CCIT Report #489 June 2004

# Transmitting to Co-located Users in Wireless Ad-Hoc and Sensory Networks *†

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**Abstract**

We consider wireless Ad-Hoc networks and sensory networks where a remotely located source is transmitting information to a destined user embedded within a group of $K$ densely packed physically co-located users enjoying outstanding SNR conditions among themselves, but suffering from quasi-static flat Rayleigh fading with respect to the source. Stringent delay constraints require that information, once available, be transmitted immediately and delivered reliably to its destination during a period of one fading block, precluding waiting until the destined user enjoy's optimal fading condition. A cooperative transmission strategy is proposed for this scenario and its expected throughput is investigated. The strategy exhibits a substantial gain in throughput especially when the co-location gain factor is high. In addition, a broadcast approach is incorporated into the transmission strategy suggesting further throughput benefits.

*Keywords*— Wireless networks, Ad-Hoc networks, Fading channels, Cooperative diversity, Relay channel, Expected throughput, Outage capacity.

*Submitted to the *IEEE Trans. Inform. Theory*, December 2003.

†This work was presented in part at CTW 2004, Capri, Italy, May 2004, and will be presented at *IEEE Int. Symp. Inform. Theory*, Chicago, Illinois, USA, June 27 – July 2, 2004.

# 1 Introduction

The tremendous advance in wireless communication technology during recent years has led to a growing interest in the performance limits of wireless systems which do not necessarily rely on pre-existing infrastructure (such as base stations in cellular communications) to facilitate the communication between the users. In this context two types of systems are often mentioned. *Wireless Ad-Hoc networks* are typically used to enable communication between a collection of mobile users, whereas *wireless sensory networks* are typically large collections of small devices spatially distributed around an environment and communicating with some control point. In both cases, the nodes in the network might suffer from severe attenuations in their received signals (fading).

In *sensor networks* the sensor devices are primarily used to gather environmental information, process it and transmit it back to some control point. On the opposite direction, queries might be used to allow efficient use of the sensors by asking only certain devices to operate for a given monitoring task. Moreover, sensor networks generally depend on *dense sensor deployment* and *physical co-location* with their targets in order to achieve their goals. This implies that hundreds or even thousands of sensor nodes are deployed in a relatively small area only a few meters apart. The *physical co-location* simplifies signal processing problems and causes the sensors to be tightly coupled with their environment.

Our paper draws its motivation primarily from *sensor networks* applications and focuses on the communication channel between the central control node and the group of physically co-located sensors. However, the results of this work can be equally applicable to *wireless Ad-Hoc networks* or any type of wireless networks where a remotely located source wishes to transfer information, and in a promptly manner, over a wireless medium to a group of co-located users. It is by now known that in a wireless network not depending on any infrastructure some form of cooperation between the users is necessary if one is to achieve large information rates. This cooperation often takes the form of multihop relaying, where packets are delivered (relayed) on a peer-to-peer basis until they reach their destinations. The achievable information rates in such networks and their behavior for large number of users were studied in [1], [2], [3], [4], [5] [6], [7], and [8], where the main result was that as the number of users $n$ increases in a fixed Ad-Hoc network, the information rate per source-destination pair diminishes as $1/\sqrt{n}$.

By introducing mobility into the model and assuming the end-to-end delay constraints are of the same time-scale as the topology changes of the network due to mobility, it was shown in [9] that the capacity per source-destination-pair does not diminish, rather it can remain constant with the increase in the number of users. The strategy was based on transmitting to a multitude of relays simultaneously and assuming that at least one of them has good enough channel conditions to relay the message successfully. This type of strategy is sometimes referred to as *multiuser diversity* (see in this respect also [10]). For more work based on the point-to-point multi-hop approach see also [11], [12], [13], and [14] where a mathematical framework for finding the capacity region of Ad-Hoc wireless networks is developed.

In all of the above mostly point-to-point communication is assumed between the nodes in the network, regarding all other transmissions as noise. In that respect these results do not fully answer the capacity question as they do not consider more elaborate coding schemes allowing arbitrary cooperation between the nodes in the network. Such an attempt is done in [15], [16] where a wireless fading AWGN network is considered. Under the assumption that the source transmits only half of the time the capacity is shown to behave asymptotically like $\log n$.

One of the main challenges in increasing the information rates in wireless networks is combatting the fading. In order to mitigate the effects of this detrimental interference several forms of diversity have been employed successfully, as temporal diversity, frequency diversity and spatial diversity (multi antenna). Since mobile transceivers preclude the use of many receiving antennas (due to their small physical size), a new form of spatial diversity was suggested [17], [18], [19] namely *cooperative diversity.* The idea is to let two mobile users share both their antennas by having each of them transmit to the base station (or destination) both their information and the information of their partner. This of course implies using the intermediate channel between the two cooperating users to let each know about the other's message. This type of cooperation in essence mimics multiple transmit antenna diversity as each user has a virtual double antenna array. Significant gains were demonstrated for both the ergodic capacity and the outage probability when employing this cooperative strategy. Codes which are suitable for user cooperation and which achieve the *cooperative diversity* gain are reported in [20], [21], [22], [23], and [24]. A similar form of user cooperation called *power combining* which aims at minimizing total transmit power is introduced in [25].

In [26], [27] the authors propose several half-duplex cooperative strategies for mobile wireless users which exploit the available diversity by relaying each the message of the other. The strategies include fixed relaying schemes such as *amplify-and-forward* where each user simply transmits to the destination a scaled version of the signal it received from its partner, and *decode-and-forward* where each user first decodes its partner's message and then re-encodes and re-transmits the decoded message (of its partner) to the destination. Dynamic schemes in which the users determine whether or not to actually relay according to channel measurements or limited feedback from the destination are also examined (*selection relaying* and *incremental relaying*, respectively). Outage probabilities and their behavior for large signals to noise ratios (SNR) are used as a figure of merit for assessing the performance of the suggested schemes. Significant outage benefits are observed although the basic approach entails an inherent bandwidth loss due to the half-duplex operation. For analysis of bit-error probabilities of specific schemes implementing the relaying protocols discussed above see [28], and also [29].

The same authors extend this approach [30], [31] to larger networks. In this strategy, called *space-time coded cooperative diversity*, and still relying on a half-duplex scheme, any user other than the source and destination can act as a potential relay. Whether a user actually operates as a relay or not in any given fading block is determined by its channel conditions with respect to the source. Outage probabilities are shown to decay as $1/\mathrm{SNR}^m$ for small rates, where $m$ is the number cooperating nodes, and a *diversity-multiplexing* trade-off similar to the one known for multi-antenna systems is discussed.

In [32], [33], a cooperative coding scheme is suggested where cooperation is done by partitioning the user's codeword and letting the user transmit part of the codeword, and letting its partner (after successful decoding of the user's message) transmit the rest of the codeword. This scheme is shown to outperform some of the previous described strategies, which involve some form of repetition coding implying inefficient use of bandwidth with respect to a coded scheme. Analysis of the probability of error of this coded cooperative scheme is done in [34].

There are many other works related to cooperation between users and implications of relaying in wireless networks, and we do not presume to include all of them. For a discussion on the optimal number of relay nodes to be used in a wireless Ad-Hoc network see [35]. *Relay regions* where multihop transmission proves more beneficial than direct transmission are treated in [36]. In [37], [38] a distributed turbo coding scheme is proposed for the relay channel by introducing interleaving at the relay prior to re-encoding and re-transmitting the message. For a discussion on the potential transmit energy savings in dense Ad-Hoc and sensory networks by employing *macro-diversity* approaches see [39]. In [40], [41] a network flooding algorithm for wireless Ad-Hoc networks based on a pure physical layer design is proposed where all the nodes act as relays or scatterers to a specific destination, generating an avalanche of signals. In [42] a cooperative scheme called *collaborative reception* is introduced. A message is sent from a source to a destination embedded in a cluster of users. Each node first receives and decodes the message, and then the nodes exchange reliability information from their SISO decoders, in order to improve their decisions. Power savings when using a two-hop relaying scheme rather than a standard one hop scheme are demonstrated in [43] for a wireless Nakagami fading channel. The idea of *accumulative broadcast* is introduced in [44] (see also [45]). As a message is forwarded through the network by nodes which have successfully received the message, other nodes which are initially outside the transmission range of the source's transmitter are allowed to collect the energy of unreliable overheard signals. These nodes will have multiple opportunities to reliably receive the message by collecting energy during each retransmission. Finally, in [46] a MIMO interpretation is used to derive upper bounds to the transport capacity of wireless networks.

In this paper we address the information throughput of a block-fading wireless network focusing on one remotely located source wishing to transfer information to a destined user embedded within a group of $K$ physically co-located users. The co-located users are assumed to be very densely packed, such that they enjoy outstanding SNR conditions relative to one another, but suffer from Rayleigh fading with respect to the source. We further assume that stringent delay constraints demand that, once information is available at the transmitter, the message be sent immediately over the channel to its destined user with minimum delay. In particular, waiting until the destined user enjoys better channel conditions is not allowed. Moreover, the message must be decoded by the destined user within the period of one fading block[1]. To that end we propose a form of user cooperation where the source starts transmitting a packet to the group of $K$ users, and each user once it successfully receives the packet, starts transmitting it to the rest of the group, until the destined user decodes the message, or until the block ends.

An important feature of this cooperative scheme is that while the actual level of cooperation is determined by the channel conditions, which vary according to the dynamics of the fading process, the source does not have to know these conditions in order for the cooperation to take place. In fact, the source transmits in the same way irrespective to whether or not cooperation actually occurs. In that respect, the strategy is robust to channel variations.

In essence, the proposed strategy can be viewed as a special case of MISO coding where the users who have successfully decoded the message sent by the source are functioning as a virtual multiple antenna array used to transmit the message to the destined user. The coding strategy can then be interpreted as a form of MISO coding where the symbols transmitted from each antenna (or user in this case) are independent, and the receiver in each user is a MISO based receiver. We therefore assume full time synchronization in the system.

In the non-ergodic regime where Shannon capacity looses its meaning the outage capacity is often used as a figure of merit. We use a variant of the outage capacity by considering the *expected throughput* which is obtained by multiplying the attempted rate

[1]Such a system could be implemented in a TDMA fashion.

by the successful decoding probability. This can be used as a long term performance measure as it reflects the average system's throughput behavior over many transmissions. For a discussion on the usage of such a measure in multi-user fading channels see [47].

The paper is organized as follows. Section 2 describes the considered model and the proposed cooperative transmission scheme, section 3 presents our main results, section 4 incorporates a broadcast approach into the cooperative scheme, section 5 presents some numerical results and concluding remarks are given in section 6. Proofs and derivations are deferred to appendices A-E.

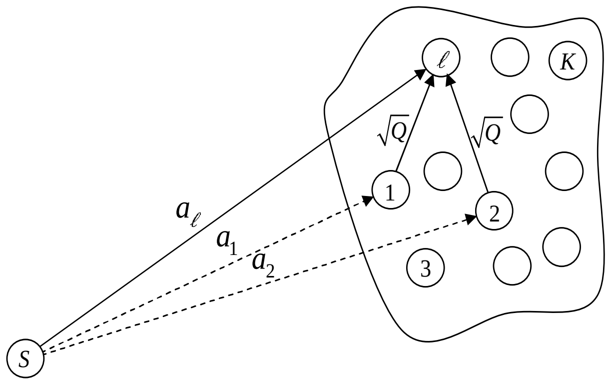


Figure 1: A Source transmitting to co-located users

## 2 A Cooperative Model

We consider a wireless Ad-Hoc or sensory network in which a remotely located source $S$ is interested in sending information to a destined user embedded within a group of $K$ co-located users as depicted in Figure 1. The information is transmitted over a shared wireless medium where all transmissions coming from the source are subject to flat Rayleigh fading. The fading coefficients between the source and the users are denoted by $\{a_m\}$, $m = 1, \ldots, K$ and are modelled by a sequence of $K$ i.i.d. zero mean unit variance circularly symmetric complex Gaussian random variables. The fading coefficients are assumed constant over a coherence time equivalent of $N$ symbols, and independent from one block of $N$ symbols to the next.

The group of $K$ co-located users are assumed to be densely packed, thus enjoying exceptionally good channel conditions relative to one another. This fact is captured by assuming a fixed and constant gain $\sqrt{Q}$ governing the transmissions between any pair of co-located users. Each co-located user is equipped with both a receiver and a transmitter, and we assume that users are capable of both transmitting and receiving using the same resources (time/frequency). We assume that each co-located user has knowledge of its channel gain with respect to the source and of the common co-located gain $Q$. However, neither the source nor any of the users have any knowledge of the channel gains of other users with respect to the source.

Prior to the beginning of every block of $N$ symbols, a destined user is chosen at random from the group of $K$ co-located users with equal probability. The source then produces a message to be sent to the destined user. We assume that stringent delay requirements demand that the message be transmitted to the destination within the fading block period. During this block period, while the source is transmitting the message to the destined user, the remaining $K-1$ users who overhear this transmission, can assist the source by relaying the message to the destined user. Say that in a given block the source transmits a message to one of the users. Then, assuming an AWGN channel, the received signal at user $m$ is:

$$Y_i^{(m)} = a_m X_i + \sqrt{Q} \sum_{\substack{j \in \mathcal{A}_i \\ j \neq m}} X_i^{(j)} + Z_i^{(m)}, \qquad i = 1, 2, \ldots, N, \qquad m = 1, 2, \ldots, K, \tag{1}$$

where $X_i$ is the $i$-th source symbol, $\mathcal{A}_i \subseteq \{1, 2, \ldots, K\}$ is the set of co-located users transmitting at time $i$, $X_i^{(j)}$ is the symbol transmitted by user $j$ in the $i$-th symbol interval and $Z_i^{(m)}$ is the AWGN at user $m$ in the $i$-th symbol interval, where $Z_i^{(m)}$ are i.i.d. circularly symmetric complex Gaussian random variables with zero mean and unit variance.

