CCIT Report No. 465
January 2004

On Achievable Rate Regions for the Gaussian Interference Channel

Igal Sason*

Department of Electrical Engineering
Technion – Israel Institute of Technology
Haifa 32000, Israel
E-mail: sason@ee.technion.ac.il

January 27, 2004

Abstract

The complete characterization of the capacity region of a two-user Gaussian interference channel is still an open problem unless the interference is strong. In this work, we derive an achievable rate region for this channel. It includes the rate region which is achieved by time/ frequency division multiplexing (TDM/ FDM), and it also includes the rate region which is obtained by time-sharing between the two rate pairs where one of the transmitters sends its data reliably at the maximal possible rate (i.e., the maximum rate it can achieve in the absence of interference), and the other transmitter decreases its data rate to the point where both receivers can reliably decode its message. The suggested rate region is easily calculable, though it is a particular case of the celebrated achievable rate region of Han and Kobayashi whose calculation is in general prohibitively complex. In the high power regime, a lower bound on the sum-capacity (i.e., the maximal achievable total rate) is derived, and we show its superiority over the maximal total rate which is achieved by the TDM/ FDM approach with moderate interference. For degraded and one-sided Gaussian interference channels, we rely on some observations of Costa and Sato, and obtain directly their sum-capacities. We conclude our discussion by pointing out two interesting open problems.

1 Model and Definition of Capacity Region

An interference channel (IC) models the situation where a number (M) of unrelated senders try to communicate their separate information to M different receivers via a common channel. Transmission of information from each sender to its corresponding receiver interferes with the communication between the other senders and their receivers.

A two-user (i.e, $M = 2$) discrete, memoryless IC consists of four finite sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$, and conditional probability distributions $p(\cdot, \cdot|x_1, x_2)$ on $\mathcal{Y}_1 \times \mathcal{Y}_2$, where $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. For coded information of block length n, the two-user discrete, memoryless IC is denoted by

$$(\mathcal{X}_1^n \times \mathcal{X}_2^n, p^n(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}_1, \mathbf{x}_2), \mathcal{Y}_1^n \times \mathcal{Y}_2^n),$$

*The paper was submitted to the *IEEE Trans. on Information Theory* on August 8, 2003, and it was revised on January 25, 2004. The paper was accepted for publication as a correspondence. The research work was done while the author was a post-doc in the Department of Computer Science and Communications at the Swiss Federal Institute of Technology (EPFL), Lausanne, Switzerland.

where

$$p^n(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}_1, \mathbf{x}_2) = \prod_{k=1}^{n} p(y_{1,k}, y_{2,k}|x_{1,k}, x_{2,k}), \quad \mathbf{x}_1 \in \mathcal{X}_1^n, \mathbf{x}_2 \in \mathcal{X}_2^n, \mathbf{y}_1 \in \mathcal{Y}_1^n, \mathbf{y}_2 \in \mathcal{Y}_2^n.$$

In the model of a two-user IC, there are two independent and uniformly distributed sources. Senders 1, 2 produce two integers: $W_1 \in \{1, \cdots, 2^{nR_1}\}$, $W_2 \in \{1, \cdots, 2^{nR_2}\}$, respectively. A $(2^{nR_1}, 2^{nR_2}, n)$ code for a two-user IC consists of two encoding functions: $e_1 : \{1, \cdots, 2^{nR_1}\} \to \mathcal{X}_1^n$, $e_2 : \{1, \cdots, 2^{nR_2}\} \to \mathcal{X}_2^n$, and two decoding functions: $d_1 : \mathcal{Y}_1^n \to \{1, \cdots, 2^{nR_1}\}$, $d_2 : \mathcal{Y}_2^n \to \{1, \cdots, 2^{nR_2}\}$. Since there is no co-operation between the two receivers in this channel, the average probabilities of error are

$$P_{e,i}^{(n)} = \frac{1}{M_1 M_2} \sum_{w_1, w_2} \text{Prob}\left\{d_i(\mathbf{y}_i) \neq w_i | W_1 = w_1, W_2 = w_2\right\}, \quad i = 1, 2.$$

A rate pair (R_1, R_2) is said to be *achievable* if there exists a sequence of $(\lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil, n)$ codes, such that $P_{e,1}^{(n)} \to 0$ and $P_{e,2}^{(n)} \to 0$ as $n \to \infty$. The rates are expressed here is terms of bits per channel use. The *capacity region* of an IC is defined as the closure of the set of all its achievable rate pairs.

In this paper, we derive an achievable rate region for the two-user Gaussian IC. The derivation of this region is based on a *modified* time (or frequency) division multiplexing approach which was originated by Sato for the degraded Gaussian IC, and which is studied here in the general setting. This achievable rate region includes the rate region which is achieved by time/ frequency division multiplexing (TDM/ FDM), and it also includes the rate region which is obtained by time-sharing between the two rate pairs where one of the transmitters sends its data reliably at the maximal possible rate (i.e., the maximum rate it can achieve in the absence of interference), and the other transmitter decreases its data rate to the point where both receivers can reliably decode its message. Yet, it is still a particular case of the Han and Kobayashi (HK) achievable rate region whose calculation is in general prohibitively complex. In the high power regime, an improved lower bound on the sum-capacity is derived as a particular case of the general HK achievable rate region. We analyze some of the properties of this lower bound, and show its superiority over the maximal total rate which is achieved by the TDM/ FDM approach. For degraded and one-sided Gaussian ICs, we rely on some observations of Costa and Sato, and derive their sum-capacities (i.e., the maximal achievable total rates in their capacity regions.) Throughout this paper, we confine ourselves to Gaussian ICs in their *standard form* (see Section 2). We assume here perfect synchronization between the transmitters and their corresponding receivers, which implies that the capacity region of the IC is convex (based on time-sharing arguments).

The structure of the paper is the following: Earlier results which are related to the derivation of the new results here are presented in Section 2. New results are provided in Section 3, and proved in Section 4. Numerical results are presented and explained in Section 5. Finally, concluding remarks appear in Section 6, where we also address two interesting open problems.

2 Earlier Results

Similar to broadcast channels, since there is no co-operation between the receivers, the capacity region of a two-user discrete, memoryless IC *only* depends on the following marginal probability distributions

$$p_1(y_1|x_1, x_2) = \sum_{y_2 \in \mathcal{Y}_2} p(y_1, y_2|x_1, x_2), \quad p_2(y_2|x_1, x_2) = \sum_{y_1 \in \mathcal{Y}_1} p(y_1, y_2|x_1, x_2).$$

Hence, the capacity region of a discrete, memoryless IC is identical to the capacity region of any other discrete, memoryless IC whose marginal probability distributions are the same.

The information-theoretic characterization of the capacity region of a discrete memoryless IC is in general unknown yet, except for some special cases (see [15] and references therein). The capacity region of a discrete memoryless IC was expressed in [1] by the following limiting expression

$$C_{\text{IC}} = \lim_{n\to\infty} \text{closure} \left(\bigcup_{X_1^n, X_2^n \text{ independent}} \left\{ (R_1, R_2) : R_1 \leq \frac{I(X_1^n; Y_1^n)}{n},\ R_2 \leq \frac{I(X_2^n; Y_2^n)}{n} \right\} \right), \quad (1)$$

which unfortunately does not lend itself to feasible computation. Unlike the case of a general discrete, memoryless MAC whose capacity region is expressible by a single letter formula, the limiting expression in Eq. (1) can not be written in general by a single-letter expression (see [11]). Unfortunately, it was also demonstrated in [4] that the restriction to Gaussian inputs in the limiting expression Eq. (1) for the capacity region of a Gaussian memoryless IC fall short of achieving capacity, even if the inputs are allowed to be dependent and non-stationary.

In 1981, Han and Kobayashi (HK) [9] have derived an achievable rate region for a general discrete memoryless IC. It encompasses the achievable rate regions that were earlier established, and is still the best one known to date. However, the computation of the full HK achievable rate region for a general discrete, memoryless IC is in general prohibitively complex, because of the huge number of degrees of freedom which are involved in the computation of its sub-regions (see [9, Theorem 3.1]). We refer the interested reader to a comprehensive survey paper on the IC which was written by van der Meulen [15].

We focus here on the Gaussian IC, which was extensively treated in the literature (e.g, [2]–[5], [9]–[15]). The input and output alphabet of a memoryless Gaussian IC is the field of real numbers ($\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{R}$), and the probability density function (p) of this channel is derived from the following linear relations between its inputs and outputs:

$$y_1^* = c_1 x_1^* + c_2 x_2^* + n_1^*, \qquad y_2^* = d_1 x_1^* + d_2 x_2^* + n_2^*$$

where $x_1^*, x_2^*, y_1^*, y_2^*$ are reals, and n_1^*, n_2^* are additive Gaussian noises with zero mean and variances N_1, N_2 respectively. The following power constraints are imposed on the transmitted signals (n-length codewords) $\mathbf{x}_1^* \in \mathcal{X}_1^n$, and $\mathbf{x}_2^* \in \mathcal{X}_2^n$:

$$\frac{1}{n} \sum_{k=1}^{n} (x_{1,k}^*)^2 \leq P_1^*, \qquad \frac{1}{n} \sum_{k=1}^{n} (x_{2,k}^*)^2 \leq P_2^*.$$

The capacity region of this Gaussian IC is identical to the capacity region of the following Gaussian IC in its *standard* form

$$y_1 = x_1 + \sqrt{a_{12}}\, x_2 + n_1, \qquad y_2 = \sqrt{a_{21}}\, x_1 + x_2 + n_2, \quad (2)$$

where $a_{12} = \frac{c_2^2 N_2}{d_2^2 N_1}, a_{21} = \frac{d_1^2 N_1}{c_1^2 N_2}$, and n_1, n_2 are additive Gaussian noises with zero mean and unit variance. This equivalence is verified by the transformation

$$y_1 = \frac{y_1^*}{\sqrt{N_1}},\ y_2 = \frac{y_2^*}{\sqrt{N_2}},\ x_1 = \frac{c_1 x_1^*}{\sqrt{N_1}},\ x_2 = \frac{d_2 x_2^*}{\sqrt{N_2}},\ n_1 = \frac{n_1^*}{\sqrt{N_1}},\ n_2 = \frac{n_2^*}{\sqrt{N_2}}.$$

Since the capacity region of an IC only depends on the marginal probability distributions, it is irrelevant whether the noise terms n_1 and n_2 are statistically dependent of each other or not. The power constraints in the standard form of the Gaussian IC are

$$\frac{1}{n} \sum_{k=1}^{n} (x_{1,k})^2 \leq P_1, \qquad \frac{1}{n} \sum_{t=1}^{n} (x_{2,k})^2 \leq P_2, \quad (3)$$

where $P_1 = \frac{a^2 P_1^*}{N_1}, P_2 = \frac{d^2 P_2^*}{N_2}$. We note that in the high SNR regime, P_1, P_2 may attain rather high values, and in the broad band and power-limited scenario, the values of P_1, P_2 are typically moderate or low.

