\ell t) \qquad \text{(LSE)}\,. \tag{8a}$$

The analysis is similar for the LSM mode. In this case, the dispersion relation is given as follows:

$$|\kappa| h\tan(hd) = \ell\tan(\ell t) \qquad \text{(LSM)}\,. \tag{8b}$$

The right-hand side of Eqs. (8a) and (8b) which apply to metamaterials has a sign opposite to the right-hand side of the corresponding equations which apply to dielectric materials.

When $\beta = 0$, $h = k_0$ and $\ell = \sqrt{|\kappa|}k_0$, so that the dispersion relation to be solved for the LSE mode becomes:

$$\sqrt{|\kappa|}\tan(k_0 d) = \tan(\sqrt{|\kappa|}k_0 t)\,. \tag{9}$$

When $\beta = k_0$, $h = 0$ and $\ell = \sqrt{|\kappa| - 1}\,k_0$, so that the dispersion relation to be solved for the LSE mode becomes:

$$\frac{\tan(\sqrt{|\kappa|-1}k_0 t)}{\sqrt{|\kappa|-1}k_0} = \left.\frac{\tan(hd)}{h}\right|_{h=0} = d\,. \tag{10}$$

3. Numerical results

The wave-guide configuration is shown in Fig. 1. For the sake of simplicity we shall suppose $m = 0$ (LSE mode). We assume $a = 2.286\,\mathrm{cm}$, $b = 1.016\,\mathrm{cm}$, and $d/t = 1/3$, 1, 3, while $|\kappa| = 4$. Graphs of $k_0 t$ vs. βt are given in Figs. 2a, 2b, 2c. The graphs are divided by the two lines $\beta = k_0$ and $\beta = \sqrt{|\kappa|}k_0$ into three ranges. For real values of β, in the first range ($0 < \beta < k_0$) h and ℓ are real, in the second range ($k_0 < \beta < \sqrt{|\kappa|}k_0$) h is imaginary and ℓ is real, while in the third range ($\sqrt{|\kappa|}k_0 < \beta$) h and ℓ are imaginary, as could be inferred by inspection of Eqs (5a) and (5b) for $m = 0$.

3.1. Real values of $\boldsymbol{\beta}$

As stated earlier, in the first range ($0 < \beta < k_0$) the solutions in the air and in the metamaterial are both trigonometric while in the second range ($k_0 < \beta < \sqrt{|\kappa|}k_0$) the solution in the air is hyperbolic and the solution in the metamaterial is trigonometric. It could be seen that the $\beta - \omega$ diagrams corresponding to different values of the width t are noticeably different in the first range ($0 < \beta < k_0$) while they show almost no change in the second range ($k_0 < \beta < \sqrt{|\kappa|}k_0$).

The $0 < \beta < k_0$ range displays interesting characteristics. For $d = a/4$, the curves exhibit generally very low group velocities ($\partial\omega/\partial\beta \sim 0$) and sometimes negative group velocities ($\partial\omega/\partial\beta < 0$). For $d = 3a/4$, the curves exhibit generally constant group velocities ($\partial\omega/\partial\beta \sim$ const.) and sometimes negative group velocities ($\partial\omega/\partial\beta < 0$).

3.2. Complex values of $\boldsymbol{\beta}$

In certain frequency ranges the propagation coefficient β may become complex, for instance for $d/t = 3, |\kappa| = 9$ (see Fig. 3). We can observe in this figure two conjugate complex values of β, one beginning at $k_0 t = 2.75$ and ending at $k_0 t = 0$, the other beginning at $k_0 t = 1.64$ and ending at $k_0 t = 0$. This particular behaviour deserves special attention. It can be seen in the figure that the two values of the propagation coefficient are complex conjugate. Indeed, replacing h and ℓ by h^* and ℓ^* in Eqs. (8a) and (8b) easily show that h^* and ℓ^* are also solutions, indicating that the complex modes cannot exist alone and always occur in pairs (β and β^*). Moreover the propagation coefficient always occurs squared (see Eqs. (5a) and (5b)) showing that $-\beta$ and $-\beta^*$ are also solutions. The complex propagation coefficient takes therefore the following forms: $\beta_{1,2} = \pm\zeta - j\sigma$ ($\zeta > 0,\ \sigma > 0$), or $\beta_{3,4} = \pm\zeta + j\sigma$ ($\zeta > 0,\ \sigma > 0$). Complex propagation coefficients have also been observed in parallel plate chirowaveguides [13].