Let $\{u_m\} = \{|a_m|^2\},\ m = 1, \ldots, K$ be the exponentially distributed unit mean power gains between the source and the co-located users. Denote by $\{\nu_\ell\},\ \ell = 1, \ldots, K$ the arrangement of $\{u_m\}$ in descending order, i.e.

$$\begin{aligned} \nu_1 &= \max(u_1, u_2, \ldots, u_K) \\ \nu_\ell &= \max_{\substack{1 \leq m \leq K \\ u_m \leq \nu_{\ell-1}}} \{u_m\} \qquad \ell = 2, \ldots, K. \end{aligned} \tag{2}$$

From here on we refer to the user with the $\ell$-th strongest fading gain with respect to the source as user $\ell$ (see Figure 1).

Before illustrating the cooperative transmission strategy for the network described above, we introduce the following argument. Consider the point-to-point channel between the source and one of the co-located users, say user $j$. Suppose that for some value of $R$, the source sends one of $M = \lfloor e^{NR} \rfloor$ equally probable messages to user $j$ during the period of one block using a codeword of length $N$. To that end the source generates a random Gaussian code book of length $N$ and power $P$. If $N$, the length of the block/codeword, is large enough, then user $j$ can reliably decode the message sent by the source provided that the rate of the code is less than the capacity [2] of the channel, i.e. $R = \frac{\log M}{N} < \log(1 + \nu_j P)$.

Consider now another co-located user, say user $k$ who overhears the transmission coming from the source. Assume also that the channel gain of user $k$ is larger than that of user $j$ such that $\log(1 + \nu_k P) = \beta \log(1 + \nu_j P)$ for some real number $\beta > 1$. In other words, the capacity of the point-to-point channel from the source to user $k$ is $\beta$ times greater than that of the channel to user $j$. Next, observe the code book generated from the original source's code book by keeping only the first $\lceil N/\beta \rceil$ symbols of every codeword and discarding the rest of the codeword. By similar arguments, user $k$ can decode the message sent by the source (using the new code book) provided $R = \frac{\log M}{\lceil N/\beta \rceil} < \log(1 + \nu_k P)$. It turns out, therefore, that user $k$ can decode the message sent by the source (using the original code book) with high probability after only $\lceil N/\beta \rceil$ symbols, or $\beta$ times more quickly than user $j$. In this context see also [48].

With this argument in mind we proceed to describe our transmission strategy for the wireless network depicted in Figure 1. Assume again that our source starts transmitting one of its code words describing a message intended to user $\ell$. Based on the argument laid out above it can be said that users $1, 2, \ldots, \ell - 1$ will decode the message before user $\ell$ will. Therefore, every user, once it successfully decodes the message, starts transmitting the same message using another predetermined code book with an average power of $P_r$. In the sequel we will normalize the value of $P_r$ to unity, interpreting the source's power $P$ as

[2] Throughout the paper all logarithms are taken to the base $e$.

a power relative to the relaying power used by the rest of the users. Such an approach is consistent with *sensory networks* where the sensor devices are typically low power devices.

To clarify the strategy consider Figure 1. The situation illustrated in the Figure corresponds to the case where users 1 and 2 have already decoded the message intended to user $\ell$ and are transmitting together with the source. All other users are silent and are in receiving mode. When user 3 decodes the message, it too will join users 1 and 2 and the source in their transmission, using another predetermined code book. The process where more and more users begin transmitting the message continues until the destined user decodes the message or until the block ends, in which case an error event is declared.

Once the intended user decodes the message[3], there is no longer need for the source and users to keep transmitting. If feedback is available, and if the users can transmit and receive at the same time, then user $\ell$ can notify through this feedback channel all other users and source to stop their transmissions until the end of the block. We call this type of policy *full acknowledge*. If users can either transmit or receive at any given time, then acknowledgements can be issued only to those users who did not decode the message yet, and are therefore in a receiving state. Only those users can learn about the successful decoding of the message by user $\ell$, and will not initiate transmission even if they manage to decode the message before the block ends. We call this policy *partial acknowledge* or *receiving site acknowledge*. Note that in this case users who started operating as relays before the destined user decoded the message will continue transmitting until the end of the block even though the destined user already decoded the message, rendering their transmission totally redundant. If no feedback channel is available for acknowledge purposes, all users begin relaying once they decode the message intended to user $\ell$ regardless to whether or not user $\ell$ had decoded the message or not. We call this policy *no acknowledge*.

From our system model it is clear that the order in which the users decode the message and start operating as relays is exactly the order of their channel power gains. Assuming

[3]The user can determine whether or not it decoded the message correctly by means of a CRC like check. However, if it knows the channel gain, it should also know how many symbols are necessary for it to make a reliable decision.

$N \gg 1$ we can designate the decoding times of the users within a block period by a fractional time. We denote by $\varepsilon_\ell$ the fractional time within the fading block period when user $\ell$ decodes the message and starts relaying it to the rest of the group. In this context, $\varepsilon_\ell = 1$ means that user $\ell$ was unable to decode the message, and did not transmit at all throughout the block.

Let $I_\ell$ be the empirical mutual information[4] associated with user $\ell$ and given [49] by:

$$I_\ell = \sum_{k=1}^{\ell} (\varepsilon_k - \varepsilon_{k-1}) \log\bigl(1 + \nu_\ell P + (k-1)Q\bigr), \quad \ell = 1, \ldots, K, \tag{3}$$

where we have taken the value of $P_r$ to be unity, as we will be examining the impact of the co-location gain on the performance of the system, thus absorbing the relaying power into the co-location gain. Notice that until time $\varepsilon_1$, only the source is transmitting, and all other users are silent. We set $\varepsilon_0 \equiv 0$ to be the beginning of the block. Note that relays are also helped by other already transmitting relays. Every time a new user/relay decodes the message and starts operating we have an increase of $Q$ in the total received power.

The fractional times $\{\varepsilon_k\},\ k = 1, \ldots, K$ can be solved iteratively, where $\varepsilon_k$ is found by solving for $I_k = R$, since we want all users to receive the message, and restricting the value of $\varepsilon_k$ to 1 in case the solution exceeds 1. The values of $\varepsilon_k$ are thus given by:

$$\begin{aligned}
\varepsilon_1 &= \min\left(1, \frac{R}{\log(1+\nu_1 P)}\right) \\
\varepsilon_2 &= \min\left(1, \varepsilon_1 + \frac{R - \varepsilon_1 \log(1+\nu_2 P)}{\log(1+\nu_2 P + Q)}\right) \\
\varepsilon_3 &= \min\left(1, \varepsilon_2 + \frac{R - \varepsilon_1 \log(1+\nu_3 P) - (\varepsilon_2 - \varepsilon_1)\log(1+\nu_3 P + Q)}{\log(1+\nu_3 P + 2Q)}\right) \\
&\ \vdots \\
\varepsilon_\ell &= \min\left(1, \varepsilon_{\ell-1} + \frac{R - \displaystyle\sum_{k=1}^{\ell-1} (\varepsilon_k - \varepsilon_{k-1}) \log\bigl(1 + \nu_\ell P + (k-1)Q\bigr)}{\log\bigl(1 + \nu_\ell P + (\ell-1)Q\bigr)}\right), \quad \ell = 4, \cdots, K.
\end{aligned} \tag{4}$$

Let $p_\ell$ denote the probability of successful decoding by user/relay $\ell$, i.e.

$$p_\ell = \Pr\left\{\varepsilon_\ell < 1\right\}. \tag{5}$$

[4]As was mentioned in section 1, the strategy can be interpreted as a form of MISO coding. We note in this context that MISO based codes do give rise to the mutual information in (3).

Assuming that the index of the destination is uniformly distributed among all $K$ users, we can express the expected throughput by:

$$R_{av}(R,P,Q) = \frac{R}{K}\sum_{k=1}^{K} p_k(R,P,Q). \tag{6}$$

As was previously mentioned, the source and relays use during their transmission a nominal power of $P$ and 1, respectively. However, they do not all transmit throughout the whole period of the block. The average power actually used to transmit the information from the source to the destination depends on the average of the time epochs $\{\varepsilon_k\}$ and on the different relaying policies. In appendix A the average powers are shown to be

$$P_{av}^{ACK}(R,P) = \frac{P}{K}\sum_{\ell=1}^{K}\mathbb{E}\left(\varepsilon_\ell\right) + \sum_{k=2}^{K}\left(\frac{K-k+1}{K}\right)(k-1)\left[\mathbb{E}\left(\varepsilon_k\right) - \mathbb{E}\left(\varepsilon_{k-1}\right)\right] \tag{7}$$

$$P_{av}^{RACK}(R,P) = P + \frac{K-1}{2} - \sum_{k=1}^{K}\left(1-\frac{k}{K}\right)\mathbb{E}\left(\varepsilon_k\right) \tag{8}$$

$$P_{av}^{NACK}(R,P) = P + (K-1) - \left(\frac{K-1}{K}\right)\sum_{k=1}^{K}\mathbb{E}\left(\varepsilon_k\right) \tag{9}$$

where $ACK$, $RACK$ and $NACK$ stand for the full system acknowledgement, receiving sites acknowledgement and no acknowledgement policies, respectively, and where $\mathbb{E}(\cdot)$ denotes expectation.

We end this section with the following definition of the network described above and considered in the following sections.

**Definition:** *A* **one-to-many co-located network** *is a wireless network in which a source is transmitting to a group of $K$ co-located users who suffer from quasi-static Rayleigh fading with respect to the source but enjoy a fixed co-location gain among themselves.*

# 3 Main Results

## 3.1 An Example: The Single User Case

Before presenting results for the multi user case it is instructive to view first the single user case which is described in (1) by letting $K = 1$. We assume that the source is sending a message described by a codeword spanning the length of a fading block with a rate of $R$ [bits/channel use] and power $P$. The channel power gain $\nu$ is exponentially distributed with unit mean. In this case the expected throughput equals

$$\begin{aligned} R_{av}^{SU}(R,P) &= R\,\Pr\{\log(1+\nu P) > R\} \\ &= Re^{-\frac{e^R-1}{P}} \end{aligned} \tag{10}$$

As is demonstrated in [47] this throughput can be optimized by proper selection of the rate $R$. In the single user case as can be readily verified, selecting a rate $R$ which satisfies $Re^R = P$ achieves the maximum of (10).

## 3.2 Source Transmitting to Two Users

Next we consider the case where the source is transmitting information to one of two co-located users, i.e. $K = 2$. In this case there is a stronger user and a weaker user, whose fading power gains are $\nu_1$ and $\nu_2$, $\nu_1 > \nu_2$, respectively. The fractional time epochs $\varepsilon_1$ and $\varepsilon_2$ when the users decode the message successfully are given in (4). We have the following result which is proved in appendix B.

**Theorem 1** *In a one-to-many co-located network with two co-located users, a co-location gain of $Q$, source power of $P$ and a rate of $R$ nats per channel use, the following expected throughput is achievable*

$$R_{av}(R,P,Q) = \frac{Re^{\frac{2}{P}}}{P}\int_{\max(1,e^R-Q)}^{e^R} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{Q}{u}\right)}{\log(u+Q)-R}\right\} - \frac{u}{P}}\,\mathrm{d}u + Re^{-\frac{e^R-1}{P}}. \tag{11}$$

It is interesting to note that the second term in (11) is just the expected rate obtained in the single user case shown above. The first term therefore is the additional rate gained from the cooperative strategy which makes use of the co-location gain between the users.

The following result (proved in appendix C) quantifies the average power invested by the source and relay in the transmission.

**Theorem 2** *In a one-to-many co-located network with two co-located users, a co-location gain of $Q$, source power of $P$, and a rate of $R$ nats per channel use, the average total power* *invested by the system is given for all acknowledge policies by:*

$$
\begin{aligned}
P_{av}^{ACK}(R,P) &= P-(P+1)\int_0^1 e^{-\frac{e^{\frac{R}{x}}-1}{P}}\,\mathrm{d}x+\int_0^1 e^{-2\frac{e^{\frac{R}{x}}-1}{P}}\,\mathrm{d}x+ \\
&\quad -\left(1-\frac{1}{P}\right)e^{\frac{2}{P}}\int_0^1\int_{\max(1,\exp\{R/x\}-1)}^{e^{\frac{R}{x}}} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{1}{u}\right)}{x\log(u+1)-R}\right\}-\frac{u}{P}}\,\mathrm{d}u\,\mathrm{d}x \qquad (12)\\
P_{av}^{RACK}(R,P) &= P+\frac{1}{2}\int_0^1 2e^{-\frac{e^{\frac{R}{x}}-1}{P}}-e^{-2\frac{e^{\frac{R}{x}}-1}{P}}\,\mathrm{d}x \qquad (13)\\
P_{av}^{NACK}(R,P) &= P+\frac{1}{2}\int_0^1 2e^{-\frac{e^{\frac{R}{x}}-1}{P}}\,\mathrm{d}x+ \\
&\quad +\frac{e^{\frac{2}{P}}}{P}\int_0^1\int_{\max(1,\exp\{R/x\}-1)}^{e^{\frac{R}{x}}} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{1}{u}\right)}{x\log(u+1)-R}\right\}-\frac{u}{P}}\,\mathrm{d}u\,\mathrm{d}x. \qquad (14)
\end{aligned}
$$

## 3.3 Source Transmitting to Many Users

In this section we consider the case of a source transmitting to $K$ co-located users adhering to the cooperative transmission strategy described in section 2. We have the following result (proved in appendix D).