An IC is called *degraded* if there exists a conditional probability $p'(y_2|y_1)$, so that the following equality holds

$$p_2(y_2|x_1, x_2) = \sum_{y_1 \in \mathcal{Y}_1} p'(y_2|y_1) p_1(y_1|x_1, x_2), \quad \forall\, y_2 \in \mathcal{Y}_2,\ (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2.$$

The latter equality implies that the second output terminal (Y_2) is a degraded version of the first output terminal (Y_1). Since the capacity region of a degraded IC only depends on its marginal probability distributions, then the capacity region of a degraded IC is identical to the capacity region of the IC whose conditional probability distribution is

$$p(y_1, y_2|x_1, x_2) = p'(y_2|y_1)\, p_1(y_1|x_1, x_2), \quad \forall\, (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2,\ (y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2.$$

The latter IC forms a cascade channel, since it is composed of two channels $p_1(y_1|x_1, x_2)$ and $p'(y_2|y_1)$ which are combined in cascade. The full characterization of the capacity region of a discrete, memoryless and degraded IC is still an open problem. In its standard form (2), a two-user Gaussian IC is degraded if and only if $a_{12} \cdot a_{21} = 1$ [3]. Inner and outer bounds on the capacity region of a degraded Gaussian IC were derived in [13].
A two-user Gaussian IC is called *one-sided* if either $a_{12} = 0$ or $a_{21} = 0$. In [5], Costa has observed and proved that the class of degraded Gaussian IC are equivalent to the class of the one-sided Gaussian IC (from the view point of their capacity regions). More specifically, if $0 < c < 1$, then it follows from [5] that the one-sided Gaussian IC whose characterization in the standard form is

$$y_1 = x_1 + n_1, \quad y_2 = cx_1 + x_2 + n_2,$$

has the same capacity region as the degraded Gaussian IC

$$y_1 = x_1 + \frac{x_2}{c} + n_1, \quad y_2 = cx_1 + x_2 + n_2.$$

In both cases, n_1 and n_2 are Gaussian random variables with zero mean and unit variance, and the same power constraints are imposed on the transmitted signals $\mathbf{x}_1, \mathbf{x}_2$ for the two channels. The degraded Gaussian IC above is the standard form of the IC which was depicted in [5, Fig. 6(d)]. Later, we will rely on the equivalence between one-sided and degraded Gaussian ICs, and derive an exact expression for the sum-capacities of both channels.

The capacity region of a Gaussian IC with strong interference was independently determined by Han and Kobayashi [9, Theorem 5.2] and Sato [14]. The capacity region of the Gaussian IC with very strong interference was determined by Carleial [2]. For the latter case of a Gaussian IC with very strong interference (which is characterized by the inequalities: $a_{12} \geq 1 + P_1$ and $a_{21} \geq 1 + P_2$), it was surprisingly demonstrated in [2] that the interference *does not* harm the capacity region. However, the capacity region of the Gaussian IC was not determined yet in the cases where a_{12} and a_{21} lie in the open interval $(0, 1)$. The complete characterization of the capacity region of a one-sided Gaussian IC whose non-vanishing interference coefficient is between zero and unity is also still unknown. For interference parameters in this interval, weak and moderate interference are defined in the symmetric case as follows: If $a_{12} = a_{21} = a$ and $P_1 = P_2 = P$ in (2) and (3), respectively, then by treating the interfering signals as an additive noise, one can achieve a total rate of $\log_2\left(1 + \frac{P}{1+aP}\right)$ bits per channel use. On the other hand, if time or frequency division multiplexing (TDM/ FDM) are applied, then the maximal total rate is independent of the interference coefficients and equals $\frac{1}{2}\log_2(1+2P)$ bits per channel use. Weak/ moderate interference is defined as the range of values of a between zero and unity for which the former/ latter approach yields a larger value of the maximal total rate. Following this definition, one easily obtains that weak interference in a two-user, symmetric Gaussian IC refers to the case where $0 < a < \frac{\sqrt{1+2P}-1}{2P}$, and moderate interference refers to the complementary condition, i.e., $\frac{\sqrt{1+2P}-1}{2P} \leq a < 1$. For a two-user Gaussian IC with weak or moderate interference, the gap between the reported upper and lower bounds on the sum-capacity is rather large (see [10]), though it vanishes for in case where $a \to 0$ (i.e., no interference) or $a \to 1$ (i.e., a Gaussian multiple-access channel).

3 New Results

Theorem 1 The set of rate pairs

$$D = \bigcup_{\alpha,\beta,\lambda\in[0,1]} \left\{ (R_1, R_2) : \begin{array}{l} R_1 \leq \lambda \cdot \gamma\left(\frac{\alpha P_1}{\lambda}\right) + (1-\lambda) \cdot \min\left\{\gamma\left(\frac{(1-\alpha)P_1}{1-\lambda+a_{12}(1-\beta)P_2}\right), \gamma\left(\frac{a_{21}(1-\alpha)P_1}{1-\lambda+(1-\beta)P_2}\right)\right\} \\ R_2 \leq (1-\lambda) \cdot \gamma\left(\frac{(1-\beta)P_2}{1-\lambda}\right) + \lambda \cdot \min\left\{\gamma\left(\frac{\beta P_2}{\lambda+a_{21}\alpha P_1}\right), \gamma\left(\frac{a_{12}\beta P_2}{\lambda+\alpha P_1}\right)\right\} \end{array} \right\} \quad (4)$$

where

$$\gamma(x) \triangleq \frac{1}{2}\log_2(1+x), \quad (5)$$

is achievable for a two-user Gaussian IC in the standard form (2) under the power constraints in (3). The achievable rate region D in Eq. (1) is included in the general Han and Kobayashi (HK) achievable rate region (see [9, Theorem 3.2]).

If $0 \leq a_{12}, a_{21} < 1$, then D has the following properties:

1. It includes the straight line which connects the two points

$$(R_1, R_2) = \left(\gamma(P_1), \gamma\left(\frac{a_{12}P_2}{1+P_1}\right)\right), \quad (R_1, R_2) = \left(\gamma\left(\frac{a_{21}P_1}{1+P_2}\right), \gamma(P_2)\right). \quad (6)$$

2. It includes the achievable rate region by TDM/ FDM, but in both cases, the maximal achievable total rate stays the same.

3. In the symmetric case where $a_{12} = a_{21} \triangleq a$, and $P_1 = P_2 \triangleq P$:
   - The calculation of the achievable rate region in (4) can be simplified, so that it only involves the two parameters $(\beta, \lambda) \in [0,1] \times [0,1]$.
   - For moderate interference (i.e., if $\frac{\sqrt{1+2P}-1}{2P} < a < 1$), the region D includes the particular achievable sub-region $\mathcal{G}'$ of the full HK achievable rate region (see [9], Eq. (5.9)). The maximal total rate which is achieved by the region D is also strictly larger than the one achieved by $\mathcal{G}'$.
   - For weak interference, the maximal total rate which is achieved by $\mathcal{G}'$ is strictly larger than the one achieved by the region D (with an equality if $a = \frac{\sqrt{1+2P}-1}{2P}$).

Theorem 2 For a *degraded* Gaussian IC which is expressed in the standard form (2) with the power constraints (3), the sum-capacity is

$$\max(R_1 + R_2) = \begin{cases} \gamma(P_1) + \gamma\left(\frac{P_2}{1+a_{21}P_1}\right) & \text{if } a_{12} \geq 1 \\ \gamma(P_2) + \gamma\left(\frac{P_1}{1+a_{12}P_2}\right) & \text{if } a_{21} \geq 1. \end{cases} \quad (7)$$

For a *one-sided* Gaussian IC, the sum-capacity in the case where $a_{12} = 0$ is

$$\max(R_1 + R_2) = \begin{cases} \gamma(P_1) + \gamma\left(\frac{P_2}{1+a_{21}P_1}\right) & \text{if } 0 \leq a_{21} \leq 1 \\ \gamma(a_{21}P_1 + P_2) & \text{if } 1 \leq a_{21} \leq 1 + P_2 \\ \gamma(P_1) + \gamma(P_2) & \text{if } a_{21} \geq 1 + P_2 \end{cases} \tag{8}$$

with a similar expression for the case where $a_{21} = 0$ (by switching the indices of users 1 and 2).

Corollary 1 For a one-sided Gaussian IC with weak or moderate interference (i.e., when the interference coefficient is not above unity), the sum-capacity is achieved if the transmitter which is not interfered sends its data at the maximal achievable rate of a single-user, and the second transmitter sends its data at the maximal possible rate where the interfering signal is treated as an additive Gaussian noise.

The reader is referred to the concluding remarks in Section 6 where we show that the sum-capacity of a one-sided Gaussian IC which is provided in Eq. (8) implies immediately the upper bound on the sum-capacity of a two-user Gaussian IC which is presented in [10, Theorem 1]. It also automatically proves that the upper bound on the sum-capacity in [10, Theorem 2] is always better than the one in [10, Theorem 1] (although we note in Section 6 that the two upper bounds on the sum-capacity of a two-user Gaussian IC in [10] asymptotically coincide in the limit where P tends to infinity). For more details, the reader is referred to items 3 and 4 in Section 6.