Let us consider the first type of solutions, i.e. $\beta_1 = \zeta - j\sigma$ and $\beta_2 = -\beta_1^* = -\zeta - j\sigma$. The propagation coefficient β_1 corresponds to a wave propagating in the z-direction and decreasing in the same direction, meaning a continuous energy loss, while the propagation coefficient β_2 corresponds to a wave propagating in the $-z$ direction and increasing in the same direction, meaning a continuous energy gain. But the structure is lossless and passive. The above point would have been valid if the two modes had not been coupled, but it looks, in fact, that they are strongly coupled, so that the wave propagating in the z-direction generates by a coupling process a wave propagating in the $-z$ direction.

Let $(\hat{\mathbf{y}}E_{yi}, \hat{\mathbf{x}}H_{xi})$ be the transverse electric and magnetic field vectors for the two LSE modes (see Eqs. (1) and (2)) having propagation coefficients β_i ($i = 1, 2$). Let us study the energy connected with the electromagnetic field obtained from the superposition of these two LSE modes. The z-component of the corresponding Poynting vector is given as follows, for the air region:

$$\begin{aligned} P_z &= -\frac{1}{2}\left(\sum_{i=1}^{2} E_{yi}\right)\left(\sum_{n=1}^{2} H_{xn}^*\right) = -\frac{1}{2}\sum_{i,n} E_{yi}H_{xn}^* \\ &= \frac{|A|^2}{2\omega\mu_0}\sum_{i,n}\beta_i(\beta_n^*)^2\Big(\sin(h_i x)\Big)\Big(\sin(h_n x)\Big)^*\ e^{j(\beta_n^*-\beta_i)z} \\ &= j\frac{|A|^2 r_a^2 r_b^2}{\omega\mu_0}\Big[\sigma + r_a\sin(3\theta_a + 2\theta_b - 2\zeta z)\Big]e^{-2\sigma z} \end{aligned} \tag{11}$$

where $\zeta - j\sigma \triangleq r_a e^{j\theta_a}$, $\sin hx \triangleq r_b e^{j\theta_b}$, and supposing $A_1 = A_2 \triangleq A$. A similar expression could be found for the metamaterial region.

The physical interpretation of Eq. (11) is very interesting. First, $\mathrm{Re}\, P_z = 0$, meaning that the *net* flow of energy through any cross-section (z=const.) of the guide is equal to zero, i.e., if there is energy flowing in the z-direction, it is exactly cancelled by that flowing in the $-z$ direction, so that the two superposed coupled modes carry together no active power. Secondly, $\mathrm{Im}\, P_z \neq 0$, meaning that the energy present in the device is only stored, reactive, energy, capacitive or inductive according to the sign of P_z. On the one hand it is oscillatory because of the $\sin(3\theta_a + 2\theta_b - 2\zeta z)$ dependence on z, and on the other hand it is also evanescent because of the $e^{-2\sigma z}$ dependence on z.

It is interesting to note that there are frequency ranges in which there is no propagation and which could be advantageously used to design a band-pass filter.

4. Conclusion

Wave propagation in a rectangular metallic wave-guide loaded with a longitudinal metamaterial slab adjacent to air, has been studied. The $\beta - \omega$ diagrams have been obtained for various width ratios of the metamaterial to air regions. For real propagation coefficients, in the $0 < \beta < k_0$ range, when the air width is small, the curves exhibit generally low group velocities and sometimes negative ones while when the air width is large, the curves exhibit generally constant group velocities and sometimes negative ones. The propagation coefficients could also be complex although the media are lossless. In this case, there is no net energy flow through any cross-section of the guide, i.e., the energy flowing in one direction is exactly cancelled by the energy flowing in the opposite direction. The energy present is only stored, reactive energy. This seems to be due to strong coupling processes occurring in the device.

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Figure captions

Figure 1: The wave-guide configuration.

Figure 2: Real propagation coefficients Brillouin diagrams.
$a = 2.286\,\text{cm}$, $b = 1.016\ \text{cm}$, $|\kappa| = 4$.
a) $d/t = 1/3$, b) $d/t = 1$, c) $d/t = 3$.

Figure 3: Complex propagation coefficients Brillouin diagrams.
$a = 2.286\,\text{cm}$, $b = 1.016\,\text{cm}$, $d/t = 3$, $|\kappa| = 9$.
β: Real propagation coefficient.
ζ: Real part of complex propagation coefficient.
σ: Imaginary part of complex propagation coefficient.