**Theorem 3** *In a one-to-many co-located network with $K$ co-located users, a co-location gain of $Q$, source power of $P$ and a rate of $R$ nats per channel use, there exists a sequence of functions $\mu_\ell(R,P,Q,\beta_2,\beta_3,\ldots,\beta_\ell)$, $\ell=1,\ldots,K$, which are found iteratively, such that the following expected throughput is achievable*

$$
\begin{aligned}
&R_{av}(R,P,Q)= \\
&=\frac{R}{K}\sum_{\ell=1}^{K}\underset{\substack{\beta_2>\beta_3>\cdots>\beta_K\\0<\mu_\ell<1}}{\iint\cdots\int} K!\exp\left\{-\frac{e^{\frac{R}{\mu_\ell(R,P,Q,\beta_2,\beta_3,\ldots,\beta_\ell)}}-1}{P}-\sum_{j=2}^{K}\beta_j\right\}\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K+ \qquad (15)\\
&\quad +Re^{-\frac{e^R-1}{P}}.
\end{aligned}
$$

## 3.4 A Simple Upper Bound - Transmitting to the Strongest User

Since the expected throughput just obtained carries with it a heavy burden of computational complexity, we resort to deriving some lower and upper bounds which can shed some light on the behavior of the expected throughput at a more moderate cost of computing.

Our first approach addresses the performance of the strongest user. Recall that the source is transmitting to one of the $K$ co-located users with equal probability. Naturally, if only the best user (the user who enjoys the best channel conditions) among the group is selected as destination, the achievable throughput is greater than in the actual setting. We therefore have the following theorem.

**Theorem 4** *In a one-to-many co-located network with $K$ co-located users, a co-location gain of $Q$, source power of $P$ and a rate of $R$ nats per channel use, the achievable expected throughput using the cooperative transmission scheme described in section 2 is upper bounded by*

$$R_{av}(R,P,Q) \leq R\left(1-\left(1-e^{-\frac{e^R-1}{P}}\right)^K\right). \tag{16}$$

*Proof:* Recall from [50] that the cumulative distribution function (CDF) of the strongest fading power gain among $K$ exponentially distributed fading gains is:

$$\mathcal{F}_{\nu_1}(x) = (1-e^{-x})^K. \tag{17}$$

The probability that the strongest user decodes the message is therefore expressed by

$$\Pr\{\varepsilon_1 < 1\} = \Pr\left\{\nu_1 > \frac{e^R-1}{P}\right\} = 1-\left(1-e^{-\frac{e^R-1}{P}}\right)^K \tag{18}$$

and the expected throughput to the strongest user is

$$\begin{aligned} R_{av}(R,P) &= R\Pr\{\varepsilon_1 < 1\} \\ &= R\left(1-\left(1-e^{-\frac{e^R-1}{P}}\right)^K\right). \end{aligned} \tag{19}$$

□

Since the bound does not exhibit the behavior of the *optimal* throughput under our cooperative scheme, we obtain in the next theorem an asymptotic upper bound which relates to the optimal throughput.

**Theorem 5** *In a one-to-many co-located network with $K$ co-located users, a co-location gain of $Q$, and a source power of $P$, the maximum achievable expected throughput using* *the cooperative transmission scheme described in section 2 is asymptotically upper bounded by*

$$\lim_{K\to\infty}\left[\max_{R\geq 0} R_{av}(R,P) - \log\left(P\log\left(\frac{K}{\log 2}\right)+1\right)\right] \leq 0. \tag{20}$$

*Proof:* The idea is to consider the asymptotic distribution of the strongest fading gain which is known [50] to be

$$\mathcal{F}_{\nu_1}(x) \underset{K\to\infty}{\longrightarrow} e^{-e^{-x+\log K}}, \tag{21}$$

suggesting that for a large number of users the strongest channel gain is more or less logarithmical with the number of users. By (18), (19) and (21), we have that

$$R_{av}(R,P) \underset{K\to\infty}{\longrightarrow} R\left(1-e^{-Ke^{-\frac{e^R-1}{P}}}\right). \tag{22}$$

By examining (22) it is evident that when the number of users is large, the rate associated with a channel gain of $\log K$ is a threshold rate which distinguishes between rates which are sustainable by the system with high probability and rates which are not. Up to that rate the throughput is identical to $R$ and from that point onward the rate begins to vanish.

If we denote by $R_1(K)$ the rate for which the throughput in (22) falls to half the rate, or equivalently the rate for which the decoding probability is 1/2, then

$$R_1(K) = \log\left(P\log\left(\frac{K}{\log 2}\right)+1\right)$$

must be greater then the rate maximizing (22). If we further denote the maximizing rate by $R^\star$, then

$$R_{av}(R^\star,P) = R^\star \Pr\left\{\nu_1 > \frac{e^{R^\star}-1}{P}\right\} \leq R^\star \leq R_1$$

proving the Theorem. □

## 3.5 Tighter Bounds - Allowing Only Two Relays

As was mentioned before, equation (15) gives rise to some computational difficulties which prevent learning the actual behavior of the performance for a large number of users. Since the upper bound (20) can not be too tight, as it relies only on the strongest user and fails to take into account the true contribution of the co-location gain, we develop in what follows lower and upper bounds which better reflect the influence of the co-location gain.

The idea underlying the lower bound is based on three ingredients. The first is to impose a restriction on the number of active relays. Instead of letting all users function as relays, we allow only two users to act as relays for each user. To clarify the restriction, assume that the source is transmitting to user $\ell$. Then, that user can select two relaying nodes, $k$ and $m$ ($k < m < \ell$), which have smaller indices (i.e. have better channel conditions) to assist it by relaying the message from the source using the strategy outlined in section 2. That is until user $k$ decodes the message no relaying is carried out at all. Once user $k$ decodes the message it starts transmitting the message to the rest of the users. Users $k+1, k+2, \ldots, m-1$ who decode the message before user $m$ remain silent. Once user $m$ decodes the message, it starts relaying the message to user $\ell$ - the destination.

The second ingredient is to scale the power of the two relays by an appropriate scaling factor. Specifically, we scale the power of user $k$ by $k$ and scale the power of user $m$ by $(m-k)$. To see why this scaling in fact yields a lower bound to the performance of the actual strategy, namely, equation (15), consider first user $k$. Here, user $k$ decodes the message later than in the actual scheme because by not allowing any relaying to be carried out before user $k$ decodes the message we are making things worse. Moreover, after user $k$ decodes the message, we limit the received power to be only $k$ times the power of a single relay until user $m$ decodes the message, whereas in reality more relays $(k+1, k+2, \ldots, m-1)$ will be joining in before user $m$ decodes the message transmitting with a total power greater than that. So again we are making things worse. Similar considerations apply to user $m$.

For the third and last ingredient note that this is a lower bound for all $k$ and $m$. We therefore need to maximize these probabilities in order to make the most of this bound. This result is stated in the following Theorem (proved in appendix E).

**Theorem 6** *In a one-to-many co-located network with $K$ co-located users, a co-location gain of $Q$, source power of $P$ and a rate of $R$ nats per channel use, the achievable expected throughput using the cooperative transmission scheme described in section 2 is lower* *bounded by*

$$R_{av}(R,P,Q) \geq Re^{-\frac{e^R-1}{P}} + \frac{R}{K}\int_{\max(0,\frac{e^R-1-Q}{P})}^{\frac{e^R-1}{P}} e^{-\frac{e^{\frac{R}{\mu_2}}-1}{P}+\beta_2} p_{\nu_2}(\beta_2)\,\mathrm{d}\beta_2 + \tag{23}$$
$$+\frac{R}{K}\sum_{\ell=3}^{K}\max_{\substack{k,m\\k<m<\ell}}\left\{\int_0^{\frac{e^R-1}{P}}\mathrm{d}\xi\int_{\xi}^{\infty}\mathrm{d}\eta\sum_{j=k}^{m-1}C_{m-1}^{j}e^{(-\frac{e^{\frac{R}{\mu}}-1}{P}+\eta)j}(1-e^{-\frac{e^{\frac{R}{\mu}}-1}{P}+\eta})^{m-1-j}p_{\nu_m,\nu_\ell}(\eta,\xi)\right\}$$

*where*

$$\mu = \frac{1-R\left(\frac{1}{\log(1+\nu_m P+Q_0+Q_1)}+\frac{\log(1+\nu_m P+Q_0+Q_1)-\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_m P+Q_0+Q_1)\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}\right)}{1-\frac{\log(1+\nu_m P+Q_0)}{\log(1+\nu_m P+Q_0+Q_1)}-\frac{\log(1+\nu_\ell P+Q_0)}{\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}+\frac{\log(1+\nu_m P+Q_0)\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_m P+Q_0+Q_1)\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}} \tag{24}$$

$$\mu_2 = \frac{1-R\frac{1}{\log(1+\nu_2 P+Q)}}{1-\frac{\log(1+\nu_2 P)}{\log(1+\nu_2 P+Q)}}$$

$$p_{\nu_2}(\beta_2) = K(K-1)(1-e^{-\beta_2})^{K-2}e^{-2\beta_2}$$

$$p_{\nu_m,\nu_\ell}(\eta,\xi) = \frac{K!}{(m-1)!(\ell-m-1)!(K-\ell)!}e^{-\eta m}(1-e^{-\xi})^{K-\ell}(e^{-\xi}-e^{-\eta})^{\ell-m-1}e^{-\xi}$$

$$Q_0 = 0, \quad Q_1 = kQ, \quad Q_2 = (m-k)Q.$$

For the upper bound we use almost an identical approach. There are two main differences with respect to the lower bound. The first is that we scale the powers of the two users differently, and the second is that we add a virtual relay which we position at the location of the source so that it has access to the source's message. In this way the virtual relay can both commence transmission at the beginning of the block and at the same time enjoy the same co-location gain as all the other relays. The idea here is to define a strategy employing only two relays for each user which can serve as an *upper* bound to the actual strategy. We do this by allowing the virtual relay to start operating from the beginning of the block with a power scaled by $k-1$. Thus, user $k$ enjoys a relaying power even greater than in the actual setting. This is because in reality, user $k$ receives this much power only after all $k-1$ relays already decoded the message, and this happens usually later than the beginning of the block. Similarly, by letting user $k$ work with $m-k$ times much power we ensure that user $m$ enjoys more power than it would normally get in the actual setting. Similar arguments hold for the scaling of user $m$'s power by $\ell-m$. The upper bound is best used when each user minimizes the decoding probability over all selections of $k$ and $m$. The bound is stated in the following Theorem.

**Theorem 7** *In a one-to-many co-located network with $K$ co-located users, a co-location gain of $Q$, source power of $P$ and a rate of $R$ nats per channel use, the achievable expected throughput using the cooperative transmission scheme described in section 2 is upper bounded by*

$$R_{av}(R,P,Q) \leq \frac{R}{K}\int_{\max(1,\frac{e^R-1-Q}{P})}^{\frac{e^R-1}{P}} e^{-\frac{e^{\frac{R}{\mu_2}}-1}{P}+\beta_2} p_{\nu_2}(\beta_2)\,\mathrm{d}\beta_2 + \tag{25}$$

$$+\frac{R}{K}\min_{\substack{k,m\\k<m<\ell}}\Bigg\{\sum_{\ell=1}^{K}\int_{\frac{e^R-1-Q_0}{P}}^{\infty}\ell C_K^{\ell}(1-e^{-x})^{K-\ell}e^{-x\ell}\,\mathrm{d}x+$$

$$+\sum_{\ell=3}^{K}\int_0^{\frac{e^R-1-Q_0}{P}}\mathrm{d}\xi\int_{\xi}^{\infty}\mathrm{d}\eta\sum_{j=k}^{m-1}C_{m-1}^{j}e^{(-\frac{e^{\frac{R}{\mu}}-1-Q_0}{P}+\eta)j}(1-e^{-\frac{e^{\frac{R}{\mu}}-1-Q_0}{P}+\eta})^{m-1-j}p_{\nu_m,\nu_\ell}(\eta,\xi)\Bigg\}$$

*where $\mu$, $\mu_2$, $p_{\nu_2}(\beta_2)$ and $p_{\nu_m,\nu_\ell}(\eta,\xi)$ are as defined in Theorem 6 and where $Q_0=(k-1)Q$, $Q_1=(m-k)Q$ and $Q_2=(\ell-m)Q$.*

We note that by allowing an increasing number of relays to each user tighter bounds are obtainable. Also, as the number of relays grows, say to $L$, the upper and lower bounds coincide with (15) for every $K$ not larger than $L$. This is simply because for $K \leq L$ there is only one way to select the relays, namely letting all the users operate sequentially as in the actual strategy.

# 4 A Broadcast Approach

In this section we incorporate into our cooperative transmission scheme a broadcast strategy [51], [52], [53]. Basically, the idea is to view the fading channel as a degraded Gaussian broadcast channel where there are many receivers, each experiencing a different signal to noise ratio, specified by the channel state. Instead of transmitting a single information stream at some rate $R$, with an associated outage probability, the transmitter sends several information streams at different rates, and corresponding powers, such that when channel conditions are poor, some of the streams are decoded, and when channel conditions are better, more information gets decoded reliably. In other words, instead of having *plenty*, some of the time, and *nothing* most of the time, we have *little* some of the time, and *a little more* the rest of the time.

## 4.1 The Single User Case

Results for the single user case can be found in [51] where a continuum of information streams is assumed to be sent by the transmitter, and in [53] where the performance gain of a broadcast strategy employing a finite number of information streams was reported. For the sake of completeness we reiterate this strategy for the single user case using only two information streams (or layers).