Theorem 3 Consider a two-user symmetric Gaussian IC in the standard form (2) where $a_{12} = a_{21} \triangleq a$, and $P_1 = P_2 \triangleq P$ is the common power constraint in Eq. (3).
Then, for high enough values of P, the sum-capacity is larger than the maximal total rate which is achieved by TDM/ FDM.
In particular, let $a = a_0$ be the *single* root of the polynomial equation

$$P^2a^4 + 2P(P+1)a^3 + (2P+1)a^2 - 2Pa - (1+P) = 0 \tag{9}$$

which lies in the interval $\frac{\sqrt{4P+1}-1}{2P} \leq a \leq 1$ (i.e., it is included in the range of values of moderate interference). Then, if $P \geq 17$ dB, the maximal total rate which is attained by this sub-region is a decreasing function of $a \in [a_0, 1]$; it decreases from its maximal value at $a = a_0$, which is at least

$$\log_2\left(\frac{a_0 + a_0P + a_0^2P}{1 - a_0 + a_0^2(Pa_0 + 1)}\right) \tag{10}$$

to the value $\frac{1}{2}\log_2(1 + 2P)$ at $a = 1$ (i.e., the maximal total rate achieved by TDM/ FDM).

Corollary 2 For a symmetric Gaussian IC with moderate interference, TDM or FDM are *not optimal* in the high power regime or the narrow-band regime.

4 Proofs of New Results

4.1 Proof of Theorem 1

In [12, Theorem 5], Sato proved that the following rate-region is achievable for a two-user IC:

$$G_{\rm B} = \text{convex hull}\left\{G_{B_1} \cup G_{B_2}\right\}$$

where

$$G_{B_1} = \text{convex hull} \bigcup_{P_{X_1}, P_{X_2}} \left\{(R_1, R_2) : \begin{array}{l} 0 \le R_1 \le I(X_1; Y_1|X_2), \\ 0 \le R_2 \le \min\{I(X_2; Y_1), I(X_2; Y_2)\end{array} \right\} \tag{11}$$

and

$$G_{B_2} = \text{convex hull} \bigcup_{P_{X_1}, P_{X_2}} \left\{(R_1, R_2) : \begin{array}{l} 0 \le R_1 \le \min\{I(X_1; Y_2), I(X_1; Y_1), \\ 0 \le R_2 \le I(X_2; Y_2|X_1)\end{array} \right\}. \tag{12}$$

As was noted in [9, Corollary 3.3], the achievable rate regions in (11) and (12) form particular cases of the general Han and Kobayashi achievable rate region. For a two-user Gaussian IC in with the power constraints in (3), consider the case where during a fraction λ of the time, the symbols of $\mathbf{x}_1$ and $\mathbf{x}_2$ are Gaussian distributed with zero mean, and variances $\frac{\alpha P_1}{\lambda}$ and $\frac{\beta P_2}{\lambda}$, respectively:

$$x_{1,i} \sim N(0, \frac{\alpha P_1}{\lambda}), \quad x_{2,i} \sim N(0, \frac{\beta P_2}{\lambda}), \quad 0 \le \alpha, \beta \le 1, \quad i = 1, 2, \ldots, n\lambda \tag{13}$$

and then, during the remaining fraction $\overline{\lambda} \triangleq 1 - \lambda$ of the transmission time,

$$x_{1,i} \sim N(0, \frac{\overline{\alpha} P_1}{\overline{\lambda}}), \quad x_{2,i} \sim N(0, \frac{\overline{\beta} P_2}{\overline{\lambda}}), \quad i = n\lambda + 1, \ldots, n \tag{14}$$

which yields that the power constraints in Eq. (3) are satisfied: $\frac{1}{n}E[||\mathbf{x}_1||^2] = P_1$, $\frac{1}{n}E[||\mathbf{x}_2||^2] = P_2$. Consider two modes of work: In the first mode, receiver 1 decodes the message of the second sender, and uses it as side information for decoding his message (in the Gaussian IC model (2), receiver 1 subtracts from the received signal $\mathbf{y}_1$, a scaled version of $\mathbf{x}_2$). Receiver 2 directly decodes his message ($\mathbf{x}_2$), based on his received signal ($\mathbf{y}_2$). This mode of operation corresponds to the achievable rate region in Eq. (11), and is used here a fraction λ of the transmission time with Gaussian inputs according to Eq. (13). In the second mode (which refers to the the achievable rate region in Eq. (12)), receiver 1 decoded his message directly, and receiver 2 decodes the message of the first sender and uses it as side information to decode his message. The latter mode of operation is used during the remaining fraction $\overline{\lambda}$ of the transmission time, and the Gaussian inputs are distributed according to Eq. (14). Let $R_1^{(i)}$ and $R_2^{(i)}$ be the transmission rates in mode no. i ($i = 1, 2$), then $(R_1, R_2) = \lambda(R_1^{(1)}, R_2^{(1)}) + \overline{\lambda}(R_1^{(2)}, R_2^{(2)})$. From Eqs. (5), (11)–(14):

$$0 \le R_1^{(1)} \le \gamma\left(\frac{\alpha P_1}{\lambda}\right), \quad 0 \le R_2^{(1)} \le \min\left\{\gamma\left(\frac{\frac{a_{12}\beta P_2}{\lambda}}{1 + \frac{\alpha P_1}{\lambda}}\right), \gamma\left(\frac{\frac{\beta P_2}{\lambda}}{1 + \frac{a_{21}\alpha P_1}{\lambda}}\right)\right\} \tag{15}$$

and

$$0 \le R_1^{(2)} \le \min\left\{\gamma\left(\frac{\frac{\overline{\alpha} P_1}{\overline{\lambda}}}{1 + \frac{a_{12}\overline{\beta} P_2}{\overline{\lambda}}}\right), \gamma\left(\frac{\frac{a_{21}\overline{\alpha} P_1}{\overline{\lambda}}}{1 + \frac{\overline{\beta} P_2}{\overline{\lambda}}}\right)\right\}, \quad 0 \le R_2^{(2)} \le \gamma\left(\frac{\overline{\beta} P_2}{\overline{\lambda}}\right). \tag{16}$$

Eqs. (15), (16) provide the achievable rate region in Eq. (4).

If $0 \le a_{12}, a_{21} < 1$, then D is simplified to

$$D = \bigcup_{\alpha,\beta,\lambda\in[0,1]} \left\{ (R_1,R_2): \begin{array}{c} R_1 \le \lambda \cdot \gamma\left(\frac{\alpha P_1}{\lambda}\right) + \overline{\lambda} \cdot \gamma\left(\frac{a_{21}\overline{\alpha}P_1}{\overline{\lambda}+\overline{\beta}P_2}\right) \\ R_2 \le \overline{\lambda} \cdot \gamma\left(\frac{\overline{\beta}P_2}{\overline{\lambda}}\right) + \lambda \cdot \gamma\left(\frac{a_{12}\beta P_2}{\lambda+\alpha P_1}\right) \end{array} \right\}. \tag{17}$$

Though the achievable rate region (17) extends the one which corresponds to TDM/ FDM (since the latter region is a particular case of the region (17) for the particular case where $\alpha = 1$ and $\beta = 0$), it does not increase the maximal total rate which is achievable by TDM/ FDM. To see this, we note that based on Eq. (17) and the identity $\gamma\left(\frac{x}{y}\right) + \gamma\left(\frac{z}{x+y}\right) = \gamma\left(\frac{x+z}{y}\right)$ which is valid for all positive values of x, y and z, we obtain that

$$R_1 + R_2 \le \frac{\lambda}{2} \cdot \log_2\left(1 + \frac{\alpha P_1 + a_{12}\beta P_2}{\lambda}\right) + \frac{\overline{\lambda}}{2} \cdot \log_2\left(1 + \frac{\overline{\beta}P_2 + a_{21}\overline{\alpha}P_1}{\overline{\lambda}}\right).$$

The maximal value of the right-hand side above is achieved by $\lambda = \frac{\alpha P_1 + a_{12}\beta P_2}{(\alpha + a_{21}\overline{\alpha})P_1 + (\overline{\beta} + a_{12}\beta)P_2}$, which yields that the maximal total rate satisfies the inequality

$$\begin{aligned} R_1 + R_2 &\le \tfrac{1}{2}\log_2\left(1 + (\alpha + a_{21}\overline{\alpha})P_1 + (\overline{\beta} + a_{12}\beta)P_2\right) \\ &\le \tfrac{1}{2}\log_2(1 + P_1 + P_2) \end{aligned}$$

with equality if $\alpha = 1$ and $\beta = 0$. It therefore follows that the maximal total rate of the achievable rate region (17) is equal to the one which corresponds to TDM/ FDM. However, in contrast to the achievable rate region by TDM/ FDM, the substitution $\alpha = \beta = \lambda$ reveals that the rate region (17) also includes the achievable rate pairs

$$(R_1, R_2) = \left(\gamma(P_1), \gamma\left(\frac{a_{21}P_1}{1+P_2}\right)\right), \quad (R_1, R_2) = \left(\gamma\left(\frac{a_{12}P_2}{1+P_1}\right), \gamma(P_2)\right)$$

and the straight line connecting these two points. This shows that the achievable rate region (17) necessarily extends the achievable rate region of TDM/ FDM.

For the symmetric case where $P_1 = P_2 \triangleq P$ and $a_{12} = a_{21} \triangleq a$, the achievable rate region in (17) is simplified under the assumption of weak or moderate interference (i.e., $0 \le a \le 1$) to

$$D = \bigcup_{\alpha,\beta,\lambda\in[0,1]} \left\{ (R_1,R_2): \begin{array}{c} R_1 \le \lambda \cdot \gamma\left(\frac{\alpha P}{\lambda}\right) + (1-\lambda) \cdot \gamma\left(\frac{a(1-\alpha)P}{1-\lambda+(1-\beta)P}\right) \\ R_2 \le (1-\lambda) \cdot \gamma\left(\frac{(1-\beta)P}{1-\lambda}\right) + \lambda \cdot \gamma\left(\frac{a\beta P}{\lambda+\alpha P}\right) \end{array} \right\}. \tag{18}$$

The simplification in (18) also suggests a simplification in the numerical calculation of the achievable rate region in (4). For values of R_2 between $\gamma\left(\frac{P}{1+aP}\right)$ and $\gamma(P)$, calculate for any pair of $(\beta, \lambda) \in [0,1] \times [0,1]$, the value of α which satisfies the equation $(1-\lambda) \cdot \gamma\left(\frac{(1-\beta)P}{1-\lambda}\right) + \lambda \cdot \gamma\left(\frac{a\beta P}{\lambda+\alpha P}\right) = R_2$ which is given in closed form as

$$\alpha \triangleq \alpha(\beta,\lambda) = a\beta \left[\exp\left\{\frac{2}{\lambda}\left[R_2 - (1-\lambda)\,\gamma\left(\frac{(1-\beta)P}{1-\lambda}\right)\right]\right\} - 1\right]^{-1} - \frac{\lambda}{P}, \tag{19}$$

and then to choose the maximal value of the function

$$R_1 \triangleq R_1(\beta,\lambda) = \lambda \cdot \gamma\left(\frac{\alpha P}{\lambda}\right) + (1-\lambda) \cdot \gamma\left(\frac{a(1-\alpha)P}{1-\lambda+(1-\beta)P}\right)$$

over those values of $(\beta, \lambda) \in [0,1] \times [0,1]$ for which $\alpha(\beta, \lambda)$ in Eq. (19) is located inside the interval $[0,1]$. This procedure suggests a simplification in the calculation of the boundary of the achievable rate region (4) under the symmetry assumption (we note that based on Eq. (17), a similar procedure can be done under the assumption that $0 \leq a_{12}, a_{21} \leq 1$, even without the further symmetry assumption above.) Straightforward (though tedious) algebra shows that for moderate interference (i.e., when $\frac{\sqrt{1+2P}-1}{2P} < a < 1$), the sub-region $\mathcal{G}'$ which is given in [9, Eqs. (4.1)–(4.9), (5.9)] (with the special setting in [9, Section 5A]) is strictly included in the achievable rate region (18) (as is also exemplified later in Section 5). However, in the case of weak interference (i.e., when $0 < a < \frac{\sqrt{1+2P}-1}{2P}$), then we have already proved that the maximal total rate which corresponds to the achievable rate region in (4) is equal to that of TDM/ FDM, but on the other hand, the maximal total rate of the region $\mathcal{G}'$ exceeds this common value. The reason for the latter statement is attributed to the fact that since the achievable rate region $\mathcal{G}'$ includes the achievable rate region where the interfering signal is regarded as an additive Gaussian noise, and also in the latter case, the total rate is equal to $\log_2\left(1 + \frac{P}{1+aP}\right)$, then for weak interference (i.e., if $0 < a < \frac{\sqrt{1+2P}-1}{2P}$), it exceeds the total rate of optimal TDM/ FDM which is equal to $\frac{1}{2}\log_2(1 + 2P)$. It then follows directly that for weak interference, the maximal total rate of the achievable rate region $\mathcal{G}'$ is strictly larger than the maximal total rate which is obtained by (17).

4.2 Proof of Theorem 2

The sum-capacity of a degraded Gaussian IC is obtained as a direct consequence of the discussion in [13] on degraded Gaussian ICs(though it was not stated there explicitly). By referring to Fig. 1 in [13], the point A_1 in this figure was shown to be on the boundary of the capacity region of degraded Gaussian ICs (by showing that this point lies on the boundary of inner and outer bounds on the capacity region). Relying on observations in [13], it is a simple matter to show that the point A_1 of Fig. 1 in [13] achieves the sum-capacity of degraded Gaussian IC (we note that the capacity region of this channel is unknown yet). To see this, we refer to the curve connecting points A_1 and A_3 in Fig. 1 of [13], which together with the straight lines $R_1 = C_1$ and $R_2 = C_2$ forms an outer bound on the capacity region of a degraded Gaussian IC [13]. The point A_1 achieves the maximal total rate w.r.t. the outer bound on the capacity region (and hence, it also achieves the sum-capacity). The reason is that since it was shown in [13] that for the curve which connects the points A_1 and A_3 in Fig. 1 of [13], the inequality $-1 \leq \frac{dR_2}{dR_1} < 0$ is satisfied, then it yields that in the case where the output Y_2 is degraded w.r.t. the output Y_1, the point A_1 achieves the maximal value of $R_1 + R_3$ on this curve. By symmetry, if the output Y_1 is degraded w.r.t. Y_2, then the point which achieves the maximal total rate is the one which is symmetric to the point A_1 w.r.t. to the line $R_1 = R_2$ (so, in order to get the maximal total rate in the latter case, we just switch between the indices of the expression $R_1 + R_2$ for the former case, as in Eq. (7)).
The sum-capacity of a one-sided Gaussian IC follows from the observation which was made by Costa [5] about the equivalence between the one-sided and the degraded Gaussian IC (where this equivalence between these two Gaussian ICs was also stated explicitly in the preliminary material which is provided in Section 2). Based on this equivalence, the part of Eq. (8) which provides the sum-capacity of a one-sided Gaussian IC with weak or moderate interference follows directly from Eq. (7) which corresponds to a degraded Gaussian IC. The part of Eq. (8) which provides the sum-capacity of a one-sided Gaussian IC with strong or very strong interference is *not new* (and it is only given in (8) for the sake of completeness); it follows directly from the discussion in Section 3 of [14].

4.3 Proof of Theorem 3

Based on [3, Theorem 6], by dividing the transmitter powers in three parts, generating random sub-codewords which are Gaussian distributed, and arguing with the general receivers in [3, Fig. 10] (which perform successive decoding), the following rate region is achievable for a two-user Gaussian IC in the standard form (2)

$$\begin{aligned}
R_1 &\le \min \left\{ \begin{array}{l} \gamma\left(\frac{\delta_1 P_1}{1+(\alpha_1+\beta_1)P_1+a_{12}P_2}\right) + \gamma\left(\frac{(\alpha_1+\beta_1)P_1}{1+a_{12}\alpha_2 P_2}\right), \\ \gamma\left(\frac{a_{21}(\delta_1+\beta_1)P_1}{1+a_{21}\alpha_1 P_1+(\alpha_2+\beta_2)P_2}\right) + \gamma\left(\frac{\alpha_1 P_1}{1+a_{12}\alpha_2 P_2}\right) \end{array} \right\} \\
R_2 &\le \min \left\{ \begin{array}{l} \gamma\left(\frac{\delta_2 P_2}{1+(\alpha_2+\beta_2)P_2+a_{21}P_1}\right) + \gamma\left(\frac{(\alpha_2+\beta_2)P_2}{1+a_{21}\alpha_1 P_1}\right), \\ \gamma\left(\frac{a_{12}(\delta_2+\beta_2)P_2}{1+a_{12}\alpha_2 P_2+(\alpha_1+\beta_1)P_1}\right) + \gamma\left(\frac{\alpha_2 P_2}{1+a_{21}\alpha_1 P_1}\right) \end{array} \right\}
\end{aligned} \tag{20}$$

where $\alpha_i, \beta_i, \delta_i$ (where $i = 1, 2$) are non-negative numbers so that $\alpha_i + \beta_i + \delta_i = 1$, and the function $\gamma(\cdot)$ is introduced in (5). We note that the interference coefficients a_{12} and a_{21} in the standard form (2) are flipped as compared to the notation in [3]. In our case, since we assume that the two-user Gaussian IC is symmetric (i.e., $a_{12} = a_{21} \triangleq a$ and $P_1 = P_2 \triangleq P$), then it follows that the maximal value of $R_1 + R_2$ in the achievable rate region (20) is attained in the symmetric case where $\alpha_1 = \alpha_2 \triangleq \alpha$, $\beta_1 = \beta_2 \triangleq \beta$, $\delta_1 = \delta_2 \triangleq \delta$, and α, β and δ are non-negative numbers whose sum is 1 (i.e., $\alpha + \beta + \delta = 1$.).

We intend to show that for values of P which are *above* a certain threshold, the maximal total rate which is achieved by Carleial's region (20) is above the maximal total rate which is achieved by optimal TDM/ FDM (where the latter value is equal to $\frac{1}{2}\log_2(1 + 2P)$ bits per channel use). To this end, we will simplify the calculation by showing that for high enough values of P, even the particular choice where $\beta = 0$ in (20) yields that the corresponding maximal total rate of the latter achievable rate region exceeds the corresponding value which is obtained by optimal TDM/ FDM. Under the assumption that $\beta = 0$, one obtains that $\delta = \overline{\alpha} \triangleq 1 - \alpha$, which yields that the maximization of $R_1 + R_2$ for the achievable rate region in (20) with $\beta = 0$ is performed by maximizing the function

$$f(\alpha) \triangleq \min\left\{\log_2\left(1 + \frac{\overline{\alpha}P}{1 + \alpha P + aP}\right), \log_2\left(1 + \frac{a\overline{\alpha}P}{1 + a\alpha P + \alpha P}\right)\right\} + \log_2\left(1 + \frac{\alpha P}{1 + a\alpha P}\right) \tag{21}$$

over the interval $0 \le \alpha \le 1$. The ratio between the two arguments of the function $\log_2(1+x)$ inside the minimization in the right-hand side of (21) is

$$\frac{\frac{a\overline{\alpha}P}{1+a\alpha P+\alpha P}}{\frac{\overline{\alpha}P}{1+\alpha P+aP}} = \frac{a + a^2P + a\alpha P}{1 + \alpha P + a\alpha P}$$

so if $a + a^2P \le 1$ (i.e., if $0 \le a \le \frac{\sqrt{1+4P}-1}{2P}$), then since $\alpha \ge 0$, then

$$\begin{aligned}
f(\alpha) &= \log_2\left(1 + \frac{a\overline{\alpha}P}{1 + a\alpha P + \alpha P}\right) + \log_2\left(1 + \frac{\alpha P}{1 + a\alpha P}\right) \\
&= \log_2\left(1 + \frac{a\overline{\alpha}P + \alpha P}{1 + a\alpha P}\right).
\end{aligned}$$