Assume $K = 1$, i.e. we have the source transmitting towards one unique destination. Let $\eta_1$ and $\eta_2$ (assume $\eta_1 < \eta_2$) be the channel states for which the receiver is able to decode the first and second streams of information, respectively. Also, let $\alpha P$ be the power allotted to the first stream and $\bar{\alpha} P$ be the power allotted to the second information stream, where $\alpha = 1 - \bar{\alpha} \in [0, 1]$. Then the rates of the two information streams are given by:

$$R_1 = \log\left(1 + \frac{\eta_1 \alpha P}{1 + \eta_1 \bar{\alpha} P}\right) \tag{26}$$

$$R_2 = \log\left(1 + \eta_2 \bar{\alpha} P\right). \tag{27}$$

Let $\nu$ be the fading gain between the transmitter and the receiver. If $\nu > \eta_1$, then because the RHS of (26) is increasing with $\eta_1$, the receiver is able to decode the first information stream, regarding the second information stream as Gaussian noise. If in addition $\nu > \eta_2$, then surely the receiver can decode the first stream, subtract it from the input and decode the second stream. The expected throughput is given by:

$$\begin{aligned} R_{av} &= R_1 \Pr\{\eta_1 < \nu < \eta_2\} + (R_1 + R_2) \Pr\{\nu > \eta_2\} \\ &= R_1 e^{-\eta_1} + R_2 e^{-\eta_2} \end{aligned}$$

where $R_{av}$ can be optimized, for each P, over $\eta_1$, $\eta_2$ and $\alpha$.

### 4.2 Two Users

We now extend the layer approach to the case where the source is transmitting a message to one of two co-located users. Let $\eta_1$, $\eta_2$, $\alpha$, $R_1$ and $R_2$ be as defined in subsection 4.1. Note that upon receiving the first information stream, the strong user can start relaying that stream, while still receiving the second information stream. Therefore, we denote by $\varepsilon_i^{(j)}$ the fractional time when user $i$ ($i = 1, 2$) decodes layer $j$ ($j = 1, 2$). These times are given by:

$$\varepsilon_1^{(1)} = \min\left(1, \frac{R_1}{\log\left(1 + \frac{\nu_1 \alpha P}{1+\nu_1 \bar{\alpha} P}\right)}\right) \tag{28}$$

$$\varepsilon_1^{(2)} = \min\left(1, \max\left(\varepsilon_1^{(1)}, \frac{R_2}{\log(1 + \nu_1 \bar{\alpha} P)}\right)\right) \tag{29}$$

$$\begin{aligned}\varepsilon_2^{(1)} = \min\Bigg(1, \max\Bigg(&\varepsilon_1^{(2)} + \frac{R_1 - \varepsilon_1^{(1)} \log\left(1 + \frac{\nu_2 \alpha P}{1+\nu_2 \bar{\alpha} P}\right) - (\varepsilon_1^{(2)} - \varepsilon_1^{(1)}) \log\left(1 + \frac{\nu_2 \alpha P + Q}{1+\nu_2 \bar{\alpha} P}\right)}{\log\left(1 + \frac{\nu_2 \alpha P + \alpha Q}{1+\nu_2 \bar{\alpha} P + \bar{\alpha} Q}\right)}, \\ &\varepsilon_1^{(1)} + \frac{R_1 - \varepsilon_1^{(1)} \log\left(1 + \frac{\nu_2 \alpha P}{1+\nu_2 \bar{\alpha} P}\right)}{\log\left(1 + \frac{\nu_2 \alpha P + Q}{1+\nu_2 \bar{\alpha} P}\right)}\Bigg)\Bigg)\end{aligned} \tag{30}$$

$$\varepsilon_2^{(2)} = \min\left(1, \max\left(\varepsilon_2^{(1)}, \varepsilon_1^{(2)} + \frac{R_2 - \varepsilon_1^{(2)} \log(1 + \nu_2 \bar{\alpha} P)}{\log(1 + \nu_2 \bar{\alpha} P + \bar{\alpha} Q)}\right)\right). \tag{31}$$

Note that in (29) the inner maximization ensures that the second information streams is decoded only after the first, because in order to decode the second information stream the cancellation of the codeword from the first stream is required. Equation (30) reflects two possible situations. In the first case, the weak user decodes the first stream *after* the strong user decodes the second stream, i.e. $\varepsilon_2^{(1)} > \varepsilon_1^{(2)}$. In this case the strong user (or relay in the two user case) devotes all of its power to send the first information stream from the minute it decodes it, and until it decodes the second stream. From the minute it decodes the second stream, it continues to transmit both streams, dividing its power between them according to $\alpha$. In the second situation, the weak user decodes the first information stream *before* the strong user decodes the second information stream, i.e. $\varepsilon_2^{(1)} < \varepsilon_1^{(2)}$. The inner maximization ensures that the correct situation is selected.

Finally, the maximization in (31) ensures that the second stream is decoded by the weak user not before the first.

Following the lines of section 4.1 the expected throughput can be expressed as:

$$R_{av} = \frac{1}{2} \sum_{i=1,2} R_1 \Pr\left\{ \left( \varepsilon_i^{(1)} < 1 \right) \cap \left( \varepsilon_i^{(2)} \geq 1 \right) \right\} + (R_1 + R_2) \Pr\left\{ \varepsilon_i^{(2)} < 1 \right\}. \tag{32}$$

For mathematical tractability, we make the following simplifying assumption. At the moment the strong receiver decodes the first information stream, it starts transmitting the first layer until it receives feedback from the weak user indicating correct decoding of the message. From that point onward, it transmits the second information stream. The corresponding decoding times are now given by:

$$\varepsilon_1^{(1)} = \min\left(1, \frac{R_1}{\log\left(1 + \frac{\nu_1 \alpha P}{1+\nu_1 \bar{\alpha} P}\right)}\right) \tag{33}$$

$$\varepsilon_1^{(2)} = \max\left(\varepsilon_1^{(1)}, \min\left(1, \frac{R_2}{\log(1 + \nu_1 \bar{\alpha} P)}\right)\right) \tag{34}$$

$$\varepsilon_2^{(1)} = \min\left(1, \varepsilon_1^{(1)} + \frac{R_1 - \varepsilon_1^{(1)} \log\left(1 + \frac{\nu_2 \alpha P}{1+\nu_2 \bar{\alpha} P}\right)}{\log\left(1 + \frac{\nu_2 \alpha P + Q}{1+\nu_2 \bar{\alpha} P}\right)}\right) \tag{35}$$

$$\varepsilon_2^{(2)} = \min\left(1, \max\left(\varepsilon_2^{(1)}, \varepsilon_1^{(2)}\right) + \frac{R_2 - \max\left(\varepsilon_2^{(1)}, \varepsilon_1^{(2)}\right) \log\left(1 + \nu_2 \bar{\alpha} P\right)}{\log\left(1 + \nu_2 \bar{\alpha} P + Q\right)}\right) \tag{36}$$

Under these assumptions we compute the decoding probabilities of the two information streams by the two users. We begin with the probability that the strong user decodes only the first information stream. This probability is easily given by:

$$\begin{aligned}
\Pr\left\{ \left( \varepsilon_1^{(1)} < 1 \right) \bigcap \left( \varepsilon_1^{(2)} \geq 1 \right) \right\} = & \\
&= \Pr\left\{ \left( \frac{R_1}{\log\left(1 + \frac{\nu_1 \alpha P}{1+\nu_1 \bar{\alpha} P}\right)} < 1 \right) \bigcap \left( \frac{R_2}{\log(1 + \nu_1 \bar{\alpha} P)} \geq 1 \right) \right\} \\
&= F_{\nu_1}\left( \frac{e^{R_2} - 1}{\bar{\alpha} P} \right) - F_{\nu_1}\left( \frac{e^{R_1} - 1}{\alpha P - \bar{\alpha} P (e^{R_1} - 1)} \right) \\
&= F_{\nu_1}(\eta_2) - F_{\nu_1}(\eta_1)
\end{aligned} \tag{37}$$

where $F_{\nu_1}(x) = \Pr\{\nu_1 \leq x\} = 1 - 2e^{-x} + e^{-2x}$ is the CDF of the strong fading gain. The probability that the strong user decodes both information streams is similarly derived as:

$$
\begin{aligned}
\Pr\left\{\varepsilon_1^{(2)} < 1\right\} &= \Pr\left\{\left(\frac{R_1}{\log\left(1+\frac{\nu_1\alpha P}{1+\nu_1\bar{\alpha}P}\right)} < 1\right)\bigcap\left(\frac{R_2}{\log(1+\nu_1\bar{\alpha}P)} < 1\right)\right\} \\
&= 1 - F_{\nu_1}\left(\max\left(\frac{e^{R_1}-1}{\alpha P - \bar{\alpha}P(e^{R_1}-1)}, \frac{e^{R_2}-1}{\bar{\alpha}P}\right)\right) \\
&= 1 - F_{\nu_1}\left(\max\left(\eta_1, \eta_2\right)\right) \\
&= 1 - F_{\nu_1}\left(\eta_2\right)
\end{aligned}
$$

Next, we address the probability that the weak user decodes only the first information stream. For convenience, we first condition this probability on the value of the weak fading gain $\nu_2$, and take into account (33), (34), (35) and (36). After some simple manipulations one gets

$$
\begin{aligned}
&\Pr\left\{\left(\varepsilon_2^{(1)} < 1\right)\bigcap\left(\varepsilon_2^{(2)} \geq 1\right)\middle|\nu_2 = \xi\right\} = \\
&\quad = \Pr\Bigg\{\left(\varepsilon_1^{(1)} < \rho_1(\nu_2, R_1, P, Q, \alpha)\right)\bigcap \\
&\qquad\quad \Bigg(\left(\varepsilon_1^{(1)} < \rho_3(\nu_2, R_1, R_2, P, Q, \alpha)\right)\bigcap \\
&\qquad\qquad \Bigg(\left(\varepsilon_1^{(1)} < \rho_2(\nu_2, R_2, P, Q, \alpha)\right)\bigcap \\
&\qquad\qquad\quad \left(\min\left(1, \frac{R_2}{\log(1+\nu_1\bar{\alpha}P)}\right) < \rho_2(\nu_2, R_2, P, Q, \alpha)\right)\Bigg)\Bigg)^C\Bigg|\nu_2 = \xi\Bigg\} \qquad (38)
\end{aligned}
$$

where we have denoted

$$
\begin{aligned}
\rho_1(\xi, R_1, P, Q, \alpha) &= \frac{\log\left(1+\frac{\xi\alpha P+Q}{1+\xi\bar{\alpha}P}\right) - R_1}{\log\left(1+\frac{Q}{1+\xi P}\right)} \\
\rho_2(\xi, R_2, P, Q, \alpha) &= \frac{\log\left(1+\xi\bar{\alpha}P+Q\right) - R_2}{\log\left(1+\frac{Q}{1+\xi\bar{\alpha}P}\right)} \\
\rho_3(\xi, R_1, R_2, P, Q, \alpha) &= \frac{\rho_2(\xi, R_2, P, Q, \alpha)\log\left(1+\frac{\xi\alpha P+Q}{1+\xi\bar{\alpha}P}\right) - R_1}{\log\left(1+\frac{Q}{1+\xi P}\right)}.
\end{aligned}
$$

Note that the conditional probability is given in terms of events depending on the value of $\nu_1$. It is therefore easy to obtain this probability as a function of the weak fading gain, namely

$$\Pr\left\{\left(\varepsilon_2^{(1)} < 1\right) \cap \left(\varepsilon_2^{(2)} \geq 1\right) \middle| \nu_2 = \xi\right\} =$$

$$= \begin{cases} 1, & (\rho_1 \geq 1) \cap [(\rho_2 < b) \cup (\rho_3 < b)]\,, \\ 1 - e^{-\max[G(\rho_2), H(\rho_2)] + \xi}, & (\rho_1 \geq 1) \cap (b < \rho_2 < 1) \cap (\rho_3 \geq 1) \\ 1 - e^{-\max[G(\rho_3), G(\rho_2), H(\rho_2)] + \xi}, & (\rho_1 \geq 1) \cap (b < \rho_2 < 1) \cap (b < \rho_3 < 1) \\ e^{-G(\rho_1)+\xi}, & (b < \rho_1 < 1) \cap [(\rho_2 < b) \cup (\rho_3 < b)] \\ e^{-G(\rho_1)+\xi} - e^{-\max[G(\rho_3), G(\rho_2), H(\rho_2)] + \xi}, & (b < \rho_1 < 1) \cap (b < \rho_2 < 1) \cap (b < \rho_3 < 1) \\ 0, & \text{otherwise} \end{cases}$$

where $b \triangleq \frac{R_1}{\log\left(1+\frac{\alpha}{\bar{\alpha}}\right)}$, $G(x) = \frac{e^{\frac{R_1}{x}}-1}{\alpha P - \bar{\alpha} P (e^{\frac{R_1}{x}}-1)}$, for $x \in (b, 1]$ and $H(x) = \frac{e^{\frac{R_2}{x}}-1}{\bar{\alpha} P}$ for $x \in (0, 1]$.