Since $a + a^2P \le 1$, then the maximum value of the function $f(\alpha)$ over the interval $\alpha \in [0, 1]$ is achieved at $\alpha = 1$, and is equal to $\log_2\left(1 + \frac{P}{1+aP}\right)$. The latter value corresponds to the maximal total rate which is achieved when the interfering signal is Gaussian distributed, and when the detector treats it as part of the additive Gaussian noise of the received signal (i.e., in addition to the AWGN of the channel). If on the other hand $a + a^2 P \geq 1$ and $a \leq 1$ (i.e., $\frac{\sqrt{1+4P}-1}{2P} \leq a \leq 1$), then we can separate the latter condition into two sub-cases: If $1 \leq a + a^2 P \leq 1 + \alpha P$, then same as we had before

$$f(\alpha) = \log_2 \left(1 + \frac{a\overline{\alpha}P + \alpha P}{1 + a\alpha P} \right).$$

Otherwise, if $1 \leq 1 + \alpha P \leq a + a^2 P$, then it follows from Eq. (21) that

$$f(\alpha) = \log_2 \left(1 + \frac{\overline{\alpha}P}{1 + \alpha P + aP} \right) + \log_2 \left(1 + \frac{\alpha P}{1 + a\alpha P} \right).$$

Let $\alpha_0 \triangleq \frac{a+a^2P-1}{P}$ (so that $0 \leq \alpha_0 \leq 1$ (since $a \leq 1$), and $1 + \alpha_0 P = a + a^2 P$). A short calculation reveals that the maximal total rate which is achieved by the rate region (20) is at least equal to

$$f(\alpha_0) = \log_2 \left(\frac{a + aP + a^2 P}{1 - a + a^2(aP + 1)} \right).$$

Let us define the function

$$g(a) \triangleq \frac{a + aP + a^2 P}{1 - a + a^2(aP + 1)}, \quad \frac{\sqrt{1 + 4P} - 1}{2P} \leq a \leq 1. \tag{22}$$

From the monotonicity of the logarithm function, the maximal value of the total rate in the achievable rate region (20) is lower bounded by the logarithm of $g(a)$ for all values of a within the interval $\frac{\sqrt{1+4P}-1}{2P} \leq a \leq 1$. We intend to find the maximum of $g(\cdot)$ on the latter interval, and also to find a range of values for a, so that the lower bound on the total rate $(R_1 + R_2)$ exceeds $\frac{1}{2}\log_2(1 + 2P)$; the latter value is the maximal total rate which is obtained by optimal TDM/ FDM.

It can be easily shown that the derivative of $g(\cdot)$ is a monotonic decreasing function in the interval $\frac{\sqrt{1+4P}-1}{2P} \leq a \leq 1$, and it has opposite signs at the endpoints of this interval. Therefore, there exits a single point inside this interval where the derivative of $g(\cdot)$ is equal to zero, which also achieves the maximal value of $g(\cdot)$ inside this interval. By differentiating the function $g(\cdot)$, and setting the derivative to zero, straightforward algebra shows that the maximal value of $g(\cdot)$ inside this interval can be calculated by solving the polynomial equation (9). Let $a = a_0$ be the solution of the polynomial equation (9) which is in the required interval (see Theorem 2). Then, a lower bound on the maximal $R_1 + R_2$ which is attained by the achievable rate region (20) is above the value which corresponds to TDM/ FDM if

$$\log_2 \left(\frac{a_0 + a_0 P + a_0^2 P}{1 - a_0 + a_0^2(a_0 P + 1)} \right) \geq \frac{1}{2}\log_2(1 + 2P) \tag{23}$$

where it can be verified numerically that inequality (23) is satisfied for values of P above 17 dB.

Discussion: We will now consider a generalization of the achievable rate region of Carleial considered above (based on [3, Theorem 6]), which also includes the two approaches of treating the weak signal as additive noise (for weak interference) or the TDM/ FDM approach (for moderate interference). To this end, we will consider the particular case of the achievable rate region of Han and Kobayashi [9] where the size of the alphabet of the time-sharing parameter Q is four, the random variables U_1, U_2, W_1 and W_2 are conditionally independent given Q (their distributions are given in Table 1 for every possible value of Q), and where the random variables X_1 and X_2 are given by

$$X_1 = U_1 + W_1, \quad X_2 = U_2 + W_2. \tag{24}$$

Based on Table 1 and Eq. (24)

$$E[x_1^2] = \delta \cdot 2\delta P + \delta \cdot 2\delta P + \lambda(1-2\delta) \cdot \frac{(1+2\delta)P}{\lambda} = P$$

and similarly $E[x_2^2] = P$, so both inputs of the two-user Gaussian IC satisfy the common power constraint (3) where $P_1 = P_2 \triangleq P$. By symmetry considerations, it is clear that the maximal total rate of the considered achievable rate region is obtained with $\lambda = \frac{1}{2}$.

The achievable rate region by TDM/ FDM results in as a particular case of the HK region with the setting in Table 1 when $\delta = 0$. The special achievable rate region where the interfering signal is treated as an additive Gaussian noise results in as a particular case of the HK region with the setting in Table 1 for $\delta = \frac{1}{2}$ and $\alpha = \beta = 1$. Finally, The particular achievable rate region of Carleial (see [3, Theorem 6]) for the symmetric two-user Gaussian IC results in from the setting in Table 1 when $\delta = \frac{1}{2}$ (which is verified from the superposition in Eq. (24)). These simple observations yield that the maximal total rate of the achievable rate region of Han and Kobayashi in the setting of Table 1 is not below the maximal total rate which one obtains by treating the interfering signal as an additive noise (in the case of weak interference), and it is also not below the maximal total rate which is obtained by optimal TDM/ FDM (in the case of moderate interference). From the discussion above regarding Carleial's achievable rate region (20), we obtained that for high values of P, the maximal total rate of the latter region (and hence, the maximal total rate which corresponds to the Han and Kobayashi (HK) region with the setting in Table 1) exceeds the maximal total rate which is obtained by the two aforementioned methods (for weak and moderate interference). In the sequel, we will calculate the maximal total rate which corresponds to the HK region with the specific setting in Table 1, and then present numerical results of the resulting expression in Section 5. The discussion in [9] yields that the maximal total rate of the achievable rate region which is defined by Han and Kobayashi is calculated by the maximization of

$$\max(R_1 + R_2) = \rho_{12} \tag{25}$$

where

$$\rho_{12} \triangleq \sigma_{12} + I(Y_1; U_1 | W_1, W_2, Q) + I(Y_2; U_2 | W_1, W_2, Q) \tag{26}$$

and

$$\sigma_{12} \triangleq \min\Big\{ \begin{array}{l} I(Y_1; W_1, W_2|Q),\ I(Y_2; W_1 W_2|Q), \\ I(Y_1; W_1|W_2, Q) + I(Y_2; W_2|W_1, Q), \\ I(Y_2; W_1|W_2, Q) + I(Y_1; W_2|W_2, Q) \end{array} \Big\}. \tag{27}$$

Based on the definition of the function $\gamma(\cdot)$ in Eq. (5), we obtain that for $\lambda = \frac{1}{2}$

$$\begin{aligned} & I(Y_1; U_1|W_1, W_2, Q) = I(Y_2; U_2|W_1, W_2, Q) \\ & \quad = \delta \cdot \gamma\left(\frac{2\alpha\delta P}{1 + 2a\beta\delta P}\right) + \delta \cdot \gamma\left(\frac{2\beta\delta P}{1 + 2a\alpha\delta P}\right) + \left(\frac{1-2\delta}{2}\right) \cdot \gamma\big(2(1+2\delta)P\big), \end{aligned} \tag{28}$$

$$I(Y_1; W_1, W_2|Q) = I(Y_2; W_1, W_2|Q) = \delta \cdot \gamma\left(\frac{2\overline{\alpha}\delta P + 2a\overline{\beta}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P}\right) + \delta \cdot \gamma\left(\frac{2\overline{\beta}\delta P + 2a\overline{\alpha}\delta P}{1 + 2\beta\delta P + 2a\alpha\delta P}\right) \tag{29}$$

$$I(Y_1; W_1|W_2, Q) = I(Y_2; W_2|W_1, Q) = \delta \cdot \gamma\left(\frac{2\overline{\alpha}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P}\right) + \delta \cdot \gamma\left(\frac{2\overline{\beta}\delta P}{1 + 2\beta\delta P + 2a\alpha\delta P}\right) \tag{30}$$

$$I(Y_2; W_1|W_2, Q) = I(Y_1; W_2|W_1, Q) = \delta \cdot \gamma\left(\frac{2a\overline{\alpha}\delta P}{1 + 2a\alpha\delta P + 2\beta\delta P}\right) + \delta \cdot \gamma\left(\frac{2a\overline{\beta}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P}\right). \quad (31)$$

By substituting Eqs. (26)–(31) in (25), we obtain that the maximal total rate of the achievable rate region which is defined in Table 1 is equal to the maximum of the function

$$\begin{aligned}
\rho_{12} &\triangleq \rho_{12}(\alpha, \beta, \delta) \\
&= 2\delta \cdot \gamma\left(\frac{2\alpha\delta P}{1 + 2a\beta\delta P}\right) + 2\delta \cdot \gamma\left(\frac{2\beta\delta P}{1 + 2a\alpha\delta P}\right) + (1 - 2\delta) \cdot \gamma\big(2(1 + 2\delta)P\big) \\
&\quad + \min\Bigg\{\delta \cdot \gamma\left(\frac{2\overline{\alpha}\delta P + 2a\overline{\beta}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P}\right) + \delta \cdot \gamma\left(\frac{2\overline{\beta}\delta P + 2a\overline{\alpha}\delta P}{1 + 2\beta\delta P + 2a\alpha\delta P}\right), \\
&\qquad 2\delta \cdot \gamma\left(\frac{2\overline{\alpha}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P}\right) + 2\delta \cdot \gamma\left(\frac{2\overline{\beta}\delta P}{1 + 2\beta\delta P + 2a\alpha\delta P}\right), \\
&\qquad 2\delta \cdot \gamma\left(\frac{2a\overline{\alpha}\delta P}{1 + 2a\alpha\delta P + 2\beta\delta P}\right) + 2\delta \cdot \gamma\left(\frac{2a\overline{\beta}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P}\right)\Bigg\}
\end{aligned} \quad (32)$$

where the function $\gamma(\cdot)$ is defined in Eq. (5), and the maximization of ρ_{12} is carried over the parameters (α, β, δ) where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \delta \leq \frac{1}{2}$.

We note that Corollary 2 follows from the scaling of the input signals in order to obtain the standard form (2) for the two-user Gaussian IC.

5 Numerical Results

We present here numerical results which illustrate the theorems in Section 3, regarding achievable rate regions and bounds on the sum-capacity of the two-user Gaussian IC.

Fig. 1 compares the achievable rate region for a two-user Gaussian IC where $a_{12} = a_{21} = 0.5$ in the standard form (2), and with a common power constraint $P_1 = P_2 = 6$ in (3). As expected from Theorem 1, the achievable rate region by TDM/ FDM (curve 1) is included in the achievable rate region which is specified in Theorem 1 (curve 4), but the maximal total rate in both cases stays the same. Based on Theorem 1, the achievable rate region whose boundary is curve 2 is included in the achievable rate region which is obtained in Theorem 1. Since in our setting $P = 6$, $a = 0.5$, and $a > \frac{\sqrt{1+2P}-1}{2P} = 0.2171$, then it follows from Theorem 1 that the particular achievable rate region $\mathcal{G}'$ (see curve 3) is included in the achievable rate region of Theorem 1 (see curve 4), as is illustrated in Fig. 1. Moreover, the maximal achievable total rate in the former region is strictly smaller than the one which corresponds to the latter achievable rate region.

Fig. 2 compares upper and lower bounds on the sum-capacity of a two-user symmetric Gaussian IC. We consider here the symmetric case where in the standard form, $P_1 = P_2 \triangleq P$ dB, and $a_{12} = a_{21} \triangleq a$ (where a designates the square of the magnitude of the interference coefficient in (2), and it is the horizontal axis in Fig. 2). As an example, for the case where $P = 30$ dB, it follows that $a_0 = 0.0821$ is the appropriate solution of the polynomial equation in Theorem 3. As can be verified from Fig. 2, if the common value of a_{12} and a_{21} (under the symmetry assumption) is between 0.0821 and 1, then the maximal total rate which is achieved by curve 5 is strictly larger than the one which refers to TDM/ FDM (curve 2). In particular, if $a_{12} = a_{21} = 0.0821$, the values of the maximal total rate which are achieved by curves 2 and 5 are 5.483 and 5.911 bits per channel use, respectively (interestingly, the latter value coincides with the lower bound on the maximal total rate which is given in Eq. (10), as compared to an upper bound of 6.774 bits per channel use, which follows from Kramer's upper bound [10] and is depicted in curve 1 of Fig. 2). Moreover, the solution of inequality (23) (where a_0 in (23) is now replaced with the arbitrary common value of the interference coefficient a) yields that $0.0448 \leq a \leq 0.1382$. Clearly, as reflected in Fig. 2, it is a partial interval as compared to the interval for which curve 5 in Fig. 2 gets values above the horizontal line in curve 2 (where the latter corresponds to the maximal total rate which is obtained by TDM/ FDM). The reason for this observation is related to the simplifications which finally led to the derivation of inequality (23), referring to a looser achievable rate region as compared to one which corresponds to the particular Han and Kobayashi rate region with the setting in Table 1 and Eq. (24). However, it is interesting to note the lower bound to a (i.e., 0.0448) approximates well the lower limit of a, for which the total rate in curves 3, 4, and 5 is above the value which corresponds to curve 2 (see Fig. 2). This phenomenon was verified also for other values of P above 17 dB. This value is the threshold on P for which there exists a bump in Fig. 2 which in turn signals a total rate above the maximal total rate which is obtained by TDM/FDM.

The sum-capacity of a degraded or a one-sided Gaussian IC is provided in Eqs. (7) and (8), respectively. The sum-capacity of a one-sided Gaussian IC is shown in Fig. 3, where we assume a common power constraint of $P = 6$. It is a monotonic decreasing function of the interference coefficient (a) in the range $0 \leq a \leq 1$, and a monotonic increasing function of a for strong interference (i.e., for $1 \leq a \leq 1 + P$) [14]. For very strong interference, the sum-capacity stays constant because the interference does not harm, as was demonstrated by Carleial [2]. The two limit cases in Fig. 3 where $a = 0$ and $a = 1$ correspond to two separate AWGN channels and to a Gaussian multiple-access channel, and therefore the sum-capacity is equal to $\log_2(1 + P) = 2.807$ and $\frac{1}{2}\log_2(1 + 2P) = 1.850$ bits per channel use, respectively (see Fig. 3).

6 Concluding Remarks

1. In [9, Theorem 3.1], Han and Kobayashi derived an achievable rate region for a general discrete memoryless IC. It is the best reported achievable rate region; it yields as particular cases all the previously reported achievable rate regions by Sato, Carleial and others (including the achievable rate region which is specified in Theorem 1 here), but it is in general prohibitively complex to calculate this rate region. For a two-user Gaussian IC with weak or moderate interference, the calculation of their achievable region in [9, Theorem 3.1] is not feasible. However, the achievable rate region D which is specified in Theorem 1 is feasible for calculation, and for a two-user Gaussian IC with *moderate* interference (i.e., if $\frac{\sqrt{1+2P}-1}{2P} < a \leq 1$), it includes the particular achievable rate region $\mathcal{G}'$ which was derived by Han and Kobayashi in [9, Section 5A] (as opposed to their general achievable rate region, the particular rate region $\mathcal{G}'$ is feasible for calculation, and it also gives the exact capacity region of a two-user Gaussian IC with *strong* interference). For moderate interference, the maximal total rate which is achieved by the rate region D in Theorem 1 is strictly larger than the value of the maximal total rate which corresponds to $\mathcal{G}'$. We note that although for a two-user Gaussian IC with *weak* interference (i.e., for $0 < a < \frac{\sqrt{1+2P}-1}{2P}$), the maximal total rate which is obtained by $\mathcal{G}'$ is strictly larger than the one which corresponds to the achievable rate region D in Theorem 1, the region D is not necessarily included in $\mathcal{G}'$ for the case of weak interference; on the other hand, Theorem 1 ensures that $\mathcal{G}' \subset D$ for a two-user Gaussian IC with moderate interference.

2. It is not clear whether the sum-capacity should be necessarily a decreasing function of the

common interference coefficient when its value varies between zero and unity (since there is no co-operation between the receivers). Nonetheless, we believe that the bump which is observed in curves 3, 4 and 5 of Fig. 2 is an artifact of the inner bounds on the capacity region: for one-sided or degraded Gaussian IC, the sum-capacity is a monotonic decreasing function in the range of weak/ moderate interference (as follows from Theorem 2, and exemplified in Fig. 3), so we believe that it is likely to be the case in general for a two-user Gaussian IC.

3. We note that the sum-capacity of a one-sided Gaussian IC which is provided in Eq. (8) implies immediately the upper bound on the sum-capacity of a two-user Gaussian IC which is presented in [10, Theorem 1]. To see that, let us consider a two-user Gaussian IC in the standard form (see Eq. (2)) where the power constraints on the first and the second inputs are P_1 and P_2, respectively. For simplicity, we will only consider the case where the interference coefficients a_{12} and a_{21} in the standard form (2) are both between zero and one, though the bound will follow easily in the same way for all the other cases. Clearly, if we cancel one of the interferences between the two senders (i.e., if we set $a_{12} = 0$ or $a_{21} = 0$), then the capacity region of the resulting one-sided Gaussian IC cannot shrink as compared to the capacity region of the original Gaussian IC. Hence, the minimal value between the sum-capacities of the two resulting one-sided Gaussian ICs forms an upper bound on the sum-capacity of the original two-user Gaussian IC (whose interference coefficients a_{12} and a_{21} are both non-zero). From the expression for the exact sum-capacity of a one-sided Gaussian IC in Eq. (8), it follows that the sum-capacity of the original two-user Gaussian IC satisfies the inequality

$$R_1 + R_2 \leq \min\left\{\gamma(P_1) + \gamma\left(\frac{P_2}{1 + a_{21}P_1}\right),\ \gamma(P_2) + \gamma\left(\frac{P_1}{1 + a_{12}P_2}\right)\right\}. \tag{33}$$

Since $\gamma(x) = \frac{1}{2}\,\log_2(1 + x)$ and since we examine here the case where $0 \leq a_{12}, a_{21} \leq 1$, then it follows immediately that

$$\gamma(P_1) + \gamma\left(\frac{P_2}{1 + a_{21}P_1}\right) = \frac{1}{2}\,\log_2\left((P_2 + a_{21}P_1 + 1)\left(\frac{P_1 + 1}{\min(1, a_{21})P_1 + 1}\right)\right) \tag{34}$$

and

$$\gamma(P_2) + \gamma\left(\frac{P_1}{1 + a_{12}P_2}\right) = \frac{1}{2}\,\log_2\left((P_1 + a_{12}P_2 + 1)\left(\frac{P_2 + 1}{\min(1, a_{12})P_2 + 1}\right)\right) \tag{35}$$

so the combination of Eqs. (33), (34) and (35) provides the upper bound on the sum-capacity of a two-user Gaussian IC which appears in [10, Theorem 1]. It also automatically proves that the upper bound on the sum-capacity in [10, Theorem 2] is always better than the one in [10, Theorem 1]. It is noted here that our notation for the interference coefficients in Eq. (2) are consistent with those in the paper of Han and Kobayashi [9, Eqs. (5.3), (5.4)], but a_{12} and a_{21} in [10] should be reversed as compared to the notation here.