The probability that the weak user has successfully decoded only the first information stream is given therefore by

$$\begin{aligned} \Pr\left\{\left(\varepsilon_2^{(1)} < 1\right) \cap \left(\varepsilon_2^{(2)} \geq 1\right)\right\} &= \int_0^\infty \Pr\left\{\left(\varepsilon_2^{(1)} < 1\right) \cap \left(\varepsilon_2^{(2)} \geq 1\right) \middle| \nu_2 = \xi\right\} p_{\nu_2}(\xi)\, \mathrm{d}\xi \\ &= 2\int_0^\infty \Pr\left\{\left(\varepsilon_2^{(1)} < 1\right) \cap \left(\varepsilon_2^{(2)} \geq 1\right) \middle| \nu_2 = \xi\right\} e^{-2\xi}\, \mathrm{d}\xi. \end{aligned}$$

Similarly, the probability that the weak user decodes both information streams conditioned on the weak fading gain is given by

$$\Pr\left\{\left(\varepsilon_2^{(2)} < 1\right) \middle| \nu_2 = \xi\right\} = \begin{cases} 1, & (\rho_2 \geq 1) \cap (\rho_3 \geq 1) \\ e^{-\max[G(\rho_2), H(\rho_2)] + \xi}, & (b < \rho_2 < 1) \cap (\rho_3 \geq 1) \\ e^{-G(\rho_3)+\xi}, & (\rho_2 \geq 1) \cap (b < \rho_3 < 1) \\ e^{-\max[G(\rho_3), G(\rho_2), H(\rho_2)] + \xi}, & (b < \rho_2 < 1) \cap (b < \rho_3 < 1) \\ 0, & \text{otherwise} \end{cases}$$

and thus the probability that the weak user decoded both information streams is given by

$$\Pr\left\{\varepsilon_2^{(2)} < 1\right\} = 2\int_0^\infty \Pr\left\{\varepsilon_2^{(2)} < 1 \middle| \nu_2 = \xi\right\} e^{-2\xi}\, \mathrm{d}\xi.$$

### 4.3 Many Users - Strongest User Upper Bound

In order to gain insight into the potential benefit of the broadcast approach when there are many users it is instructive to view the strongest users's performance under this approach.

The analysis parallels that of sub section 4.1. The difference is in the fact that the statistics are governed by the best out of $K$ fading gain. The expected throughput is given by

$$\begin{aligned} R_{av} &= R_1 \Pr\{\eta_1 < \nu_1 < \eta_2\} + (R_1 + R_2) \Pr\{\nu_1 > \eta_2\} \\ &= R_1 \left(1 - \left(1 - e^{-\eta_1}\right)^K\right) + R_2 \left(1 - \left(1 - e^{-\eta_2}\right)^K\right) \end{aligned} \tag{39}$$

where the expected throughput can be optimized over $\eta_1$, $\eta_2$ and $\alpha$ (see section 5).

# 5 Numerical Results

In this section we present some numerical results regarding the performance of our proposed cooperative transmission strategy. We consider first the case of two co-located users. Figure 2 depicts the expected throughput achievable in the two user case with and with-

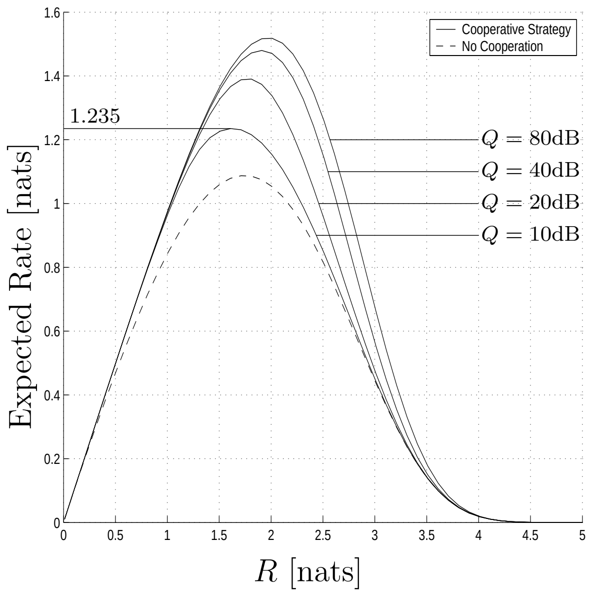


Figure 2: Expected rate with two users, $P = 10$ dB.

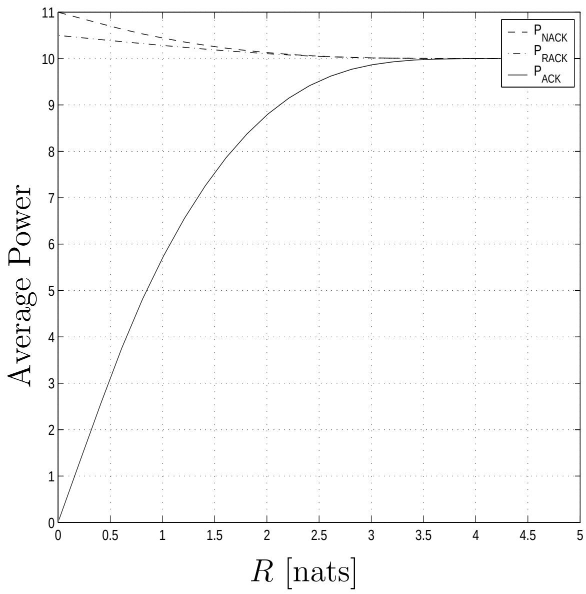


Figure 3: Expected power with two users, $P = 10$ dB.

out cooperation. The throughput is plotted versus the operating rate $R$. As is evident from these curves, for optimal performance, the rate $R$ should be chosen to maximize the expected throughput curve. When no cooperation is performed, the expected throughput is just the single user throughput (see section 3.1) which is maximized for $R$ satisfying $Re^R = P$. The exact value of $R$ which maximizes the throughput in the cooperative strategy is not known, however, the effect of the co-location gain on the maximum throughput is demonstrated by considering increasingly growing values of $Q$.

Figure 3 depicts the expected power consumed by the source and by the strong user during the transmission. Note that for small rates the power in the full acknowledge policy is minimal, as power is spent only as long as it is needed. In the receiving sites acknowledge and in the no acknowledge case more power is wasted because the strong user starts operating early and operates until the end of the block. For very high rates, the power consumption in all policies coincides, as in those cases the strong user never decodes the message, and the source is the only one transmitting throughout the block period. However, in this case the associated throughput is small, as the message gets decoded with very small probability.

Next, we address the case of $K$ co-located users, and plot in Figure 4 some upper and lower bounds which predict the behavior of the maximum achievable expected rate under the cooperative strategy described in section 2. The *UB to strongest user* curve corresponds to (20) and is an upper bound to the performance of the strongest user. The benefit of this bound lies in its simple analytical formulation as opposed to most of the bounds presented in the sequel. However, considering the exact expression for the best users' performance, namely, equation (19) and optimizing it for all $K$ over $R \geq 0$, we obtain the *Strongest User - 1 layer* curve. The gap between the *UB to strongest user* curve and this curve lies in the fact that for the former, we have chosen the value of $R$ for which the throughput is halved which occurs strictly after the actual peak of the rate curve.

Next shown in Figure 4 are six groups of curves corresponding to six values of the co-location gain factor $Q$. Each group consists of three types of curves. The curves *"LB – 2 Relays"* and *"UB – 2 Relays"* are obtained by optimizing the bounds in Theorems 6 and

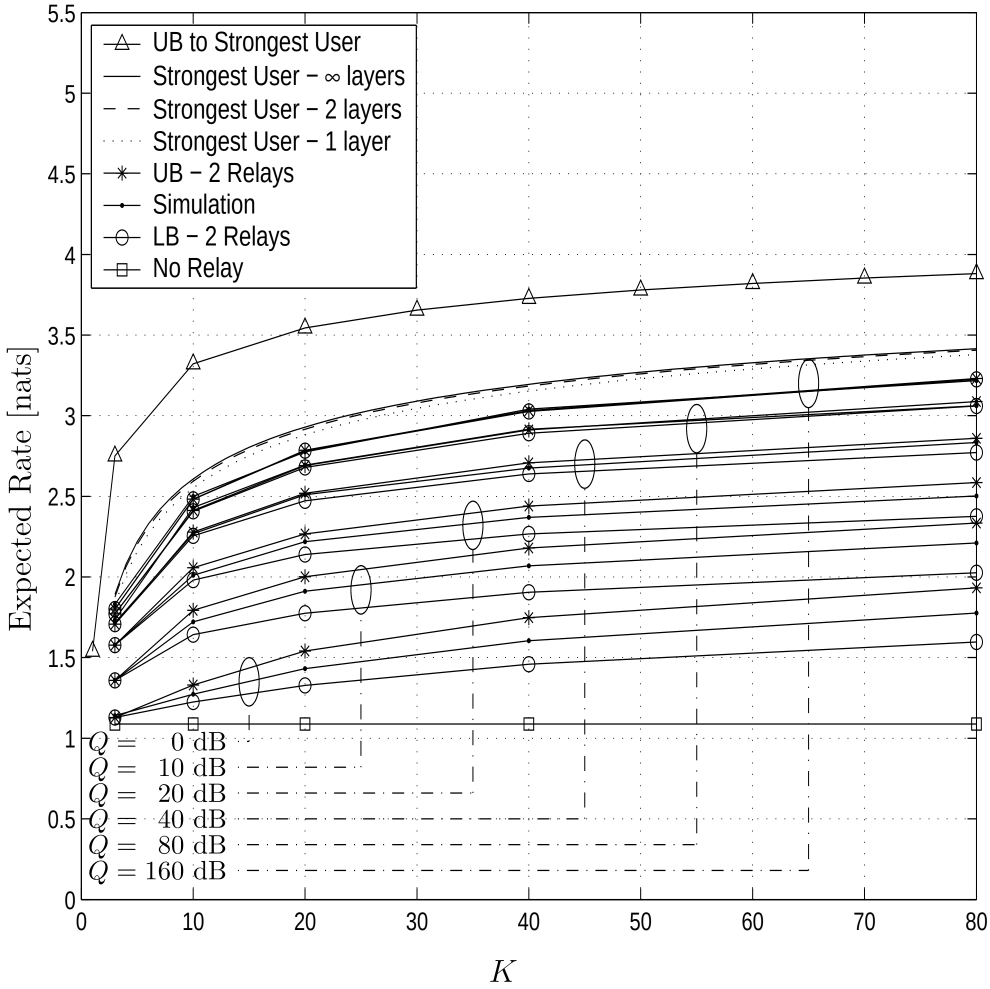


Figure 4: Expected rate for $K$ users, $P = 10$ dB.

7, respectively, over all $R \geq 0$. The third curve *"Simulation"* is obtained by empirically averaging the throughput over many drawings of the fading coefficients. Our baseline for comparison is the *"No Relay"* curve which corresponds to the performance of a system where no relaying is performed, and only direct transmission is allowed.

The curves reveal a substantial throughput benefit over direct transmission. This benefit increases as the co-location gain factor becomes increasingly large. At the extreme, for very high co-location gains, the performance of the system approaches that of the best user, implying that in a densely packed network of destination users, the proposed strategy can be beneficial in terms of throughput at a minor power expenditure by the cooperating transceivers.

Finally, we wish to examine the impact of the broadcast approach on our cooperative transmission scheme. We first consider the two user case. While in the two user case without broadcast there is only one degree of freedom, namely the rate $R$, here one can choose two possible rates $R_1$ and $R_2$ as well as the power splitting parameter $\alpha$. So, the throughput function is actually a three dimensional function. Figure 5 depicts the expected

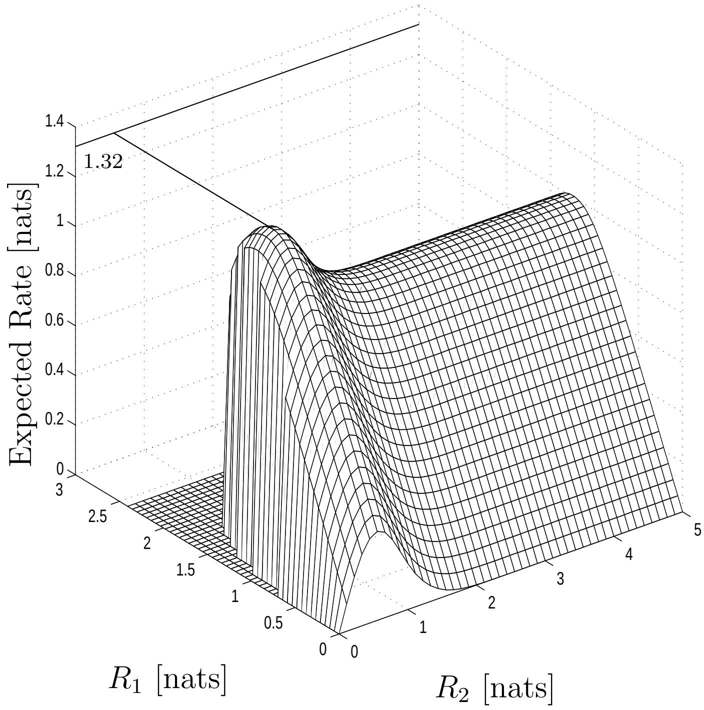


Figure 5: Expected rate with two users and a broadcast approach, $P = 10$ dB, $Q = 10$ dB, $\alpha = 0.91$.

throughput as a function of the two information rates, for given values of the source power $P$, the co-location gain $Q$ and the fraction of the power $\alpha$ allotted to the first information stream. The dead zone effect in the vicinity of the rate pair $(R_1, R_2) = (2, 0)$ is due to the fact that restricting the value of $R_2 = \log(1 + \eta_2 \bar{\alpha} P)$ to some value, poses a restriction on $\eta_2$, which by definition upper bounds the value of $\eta_1$ which determines the rate $R_1$. For $R_1$ approaching zero we essentially return to the one stream case where the optimal throughput is obtained for some value of $R_2$. Similarly, when $R_2$ approaches infinity, then the second information stream is never decoded by any of the two users, and the problem reduces again to the one stream case with optimal throughput achieved at some value of $R_1$. The relationship between the two optimal rates for the two last extreme cases is determined by the value of $\alpha$. Note that the maximum achievable throughput in this case is $R_{av} = 1.32$ [nats] which exceeds the best throughput when using a standard (one layer) cooperative approach, cf. Figure 2.

In order to asses the achievable performance when using the broadcast strategy for a large number of users we consider the performance of the strongest user. We have seen that in the standard case (only one information stream) this performance is approached when the co-location gain is high. This performance is obtained by optimizing (39) over $\eta_1$, $\eta_2$ and $\alpha$. The results of the numerical optimization appear in Figure 4 (see also Figure 6 for better resolution) as the *Strongest User - 2 layers* curve. Some improvement is noticed with respect to the standard approach involving only one information stream.

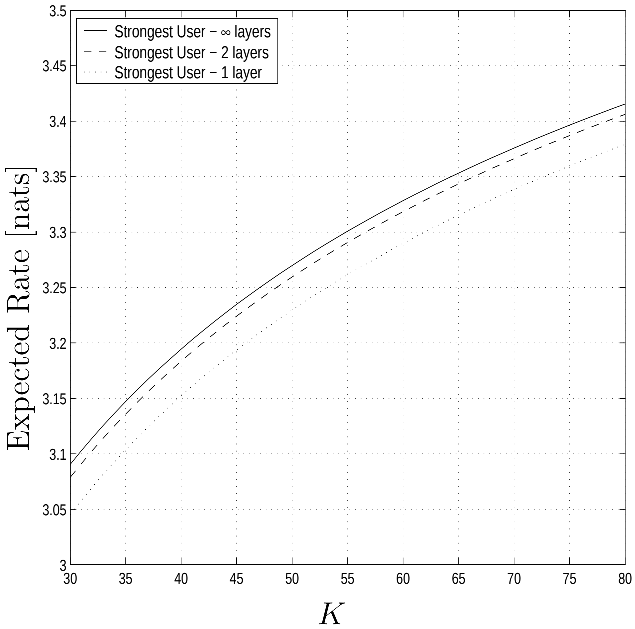


Figure 6: Expected rate when transmitting to strongest user, $P = 10$ dB.

As was demonstrated in [53] it turns out here also that introducing more information streams has little impact on the expected throughput. By using the results of [54, equation (7)], we compute the expected throughput when transmitting to the best user using the broadcast strategy with an infinite number of information streams (see the curve *Strongest*

*User - $\infty$ layers* in Figure 6). As is seen in the figure, with two-layers it is possible to achieve approximately 75% of the gap between the infinite stream broadcast scheme and the single layer standard scheme.

## 6 Conclusions

We have considered wireless networks and focused on a scenario in which a remotely located source is transmitting information to a group of co-located users enjoying extremely good channel conditions among themselves but experiencing Rayleigh fading with respect to the source's transmissions. Such a scenario can occur in wireless sensory networks as well as in some Ad-Hoc networks. We have assumed that stringent delay requirements preclude waiting until the destined users' channel condition improve, and force the source to transmit the message immediately and within the period of one fading block. We proposed a distributed cooperative transmission scheme according to which users who have better channel conditions than the destined user and decode the message sooner, join the source by transmitting the same message using an appropriate code book, creating an avalanche effect until the destined user decodes the message. Using the expected throughput as a figure of merit, we have shown that our proposed strategy achieves a substantial throughput gain over a system employing direct transmission at a relatively low power expenditure by the cooperating users, providing the inter user co-location gain is large. We have also demonstrated that incorporating a broadcast approach into the proposed strategy yields some more throughput benefit, in which case two layers of information are sufficient to retrieve most of the gain promised by an infinite layer broadcast scheme. Our proposed strategy is robust in the sense that the source need not be aware of whether or not cooperation is actually taking place. The cooperation either occurs or not depending on the channel conditions.

Future work will address cooperative transmission strategies for more complex models. One interesting example might be the case where a source is helped by one group of co-locate users in order to communicate with another remote group of co-located users.

Another extension would be to consider more than one transmitting/receiving antenna at the mobile users.

# Appendix A

# Expected Transmitted Power

In this appendix we obtain expressions for the average power used by the source plus relays in the transmission process in each of the three acknowledge policies. For the full acknowledgement policy we have that the average power used for transmission to the $\ell$-th user is given by

$$P_{av}^{ACK}(\ell) = P\mathbb{E}\left(\varepsilon_\ell\right) + Q\sum_{k=2}^{\ell}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right) - \mathbb{E}\left(\varepsilon_{k-1}\right)\right]. \tag{A.1}$$

The total average power in this case equals:

$$\begin{aligned} P_{av}^{ACK} &= \frac{1}{K}\sum_{\ell=1}^{K} P_{av}^{ACK}(\ell) \\ &= P\frac{1}{K}\sum_{\ell=1}^{K}\mathbb{E}\left(\varepsilon_\ell\right) + Q\frac{1}{K}\sum_{\ell=1}^{K}\sum_{k=2}^{\ell}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right) - \mathbb{E}\left(\varepsilon_{k-1}\right)\right] \\ &= P\frac{1}{K}\sum_{\ell=1}^{K}\mathbb{E}\left(\varepsilon_\ell\right) + Q\sum_{k=2}^{K}\left(\frac{K-k+1}{K}\right)(k-1)\left[\mathbb{E}\left(\varepsilon_k\right) - \mathbb{E}\left(\varepsilon_{k-1}\right)\right]. \end{aligned} \tag{A.2}$$

In the no-acknowledgement case, the average power used to transmit to the $\ell$-th user is

$$\begin{aligned} P_{av}^{NACK}(\ell) &= P + Q\sum_{k=2}^{\ell}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right) - \mathbb{E}\left(\varepsilon_{k-1}\right)\right] + Q\sum_{k=\ell+1}^{K}(k-2)\left[\mathbb{E}\left(\varepsilon_k\right) - \mathbb{E}\left(\varepsilon_{k-1}\right)\right] \\ &\quad + Q\left(K-1\right)\left[1 - \mathbb{E}\left(\varepsilon_K\right)\right] \end{aligned}$$

where the first term corresponds to the power of the source transmitting throughout the block period, the second term corresponds to the transmission of the relays from $t = \varepsilon_1$ until $t = \varepsilon_\ell$, the third term corresponds to the transmission of the relays from $t = \varepsilon_\ell$ until $t = \varepsilon_K$ (note that user $\ell$ does not function as a relay, and therefore does not transmit at all) and the last term refers to the transmission of all $K-1$ relays in the period $[\varepsilon_K, 1]$.

This expression can be further simplified as follows:

$$\begin{aligned}
P_{av}^{NACK}(\ell) &= P + Q\left\{\sum_{k=2}^{\ell}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right)-\mathbb{E}\left(\varepsilon_{k-1}\right)\right] + \sum_{k=\ell+1}^{K}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right)-\mathbb{E}\left(\varepsilon_{k-1}\right)\right]\right\} \\
&\quad -Q\sum_{k=\ell+1}^{K}\left[\mathbb{E}\left(\varepsilon_k\right)-\mathbb{E}\left(\varepsilon_{k-1}\right)\right] + Q\left(K-1\right)\left[1-\mathbb{E}\left(\varepsilon_K\right)\right] \\
&= P + Q\sum_{k=2}^{K}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right)-\mathbb{E}\left(\varepsilon_{k-1}\right)\right] - Q\left[\mathbb{E}\left(\varepsilon_K\right)-\mathbb{E}\left(\varepsilon_\ell\right)\right] + \\
&\quad +Q\left(K-1\right)\left[1-\mathbb{E}\left(\varepsilon_K\right)\right] \\
&= P - Q\sum_{k=1}^{K-1}\mathbb{E}\left(\varepsilon_k\right) + Q\left(K-1\right)\mathbb{E}\left(\varepsilon_K\right) - Q\left[\mathbb{E}\left(\varepsilon_K\right)-\mathbb{E}\left(\varepsilon_\ell\right)\right] + \\
&\quad +Q\left(K-1\right)\left[1-\mathbb{E}\left(\varepsilon_K\right)\right] \\
&= P - Q\sum_{k=1}^{K-1}\mathbb{E}\left(\varepsilon_k\right) + Q\left(K-1\right) - Q\left[\mathbb{E}\left(\varepsilon_K\right)-\mathbb{E}\left(\varepsilon_\ell\right)\right] + \\
&= P - Q\sum_{k=1}^{K}\mathbb{E}\left(\varepsilon_k\right) + Q\left(K-1\right) + Q\mathbb{E}\left(\varepsilon_\ell\right)
\end{aligned}$$

After averaging over all users we get the average power for the relaying policy where no acknowledgement is issued, namely:

$$\begin{aligned}
P_{av}^{NACK} &= \frac{1}{K}\sum_{\ell=1}^{K}P_{av}^{NACK}\left(\ell\right) \\
&= \frac{1}{K}\sum_{\ell=1}^{K}\left\{P - Q\sum_{k=1}^{K}\mathbb{E}\left(\varepsilon_k\right) + Q\left(K-1\right) + Q\mathbb{E}\left(\varepsilon_\ell\right)\right\} \\
&= P - Q\sum_{k=1}^{K}\mathbb{E}\left(\varepsilon_k\right) + Q\left(K-1\right) + \frac{Q}{K}\sum_{\ell=1}^{K}\mathbb{E}\left(\varepsilon_\ell\right) \\
&= P + \left(K-1\right)Q - Q\left(\frac{K-1}{K}\right)\sum_{k=1}^{K}\mathbb{E}\left(\varepsilon_k\right).
\end{aligned}$$

Finally, under the receiving-sites acknowledgement policy, the average transmitted power to user $\ell$ is expressed by

$$\begin{aligned}
P_{av}^{RACK}(\ell) &= P + Q\sum_{k=2}^{\ell}(k-1)\left[\mathbb{E}\left(\varepsilon_k\right)-\mathbb{E}\left(\varepsilon_{k-1}\right)\right] + Q\sum_{k=\ell+1}^{K}\left(\ell-1\right)\left[\mathbb{E}\left(\varepsilon_k\right)-\mathbb{E}\left(\varepsilon_{k-1}\right)\right] \\
&\quad +Q\left(\ell-1\right)\left[1-\mathbb{E}\left(\varepsilon_K\right)\right]
\end{aligned}$$

where the first term corresponds to the power of the source transmitting throughout the block period, the second term corresponds to the transmission of the relays from $t = \varepsilon_1$ until $t = \varepsilon_\ell$, the third term corresponds to the transmission of the relays from $t = \varepsilon_\ell$ until $t = \varepsilon_K$ (note that user's $\ell, \ell+1, \ldots, K$ do not function as relays because they know that the message was already successfully received by user $\ell$, and therefore do not transmit at all) and the last term refers to the transmission of all $\ell - 1$ relays in the period $[\varepsilon_K, 1]$ (which transmit even though it is not needed). This expression can be simplified to

$$\begin{aligned} P_{av}^{RACK}(\ell) &= P - Q\sum_{k=1}^{\ell-1}\mathbb{E}(\varepsilon_k) + Q(\ell-1)\mathbb{E}(\varepsilon_\ell) + Q(\ell-1)\left[\mathbb{E}(\varepsilon_K) - \mathbb{E}(\varepsilon_\ell)\right] \\ &\quad + Q(\ell-1)\left[1 - \mathbb{E}(\varepsilon_K)\right] \\ &= P + Q(\ell-1) - Q\sum_{k=1}^{\ell-1}\mathbb{E}(\varepsilon_k). \end{aligned}$$

Therefore, the average total power used for this policy is given by:

$$\begin{aligned} P_{av}^{RACK} &= \frac{1}{K}\sum_{\ell=1}^{K} P_{av}^{RACK}(\ell) \\ &= \frac{1}{K}\sum_{\ell=1}^{K}\left\{P + Q(\ell-1) - Q\sum_{k=1}^{\ell-1}\mathbb{E}(\varepsilon_k)\right\} \\ &= P + \frac{Q}{K}\sum_{\ell=1}^{K}(\ell-1) - \frac{Q}{K}\sum_{\ell=1}^{K}\sum_{k=1}^{\ell-1}\mathbb{E}(\varepsilon_k) \\ &= P + \frac{Q}{K}\sum_{\ell=0}^{K-1}\ell - \frac{Q}{K}\sum_{\ell=2}^{K}\sum_{k=1}^{\ell-1}\mathbb{E}(\varepsilon_k) \\ &= P + \frac{Q}{K}\frac{K(K-1)}{2} - \frac{Q}{K}\sum_{k=1}^{K-1}(K-k)\mathbb{E}(\varepsilon_k) \\ &= P + Q\frac{K-1}{2} - \frac{Q}{K}\sum_{k=1}^{K}(K-k)\mathbb{E}(\varepsilon_k) \\ &= P + \frac{K-1}{2}Q - Q\sum_{k=1}^{K}\left(1 - \frac{k}{K}\right)\mathbb{E}(\varepsilon_k) \end{aligned}$$

# Appendix B

# Average Throughput for Two Users

In this appendix we compute the average throughput achieved in the case of two co-located users. Recall from (4) that the fractional time epochs when the two users decode the message successfully are:

$$\varepsilon_1 = \min\left(1,\ \frac{R}{\log\left(1+\nu_1 P\right)}\right) \tag{B.1}$$

$$\varepsilon_2 = \min\left(1,\ \varepsilon_1 + \frac{R - \varepsilon_1 \log\left(1+\nu_2 P\right)}{\log\left(1+\nu_2 P + Q\right)}\right). \tag{B.2}$$