4. In continuation to item 3, we note that the two upper bounds on the sum-capacity of a two-user Gaussian IC which are presented in [10] coincide asymptotically in the limit where we let P tend to infinity. For simplicity, it is shown in the symmetric case where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a$. In the symmetric case, the upper bounds on the sum-capacity which appear in [10, Theorem 1] and [10, Theorem 2] on the sum-capacity of a two-user Gaussian IC are

$$R_1 + R_2 \leq \frac{1}{2}\,\log_2(1 + P) + \frac{1}{2}\,\log_2\left(1 + \frac{P}{1 + aP}\right) \tag{36}$$

and

$$R_1 + R_2 \leq \log_2\left(1 + \frac{-(1 + a) + \sqrt{(1 + a)^2 + 4a(1 + a)P}}{2a}\right) \tag{37}$$

so it can be easily verified that if we let P tend to infinity, then ratio of the left sides of Eqs. (36) and (37) tends to 1, which means that these two upper bounds coincide in the limit where $P \to \infty$. For finite P, as was mentioned in [10] and in item 3 above, the upper bound on the sum-capacity in Eq. (37) is always tighter than the upper bound in Eq. (36).

5. For a Gaussian IC, it was stated in Theorem 1 of [5] that if one of the senders is transmitting at its maximal possible rate (i.e., the maximal rate which is achievable for a single-user, in the absence of interference), then the other sender is obliged to decrease its data rate to the point where both receivers can reliably decode its message. The implication of the proof of this converse theorem is that these two rate-pairs form the corner points of the capacity region of a two-user Gaussian IC. For a two-user Gaussian IC with strong interference whose characterization of its capacity region is completely known (see [9, Theorem 5.2] and [14]), this is indeed the case, as the capacity region in the latter case is the intersection of the capacity regions of the two Gaussian multiple-access channels which are induced by the two-user Gaussian interference channel. Unfortunately, it is not clear to be the case in general due to a problem in a certain step of the proof to the converse theorem in [5, Appendix B] (which was confirmed by the author, though we could not fix this problematic step in the proof). By restating this step as a pure mathematical problem (i.e., not necessarily related to Gaussian interference channels), it considers the following issue: let X and Y be two n-dimensional random vectors, where X is a Gaussian random vector with i.i.d. components of zero mean and variance σ^2. The differential entropy of X attains its is maximal value under the above power constraint, and $h(X) = \frac{n}{2} \log(2\pi e \sigma^2)$. Y is an n-dimensional random vector whose components satisfy the following constraints:

$$E[Y_i] = 0, \quad \frac{1}{n}\sum_{i=1}^{n}(Y_i)^2 \le \sigma^2. \tag{38}$$

Unlike X, the components of Y may be correlated, and Y may *not* be Gaussian. However, Y is "almost Gaussian" in the sense that

$$h(X) - h(Y) \le n\varepsilon \tag{39}$$

for some positive constant ε (we note that under the conditions in (38), $h(X) - h(Y)$ is necessarily non-negative.) We introduce now an arbitrary n-dimensional random vector Z where in probability 1, the vector Z is inside a sphere of radius $\sqrt{nP}$ for a positive constant P, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n}(z_i)^2 \le P, \tag{40}$$

and Z is known to be statistically independent of X and Y. The question is if under the assumptions in (38)–(40), one can prove an inequality of the form:

$$h(Y+Z) - h(X+Z) \le \delta(\varepsilon, P) \cdot n \tag{41}$$

where $\delta(\varepsilon, P) \to 0$ as $\varepsilon \to 0$?

In essence, the proof in [5, Appendix B] shows that

$$D(P_Y || P_X) \le n\varepsilon,$$

where P_X and P_Y designate the probability distributions of X and Y, respectively, and $D(P||Q)$ stands for the divergence between the probability distributions P and Q. Then, by the data processing theorem, it follows that

$$D(P_{Y+Z} || P_{X+Z}) \le D(P_Y || P_X) \le n\varepsilon.$$

The proof in [5, Appendix B] relies on Pinsker's inequality which is a lower bound on the divergence in terms of the L_1 distance between the two probability measures. However, the problematic issue in this proof is related to the link between the L_1 distance and the difference between the two differential entropies. Specifically, [5] utilizes a parameter β which depends exponentially in n, but is treated after [5, Eq. (B.11)] as if it was a constant. This yields that $\log(\beta)$ in [5, Eq. (B.14)] grows linearly in n, and hence the right-hand side of [5, Eq. (B.14)] grows like $n^{\frac{3}{2}}$ (and not linearly, as desired).

6. One possible direction in trying to fix this problematic issue which is addressed in item 2 relies on recent improvements on Pinsker's inequality [8]. One of these refinements (see [8, Theorem 7]) yields that

$$D(P||Q) \geq \frac{1}{2}V^2 + \frac{1}{36}V^4 + \frac{1}{270}V^6 + \frac{221}{340,200}V^8 \tag{42}$$

where V designates the L_1 distance between P and Q, i.e., $V \triangleq \int_{-\infty}^{\infty} |P(x) - Q(x)| \, \mathrm{d}x$. Eq. (42) enables to reduce the power of n in the right hand-side of [5, inequality (B.14)] from $\frac{3}{2}$ to $\frac{9}{8}$. A further refinement of Pinsker's inequality (see [8, Section 4]) enables to further reduce the power of n to $\frac{49}{48}$, but it is still not the desired *linear dependence* in n which is required to complete the proof in [5]. We also note that based on the discussion in [8, p. 1496], one cannot hope to reduce the power of n in (41) below $\frac{59}{58}$. However, a second thought reveals that since $0 \leq V \leq 2$, one automatically gets from Pinsker's inequality a looser lower bound on the divergence

$$D(P||Q) \geq \frac{V^2}{2}\left(\frac{V}{2}\right)^{k-2} = \frac{V^k}{2^{k-1}}, \quad k = 2, 3, \ldots \tag{43}$$

which implies that the power of n in (41) can be made arbitrarily close to unity (by taking the value of the integer k in (43) sufficiently large), though it will be still above 1. We note that in order to make the power of n in (41) equal to 1, we need to let k in (43) tend to infinity, but this will imply that $\sqrt[q]{\varepsilon} \to 1$ for an arbitrary positive value of ε (so in the latter case, $\delta(\varepsilon, P)$ will not tend to zero when $\varepsilon \to 0$, as required in Eq. (41)). It therefore yields that a new line of attack is needed to solve the problem.

Solving this pure mathematical problem has a direct implication on the characterization of the two corner points of the capacity region of a general Gaussian interference channel (it is likely that the two rate pairs which were specified in [5, Theorem 1] are indeed the two corner points of the capacity region of a two-user Gaussian IC.) As a consequence of the discussion here, it is not clear whether the above two rate-pairs are indeed corner points on the boundary of the capacity region of a general Gaussian interference channel. Our impression, which is based on the case of a Gaussian interference channel with strong/ very strong interference is that these two points are indeed two corner points of the capacity region, and that [5, Theorem 1] holds, though the problematic link in the proof needs a rigorous proof.

7. The derivation of an achievable rate region for the two-user Gaussian IC refers to vanishing *average decoding error probability*. The maximal-error and average-error capacity regions are *not necessarily identical* for general multi-user channels; In [7], Dueck has shown by examples that for two-way channels and for multiple-access channels, the capacity regions depend on the error concept used. Though these two capacity regions are identical for a general broadcast channel [6, 16], this is not necessarily the case for general multi-user channels. In particular, these two capacity regions are not necessarily identical for interference channels (otherwise, this would also be the case for a general multiple-access channel, in contradiction to [7]).

Acknowledgment: The author wishes to acknowledge Emre Telatar for stimulating discussions on interference channels, and for his comments on the first version of this paper. The author is also thankful to Gerhard Kramer for a discussion on his paper [10], and to Max Henrique Costa for personal communications related to his paper on the Gaussian interference channel [5]. Finally, the author is grateful to the Associate Editor, Amos Lapidoth, and to the three anonymous referees for their helpful suggestions which improved the lucidity of the presentation.

References

[1] R. Ahlswede, "The capacity region of a channel with two senders and two receivers," *Annals of Probability*, vol. 2, no. 5, pp. 805–814, October 1974.

[2] A. B. Carleial, "A case where interference does not reduce capacity," *IEEE Trans. on Information Theory*, vol. 21, no. 5, pp. 569–570, September 1975.

[3] A. B. Carleial, "Interference channels," *IEEE Trans. on Information Theory*, vol. 24, no. 1, pp. 60–70, January 1978.

[4] R. S. Cheng and S. Verdu, "On limiting characterizations of memoryless multiuser capacity regions," *IEEE Trans. on Information Theory*, vol. 39, no. 2, pp. 609–612, March 1993.

[5] M. H. M. Costa, "On the Gaussian interference channel," *IEEE Trans. on Information Theory*, vol. 31, no. 5, pp. 607–615, September 1985.

[6] I. Csiszár and J. Körner, *Information Theory: Theorems for Discrete Memoryless Systems*, Budapest, 1981.

[7] G. Dueck, "Maximal error capacity regions are smaller than average for multi-user channels," *Problems of Control and Information Theory*, vol. 7, no. 1, pp. 11–19, January 1978.