It is readily verified that the probability density function (PDF) of the fading gains $\nu_1$ and $\nu_2$ is $p_{\nu_1,\nu_2}(\alpha,\beta) = 2e^{-\alpha}e^{-\beta}$. This yields the marginal PDFs $p_{\nu_1}(\alpha) = 2e^{-\alpha} - 2e^{-2\alpha}$ and $p_{\nu_2}(\beta) = 2e^{-2\beta}$ of the strong user and weak user, respectively, and the conditional PDF $p_{\nu_1|\nu_2}(\alpha|\beta) = e^{-\alpha+\beta}$. The decoding probability for the strong user is thus given by

$$\begin{aligned} p_1(R,P) = \Pr\Big\{\varepsilon_1 < 1\Big\} &= \Pr\left\{\frac{R}{\log\left(1+\nu_1 P\right)} < 1\right\} \\ &= \Pr\left\{\nu_1 > \frac{e^R - 1}{P}\right\} \\ &= \int_{\frac{e^R-1}{P}}^{\infty} p_{\nu_1}(\alpha)\ \mathrm{d}\alpha \\ &= 2e^{-\frac{e^R-1}{P}} - e^{-2\frac{e^R-1}{P}}. \end{aligned} \tag{B.3}$$

For the weaker user we are interested in computing

$$\begin{aligned} p_2(R,P,Q) = \Pr\Big\{\varepsilon_2 < 1\Big\} &= \Pr\left\{\min\left(1,\ \varepsilon_1 + \frac{R - \varepsilon_1 \log\left(1+\nu_2 P\right)}{\log\left(1+\nu_2 P + Q\right)}\right) < 1\right\} \\ &= \Pr\left\{\varepsilon_1 + \frac{R - \varepsilon_1 \log\left(1+\nu_2 P\right)}{\log\left(1+\nu_2 P + Q\right)} < 1\right\} \\ &= \Pr\left\{\varepsilon_1 < \frac{\log\left(1+\nu_2 P + Q\right) - R}{\log\left(1 + \frac{Q}{1+\nu_2 P}\right)}\right\}. \end{aligned}$$

Note that the conditional CDF of $\varepsilon_1$ conditioned on $\nu_2$ is given by:

$$
\begin{aligned}
\Pr\left\{\varepsilon_1 \le \alpha \;\middle|\; \nu_2 = \xi\right\} &= \Pr\left\{\frac{R}{\log(1+\nu_1 P)} \le \alpha \;\middle|\; \nu_2 = \xi\right\} \\
&= \Pr\left\{\nu_1 \ge \frac{e^{\frac{R}{\alpha}}-1}{P} \;\middle|\; \nu_2 = \xi\right\} \\
&= \begin{cases} \int_{\frac{e^{\frac{R}{\alpha}}-1}{P}}^{\infty} p_{\nu_1|\nu_2}(u \mid \xi)\,\mathrm{d}u, & \xi < \frac{1}{P}\left(e^{\frac{R}{\alpha}}-1\right),\ 0 \le \alpha < 1 \\ 1, & \xi \ge \frac{1}{P}\left(e^{\frac{R}{\alpha}}-1\right) \ \text{ or } \ \alpha \ge 1 \end{cases} \\
&= \begin{cases} \int_{\frac{e^{\frac{R}{\alpha}}-1}{P}}^{\infty} e^{-u+\xi}\,\mathrm{d}u, & \xi < \frac{1}{P}\left(e^{\frac{R}{\alpha}}-1\right),\ 0 \le \alpha < 1 \\ 1, & \xi \ge \frac{1}{P}\left(e^{\frac{R}{\alpha}}-1\right) \ \text{ or } \ \alpha \ge 1 \end{cases} \\
&= \begin{cases} e^{-\frac{e^{\frac{R}{\alpha}}-1}{P}+\xi}, & \xi < \frac{1}{P}\left(e^{\frac{R}{\alpha}}-1\right),\ 0 \le \alpha < 1 \\ 1, & \xi \ge \frac{1}{P}\left(e^{\frac{R}{\alpha}}-1\right) \ \text{ or } \ \alpha \ge 1 \end{cases} .
\end{aligned}
\tag{B.4}
$$

The probability $p_2(R,P,Q)$ is therefore expressed as

$$
\begin{aligned}
p_2(R,P,Q) &= \Pr\left\{\varepsilon_2 < 1\right\} = \int_0^{\infty} \Pr\left\{\varepsilon_2 < 1 \;\middle|\; \nu_2 = \xi\right\} p_{\nu_2}(\xi)\,\mathrm{d}\xi = \\
&= \int_0^{\infty} \Pr\left\{\varepsilon_1 < \frac{\log(1+\nu_2 P + Q) - R}{\log\left(1+\frac{Q}{1+\nu_2 P}\right)} \;\middle|\; \nu_2 = \xi\right\} p_{\nu_2}(\xi)\,\mathrm{d}\xi \\
&= \int_0^{\frac{e^R-1}{P}} \Pr\left\{\varepsilon_1 < \frac{\log(1+\nu_2 P + Q) - R}{\log\left(1+\frac{Q}{1+\nu_2 P}\right)} \;\middle|\; \nu_2 = \xi\right\} p_{\nu_2}(\xi)\,\mathrm{d}\xi + \int_{\frac{e^R-1}{P}}^{\infty} p_{\nu_2}(\xi)\,\mathrm{d}\xi \\
&= \int_{\max\left(0,\frac{e^R-1-Q}{P}\right)}^{\frac{e^R-1}{P}} e^{-\frac{1}{P}\left[\exp\left\{\frac{R\log\left(1+\frac{Q}{1+\xi P}\right)}{\log(1+\xi P+Q)-R}\right\}-1\right] + \xi} 2e^{-2\xi}\,\mathrm{d}\xi + \int_{\frac{e^R-1}{P}}^{\infty} 2e^{-2\xi}\,\mathrm{d}\xi \\
&= 2\int_{\max\left(0,\frac{e^R-1-Q}{P}\right)}^{\frac{e^R-1}{P}} e^{-\frac{1}{P}\left[\exp\left\{\frac{R\log\left(1+\frac{Q}{1+\xi P}\right)}{\log(1+\xi P+Q)-R}\right\}-1\right] - \xi}\,\mathrm{d}\xi + e^{-2\frac{e^R-1}{P}} \\
&= \frac{2e^{\frac{2}{P}}}{P}\int_{\max(1,e^R-Q)}^{e^R} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{Q}{u}\right)}{\log(u+Q)-R}\right\} - \frac{u}{P}}\,\mathrm{d}u + e^{-2\frac{e^R-1}{P}}.
\end{aligned}
$$

Finally, the expected rate from source to destination is given by

$$\begin{aligned} R_{av}\left(R,P,Q\right) &= \frac{R}{2}\Big(p_1(R,P)+p_2(R,P,Q)\Big) \\ &= \frac{Re^{\frac{2}{P}}}{P}\int_{\max(1,e^R-Q)}^{e^R} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{Q}{u}\right)}{\log(u+Q)-R}\right\}-\frac{u}{P}}\,\mathrm{d}u + Re^{-\frac{e^R-1}{P}}. \end{aligned} \tag{B.5}$$

# Appendix C

# Average Power in Transmission to Two Users

As is evident from equations (7), (8) and (9) the average powers depend on $P$ and the expectations $\mathbb{E}(\varepsilon_1)$ and $\mathbb{E}(\varepsilon_2)$. We therefore compute first these expectations. Consider the CDF of $\varepsilon_1$, namely

$$\begin{aligned} \Pr\left\{\varepsilon_1 \le x\right\} &= \begin{cases} \Pr\left\{\nu_1 \ge \frac{e^{\frac{R}{x}}-1}{P}\right\}, & 0<x<1 \\ 1, & x=1 \end{cases} \\ &= \begin{cases} \int_{\frac{e^{\frac{R}{x}}-1}{P}}^{\infty} 2e^{-\nu_1}-2e^{-2\nu_1}\,\mathrm{d}\nu_1, & 0<x<1 \\ 1, & x=1 \end{cases} \\ &= \begin{cases} 2e^{-\frac{e^{\frac{R}{x}}-1}{P}}-e^{-2\frac{e^{\frac{R}{x}}-1}{P}}, & 0<x<1 \\ 1, & x=1 \end{cases}. \end{aligned} \tag{C.1}$$

Note that this CDF has a discontinuity at $x=1$, accounting for the fact that we have restricted the value of $\varepsilon_1$ to be equal to 1 whenever the fading gain is not strong enough to ensure decoding before the end of the block. The expectation follows by

$$\begin{aligned} \mathbb{E}\left(\varepsilon_1\right) &= \int_0^1 x\,\mathrm{d}\Pr\left\{\varepsilon_1\le x\right\} \\ &= x\Pr\left\{\varepsilon_1\le x\right\}\Big|_0^1 - \int_0^1 \Pr\left\{\varepsilon_1\le x\right\}\,\mathrm{d}x \\ &= 1-\int_0^1 2e^{-\frac{e^{\frac{R}{x}}-1}{P}}-e^{-2\frac{e^{\frac{R}{x}}-1}{P}}\,\mathrm{d}x. \end{aligned} \tag{C.2}$$

Next, we consider the CDF of $\varepsilon_2$, namely,

$$\begin{aligned}
\Pr\Big\{\varepsilon_2 \le x\Big\} &= \Pr\left\{\min\left(1,\ \varepsilon_1 + \frac{R - \varepsilon_1 \log(1+\nu_2 P)}{\log(1+\nu_2 P + Q)}\right) \le x\right\} \\
&= \Pr\left\{\varepsilon_1 + \frac{R - \varepsilon_1 \log(1+\nu_2 P)}{\log(1+\nu_2 P + Q)} \le x\right\} \\
&= \Pr\left\{\varepsilon_1 \le \frac{x\log(1+\nu_2 P + Q) - R}{\log\left(1 + \frac{Q}{1+\nu_2 P}\right)}\right\}, \quad x < 1 \qquad \text{(C.3)}
\end{aligned}$$

The following chain of equations leads to the desired expectation

$$\begin{aligned}
\Pr\Big\{\varepsilon_2 \le x\Big\} &= \int_0^\infty \Pr\Big\{\varepsilon_2 \le x \ \Big| \ \nu_2 = \xi\Big\}\, p_{\nu_2}(\xi)\, \mathrm{d}\xi = \\
&\stackrel{(a)}{=} \int_0^\infty \Pr\left\{\varepsilon_1 < \frac{x\log(1+\nu_2 P + Q) - R}{\log\left(1 + \frac{Q}{1+\nu_2 P}\right)} \ \middle| \ \nu_2 = \xi\right\} p_{\nu_2}(\xi)\, \mathrm{d}\xi \\
&\stackrel{(b)}{=} \int_0^{\frac{e^{\frac{R}{x}}-1}{P}} \Pr\left\{\varepsilon_1 < \frac{x\log(1+\nu_2 P + Q) - R}{\log\left(1 + \frac{Q}{1+\nu_2 P}\right)} \ \middle| \ \nu_2 = \xi\right\} p_{\nu_2}(\xi)\, \mathrm{d}\xi + \int_{\frac{e^{\frac{R}{x}}-1}{P}}^\infty p_{\nu_2}(\xi)\, \mathrm{d}\xi \\
&\stackrel{(c)}{=} \int_{\max\left(0, \frac{e^{\frac{R}{x}}-1-Q}{P}\right)}^{\frac{e^{\frac{R}{x}}-1}{P}} e^{-\frac{1}{P}\left[\exp\left\{\frac{R\log\left(1+\frac{Q}{1+\xi P}\right)}{x\log(1+\xi P + Q) - R}\right\} - 1\right] + \xi} 2e^{-2\xi}\, \mathrm{d}\xi + \int_{\frac{e^{\frac{R}{x}}-1}{P}}^\infty 2e^{-2\xi}\, \mathrm{d}\xi \\
&\stackrel{(d)}{=} 2\int_{\max\left(0, \frac{e^{\frac{R}{x}}-1-Q}{P}\right)}^{\frac{e^{\frac{R}{x}}-1}{P}} e^{-\frac{1}{P}\left[\exp\left\{\frac{R\log\left(1+\frac{Q}{1+\xi P}\right)}{x\log(1+\xi P + Q) - R}\right\} - 1\right] - \xi} \mathrm{d}\xi + e^{-2\frac{e^{\frac{R}{x}}-1}{P}} \\
&\stackrel{(e)}{=} \frac{2e^{\frac{2}{P}}}{P} \int_{\max\left(1, e^{\frac{R}{x}} - Q\right)}^{e^{\frac{R}{x}}} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{Q}{u}\right)}{x\log(u+Q) - R}\right\} - \frac{u}{P}} \mathrm{d}u + e^{-2\frac{e^{\frac{R}{x}}-1}{P}}
\end{aligned}$$

where $(a)$ follows by substitution of (C.3), $(b)$ follows trivially from $(a)$, $(c)$ follows by substitution of the conditional CDF (B.4) and the PDF of the weak user and $(d)$ and $(e)$ are trivial derivations. The expectation $\mathbb{E}(\varepsilon_2)$ can now be expressed as follows

$$\begin{aligned}
\mathbb{E}(\varepsilon_2) &= \int_0^1 x \, \mathrm{d}\Pr\{\varepsilon_2 \le x\} \\
&= x\Pr\{\varepsilon_2 \le x\}\Big|_0^1 - \int_0^1 \Pr\{\varepsilon_2 \le x\}\, \mathrm{d}x \\
&= 1 - \frac{2e^{\frac{2}{P}}}{P} \int_0^1 \int_{\max(1, \exp\{R/x\} - Q)}^{e^{\frac{R}{x}}} e^{-\frac{1}{P}\exp\left\{\frac{R\log\left(1+\frac{Q}{u}\right)}{x\log(u+Q) - R}\right\} - \frac{u}{P}} \mathrm{d}u\, \mathrm{d}x - \int_0^1 e^{-2\frac{e^{\frac{R}{x}}-1}{P}} \mathrm{d}x. \qquad \text{(C.4)}
\end{aligned}$$

Finally, substituting equations (C.2) and (C.4) in equations (7), (8) and (9) and specializing the expressions for $K = 2$ yields the desired result.