[8] A. Fedotov, P. Harremöes and F. Topsϕe, "Refinements of Pinsker's inequality," *IEEE Trans. on Information Theory*, vol. 49, no. 6, pp. 1491–1498, June 2003.

[9] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. on Information Theory*, vol. 27, no. 1, pp. 49–60, January 1981.

[10] G. Kramer, "Genie-aided outer bounds on the capacity of interference channels," *Proceedings 2001 IEEE International Symposium on Information Theory*, p. 103, Washington, DC, USA, June 24–29, 2001.

[11] M. Mandell and R. J. McEliece, "Some properties of memoryless multi-terminal interference channels," *Proceedings 1991 IEEE International Symposium on Information Theory*, p. 212, Budapest, Hungary, June 1991.

[12] H. Sato, "Two-user communication channels," *IEEE Trans. on Information Theory*, vol. 23, no. 3, pp. 295–304, May 1977.

[13] H. Sato, "On degraded Gaussian two-user channels," *IEEE Trans. on Information Theory*, vol. 24, no. 5, pp. 637–640, September 1978.

[14] H. Sato, "The capacity of the Gaussian interference channel under strong interference," *IEEE Trans. on Information Theory*, vol. 27, no. 6, pp. 786–788, November 1981.

[15] E. C. van der Meulen, "Some reflections on the interference channel," pp. 409–421, Kluwer, Boston, 1994.

[16] F. M. J. Willems, "The maximal-error and average-error capacity regions of the broadcast channel are identical: A direct proof," *Problems of Control and Information Theory*, vol. 19, no. 4, pp. 339-347, 1990.

Table Caption

Table 1: The parameters Q, U_1, W_1, U_2, W_2 which are used to define a particular sub-region of the achievable rate region of Han and Kobayashi [9]. The random variables U_1, U_2, W_1 and W_2 are conditionally independent given Q, and the random variables X_1 and X_2 are given in Eq. (24). We assume here that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \lambda \leq 1$, and $0 \leq \delta \leq \frac{1}{2}$.

Figure Captions

Figure 1: Achievable rate regions for a two-user Gaussian IC with weak or moderate interference coefficients. Curve 1 is the boundary of the achievable rate region by TDM/ FDM. Curve 2 is the boundary of the achievable rate region which is obtained by time-sharing between the two rate pairs which are specified in Eq. (6). Curve 3 is the boundary of the region $\mathcal{G}'$ which was derived by Han and Kobayashi as a particular case of their general achievable rate region (see [9, Eqs. (4.1)–(4.9), (5.9)], and the setting in [9, Section 5.A]). Curve 4 is the achievable rate region in Eq. (18) here.

Figure 2: Upper and lower bounds on the sum-capacity of a two-user Gaussian IC. The curves which are depicted in Fig. 2 correspond to the following bounds: Curve 1 is Kramer's upper bound on the sum-capacity [10, Theorem 2]. Curve 2 is the simple version of Carleial's lower bound [3] which treats the interfering signal as an additive Gaussian noise for weak interference, and which relies on the TDM/ FDM approach for moderate interference. Curve 3 is the maximal total rate which is achieved with the particular sub-region of Han and Kobayashi ($\mathcal{G}'$) in Section 5.A of [9] (where in the latter case, time-sharing is not performed). Curves 4 and 5 refer to the maximal total rates which are obtained for the particular cases of the general achievable rate region of Han and Kobayashi [9] with the specific setting in Table 6 and in Eq. (24); curve 4 refers to the case where $\delta = \frac{1}{2}$ in Table 6 (so, the time-sharing parameter Q is a binary random variable), and curve 5 of Fig. 2 corresponds to the case where the maximization of the total rate is carried over the additional parameter δ which should be in the interval $[0, \frac{1}{2}]$ (so in the latter case, Q is a random variable whose alphabet is of size four). Curves 4 and 5 are therefore calculated by the maximization of the right-hand side of Eq. (32) where for the calculation of curve 4, we set $\delta = \frac{1}{2}$ and maximize ρ_{12} over the parameters $(\alpha, \beta) \in [0,1] \times [0,1]$, and the calculation of curve 5 is performed by the numerical maximization of ρ_{12} in (32) over the three parameters α, β and δ (where $0 \leq \delta \leq \frac{1}{2}$).

Figure 3: The sum-capacity a *one-sided* Gaussian IC with a common power constraint $P_1 = P_2 = 6$.

Table

Time-sharing parameter	U_1	W_1	U_2	W_2
$\text{Prob}\,(Q=0)=\delta$	$\sim N(0, 2\alpha\delta P)$	$\sim N(0, 2\overline{\alpha}\delta P)$	$\sim N(0, 2\beta\delta P)$	$\sim N(0, 2\overline{\beta}\delta P)$
$\text{Prob}\,(Q=1)=\delta$	$\sim N(0, 2\beta\delta P)$	$\sim N(0, 2\overline{\beta}\delta P)$	$\sim N(0, 2\alpha\delta P)$	$\sim N(0, 2\overline{\alpha}\delta P)$
$\text{Prob}\,(Q=2)=\lambda(1-2\delta)$	$\sim N(0, \frac{(1+2\delta)P}{\lambda})$	0	0	0
$\text{Prob}\,(Q=3)=\overline{\lambda}(1-2\delta)$	0	0	$\sim N(0, \frac{(1+2\delta)P}{\overline{\lambda}})$	0

Table 1

Figures

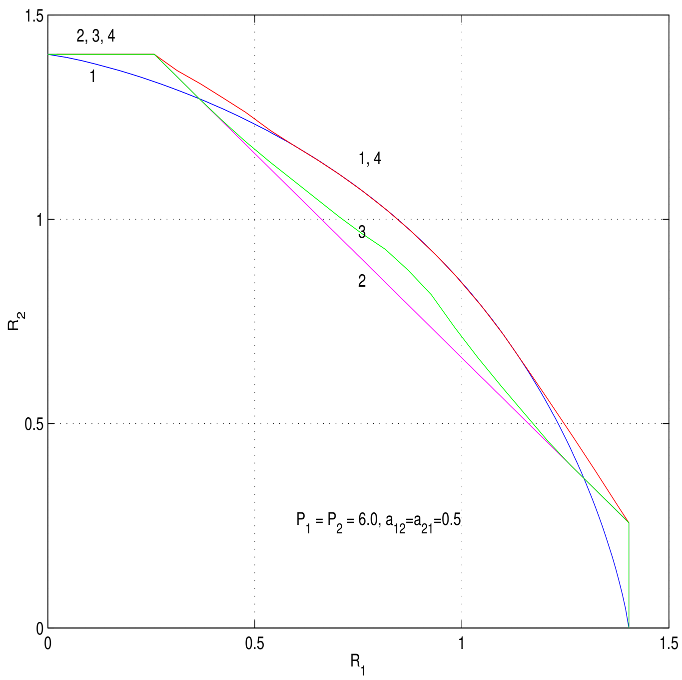

Figure 1

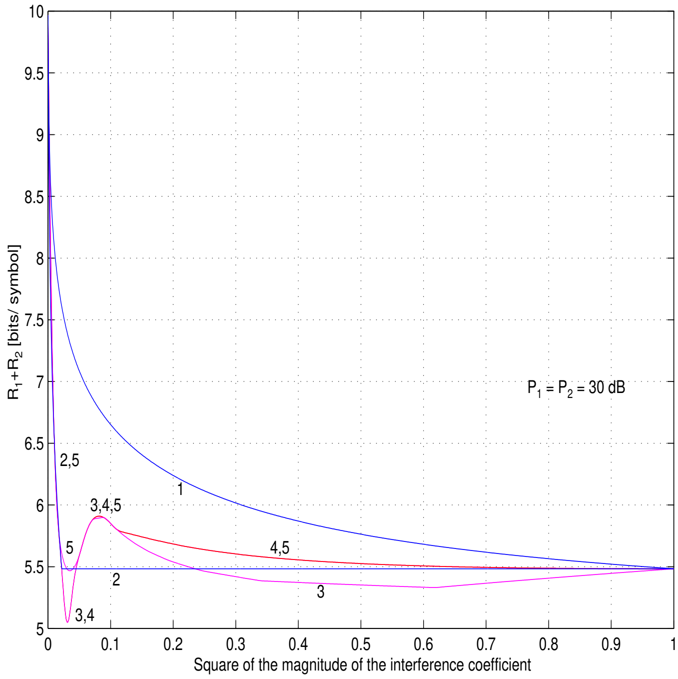

Figure 2

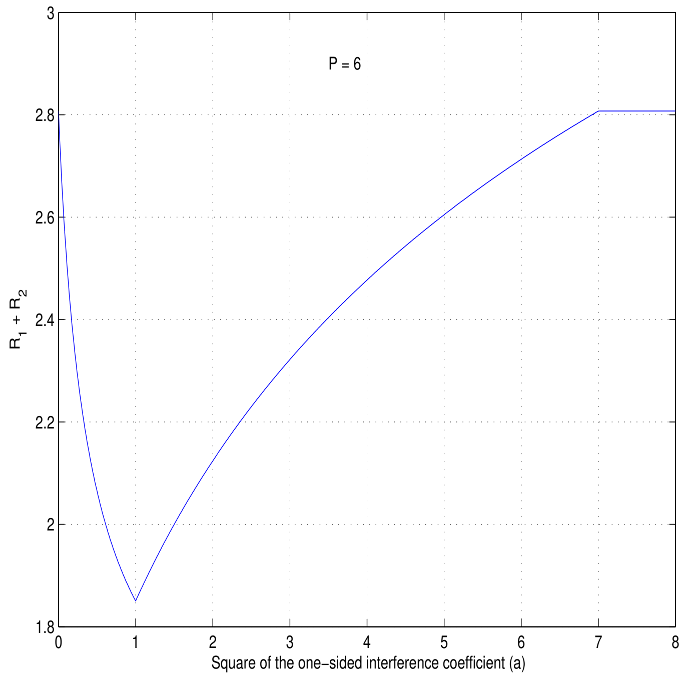

Figure 3