# Appendix D

# Expected Throughput for $K$ Users

Consider the definition of the fractional decoding times (4) and focus on the event $\{\varepsilon_\ell < 1\}$. Note that this event is contained in the event $\bigcap_{j=1}^{\ell-1} \{\varepsilon_j < 1\}$. Therefore, for the purpose of computing the probability $\Pr\{\varepsilon_\ell < 1\}$ one can assume that the fractional time epochs are defined without the restriction to unity of all preceding time epochs, namely

$$
\begin{aligned}
\varepsilon_1 &= \frac{R}{\log(1+\nu_1 P)} \\
\varepsilon_2 &= \varepsilon_1 + \frac{R - \varepsilon_1 \log(1+\nu_2 P)}{\log(1+\nu_2 P + Q)} \\
&\ \vdots \\
\varepsilon_m &= \varepsilon_{m-1} + \frac{R - \sum_{k=1}^{m-1} (\varepsilon_k - \varepsilon_{k-1}) \log\big(1+\nu_m P + (k-1)Q\big)}{\log\big(1+\nu_m P + (m-1)Q\big)}, \quad m = 3, \ldots, \ell-1 \\
\varepsilon_\ell &= \min\left(1, \underbrace{\varepsilon_{\ell-1} + \frac{R - \sum_{k=1}^{\ell-1} (\varepsilon_k - \varepsilon_{k-1}) \log\big(1+\nu_\ell P + (k-1)Q\big)}{\log\big(1+\nu_\ell P + (\ell-1)Q\big)}}_{A}\right).
\end{aligned}
\tag{D.1}
$$

Note that the term $A$ can be written as $a(R,P,Q,\nu_2,\ldots,\nu_\ell)\varepsilon_1 + b(R,P,Q,\nu_2,\ldots,\nu_\ell)$. Now, the event $\{\varepsilon_\ell < 1\}$ can be equivalently stated as $\{\varepsilon_1 < \mu_\ell(R,P,Q,\nu_2,\nu_3,\ldots,\nu_\ell)\}$ where $\mu_\ell(R,P,Q,\nu_2,\ldots,\nu_\ell) \triangleq \frac{1-b(R,P,Q,\nu_2,\ldots,\nu_\ell)}{a(R,P,Q,\nu_2,\ldots,\nu_\ell)}$. Now, whenever $\mu_\ell(R,P,Q,\nu_2,\ldots,\nu_\ell) < 0$ this implies that the probability vanishes, whereas a value greater than unity implies that the event occurs with probability one. The former case happens when even users stronger than the $\ell$-th user could not decode the message, while the latter case occurs when the $\ell$-th user was strong enough to decode the message even with no help from other users, simply because its channel was strong enough i.e. $\log(1+\nu_\ell P) > R$.

The sequence of functions $\mu_\ell(R, P, Q, \nu_2, \ldots, \nu_\ell)$ can in principal be computed for all $\ell$. The first three terms of this sequence are

$$
\begin{aligned}
&\mu_1 = 1\\
&\mu_2(R, P, Q, \nu_2) = \frac{1 - R\frac{1}{\log(1+\nu_2 P+Q)}}{1 - \frac{\log(1+\nu_2 P)}{\log(1+\nu_2 P+Q)}}\\
&\mu_3(R, P, Q, \nu_2, \nu_3) = \frac{1 - R\left(\frac{1}{\log(1+\nu_2 P+Q)} + \frac{\log(1+\nu_2 P+Q)-\log(1+\nu_3 P+Q)}{\log(1+\nu_2 P+Q)\log(1+\nu_3 P+2Q)}\right)}{1 - \frac{\log(1+\nu_2 P)}{\log(1+\nu_2 P+Q)} - \frac{\log(1+\nu_3 P)}{\log(1+\nu_3 P+2Q)} + \frac{\log(1+\nu_2 P)\log(1+\nu_3 P+Q)}{\log(1+\nu_2 P+Q)\log(1+\nu_3 P+2Q)}}\\
&\vdots \\
&\mu_\ell(R, P, Q, \nu_2, \nu_3, \ldots, \nu_\ell) = \cdots, \;\; \ell = 4, \cdots, K.
\end{aligned}
\tag{D.2}
$$

Next, observe that the conditional CDF of $\nu_1$ given $\nu_2, \nu_3, \ldots, \nu_\ell, \ldots, \nu_K$ can be easily written due to the Markovian property of the ordered statistics [50] as follows

$$
\begin{aligned}
\Pr\{\nu_1 \le a \big| \nu_2 = \beta_2, \nu_3 = \beta_3, \ldots, \nu_K = \beta_K\} &= \int_\beta^a p_{\nu_1|\nu_2,\nu_3,\ldots,\nu_K}(\alpha|\beta_2, \beta_3, \ldots, \beta_K)\,\mathrm{d}\alpha\\
&= \int_\beta^a p_{\nu_1|\nu_2}(\alpha|\beta_2)\,\mathrm{d}\alpha\\
&= \int_\beta^a e^{-\alpha+\beta_2}\,\mathrm{d}\alpha\\
&= 1 - e^{-a+\beta_2}.
\end{aligned}
\tag{D.3}
$$

So, the probability of successful decoding by user $\ell$ conditioned on $\{\nu_2, \nu_3, \ldots, \nu_\ell\}$ is

$$
\begin{aligned}
&\Pr\{\varepsilon_\ell < 1|\nu_2 = \beta_2, \ldots, \nu_K = \beta_K\} =\\
&\qquad = \Pr\{\varepsilon_1 < \mu_\ell(R, P, Q, \nu_2, \nu_3, \ldots, \nu_\ell)|\nu_2 = \beta_2, \ldots, \nu_K = \beta_K\}\\
&\qquad = \begin{cases} 0, & \mu_\ell \le 0\\ \Pr\left\{\nu_1 > \frac{e^{\frac{R}{\mu_\ell(R,P,Q,\nu_2,\nu_3,\ldots,\nu_\ell)}}-1}{P}\middle|\nu_2 = \beta_2, \ldots, \nu_K = \beta_K\right\}, & 0 < \mu_\ell < 1\\ 1, & \mu_\ell \ge 1 \end{cases}\\
&\qquad = \begin{cases} 0, & \mu_\ell \le 0\\ \exp\left\{-\frac{e^{\frac{R}{\mu_\ell(R,P,Q,\nu_2,\nu_3,\ldots,\nu_\ell)}}-1}{P} + \beta_2\right\}, & 0 < \mu_\ell < 1\\ 1, & \mu_\ell \ge 1 \end{cases}.
\end{aligned}
\tag{D.4}
$$

Now we can express the decoding probability of the $\ell$-th user as

$$
\begin{aligned}
&\Pr\{\varepsilon_\ell < 1\} = \\
&= \underset{\beta_2>\beta_3>\cdots>\beta_K}{\iint\!\cdots\!\int} \Pr\{\varepsilon_\ell < 1 | \nu_2 = \beta_2, \ldots, \nu_K = \beta_K\}\, p_{\nu_2,\ldots,\nu_k}(\beta_2,\ldots,\beta_K)\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K \\
&= \underset{\substack{\beta_2>\beta_3>\cdots>\beta_K\\ 0<\mu_\ell<1}}{\iint\!\cdots\!\int} K!\exp\left\{-\frac{e^{\frac{R}{\mu_\ell(R,P,Q,\nu_2,\nu_3,\ldots,\nu_\ell)}}-1}{P}+\beta_2\right\} e^{-2\beta_2-\beta_3-\cdots-\beta_K}\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K + \\
&\quad + \underset{\substack{\beta_2>\beta_3>\cdots>\beta_K\\ \mu\geq 1}}{\iint\!\cdots\!\int} p_{\nu_2,\ldots,\nu_k}(\beta_2,\ldots,\beta_K)\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K \\
&= \underset{\substack{\beta_2>\beta_3>\cdots>\beta_K\\ 0<\mu_\ell<1}}{\iint\!\cdots\!\int} K!\exp\left\{-\frac{e^{\frac{R}{\mu_\ell(R,P,Q,\beta_2,\beta_3,\ldots,\beta_\ell)}}-1}{P}-\sum_{j=2}^{K}\beta_j\right\}\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K + \\
&\quad + \underset{\substack{\beta_2>\beta_3>\cdots>\beta_K\\ \nu_\ell>\frac{e^R-1}{P}}}{\iint\!\cdots\!\int} p_{\nu_2,\ldots,\nu_k}(\beta_2,\ldots,\beta_K)\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K \\
&= \underset{\substack{\beta_2>\beta_3>\cdots>\beta_K\\ 0<\mu_\ell<1}}{\iint\!\cdots\!\int} K!\exp\left\{-\frac{e^{\frac{R}{\mu_\ell(R,P,Q,\beta_2,\beta_3,\ldots,\beta_\ell)}}-1}{P}-\sum_{j=2}^{K}\beta_j\right\}\mathrm{d}\beta_2\,\mathrm{d}\beta_3\cdots\mathrm{d}\beta_K + \\
&\quad + \Pr\left\{\nu_\ell > \frac{e^R-1}{P}\right\}
\end{aligned}
$$

Finally, by averaging over all the users one gets the desired result.

# Appendix E

# Lower and Upper Bounds Involving Only Two Relays

In this appendix we provide lower and upper bounds to the expected throughput by considering two relaying users for each destination user. To this end we will need the PDF of the $k$-th strongest fading gain conditioned on the $m$-th and $\ell$-th strongest fading gains

($\ell > m > k$), which with the help of [50] is given by:

$$\begin{aligned} p_{\nu_k|\nu_m,\nu_\ell}(\alpha|\beta,\gamma) &= p_{\nu_k|\nu_m}(\alpha|\beta) \\ &= \frac{(m-1)!}{(m-k-1)!(k-1)!} e^{-\alpha k+(m-1)\beta}(e^{-\beta}-e^{-\alpha})^{m-k-1} \end{aligned} \tag{E.1}$$

Say the source transmits to user $\ell$. Let $k \in \{0, 1, 2, \ldots, \ell-2\}$ and $m \in \{0, 1, 2, \ldots, \ell-1\}$, $m \geq k$, be the indices of the relays for the $\ell$-th user. Then the fractional time epochs of the decoding times are given by:

$$\begin{aligned} \varepsilon_1 &= \min\left(1, \frac{R}{\log(1+\nu_1 P)}\right) \\ \varepsilon_2 &= \min\left(1, \varepsilon_1 + \frac{R-\varepsilon_1\log(1+\nu_2 P)}{\log(1+\nu_2 P+Q_1)}\right) \\ \varepsilon_3 &= \min\left(1, \varepsilon_2 + \frac{R-\varepsilon_1\log(1+\nu_3 P)-(\varepsilon_2-\varepsilon_1)\log(1+\nu_3 P+Q_1)}{\log(1+\nu_3 P+Q_1+Q_2)}\right) \\ &\vdots \\ \varepsilon_\ell &= \min\left(1, \varepsilon_m + \frac{R-\varepsilon_k\log(1+\nu_\ell P+Q_0)-(\varepsilon_m-\varepsilon_k)\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}\right), \\ &\qquad \ell = 4, \cdots, K \end{aligned} \tag{E.2}$$

Where $\varepsilon_k = \min\left(1, \frac{R}{\log(1+\nu_k P+Q_0)}\right)$ and $\varepsilon_m = \min\left(1, \varepsilon_k + \frac{R-\varepsilon_k\log(1+\nu_m P+Q_0)}{\log(1+\nu_m P+Q_0+Q_1)}\right)$. For the moment we assume that there are three co-location gains. $Q_0$ is the fixed gain from a virtual relay located next to the source to user $\ell$. $Q_1$ is the gain from user $k$ to the destination $\ell$, and $Q_2$ is the gain from user $m$ to the destination. Their specific values will be determined later on. Under these assumptions the decoding probability of the $\ell$-th user, $\ell = 3, \ldots, K$, is obtained as follows.

$$\begin{aligned} & p_\ell(R, P, Q_0, Q_1, Q_2) = \Pr\Big\{\varepsilon_\ell < 1\Big\} = \\ &= \Pr\left\{\varepsilon_m + \frac{R-\varepsilon_k\log(1+\nu_\ell P+Q_0)-(\varepsilon_m-\varepsilon_k)\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_\ell P+Q_0+Q_1+Q_2)} < 1\right\} \\ &= \Pr\Bigg\{\varepsilon_k + \frac{R-\varepsilon_k\log(1+\nu_m P+Q_0)}{\log(1+\nu_m P+Q_0+Q_1)} + \\ &\quad + \frac{R-\varepsilon_k\log(1+\nu_\ell P+Q_0) - \frac{R-\varepsilon_k\log(1+\nu_m P+Q_0)}{\log(1+\nu_m P+Q_0+Q_1)}\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_\ell P+Q_0+Q_1+Q_2)} < 1\Bigg\} \\ &= \Pr\Big\{\varepsilon_k < \mu(R, P, Q_0, Q_1, Q_2, \nu_m, \nu_\ell)\Big\}. \end{aligned}$$

where

$$\mu(R,P,Q_0,Q_1,Q_2,\nu_m,\nu_\ell)=$$
$$=\frac{1-R\left(\frac{1}{\log(1+\nu_m P+Q_0+Q_1)}+\frac{\log(1+\nu_m P+Q_0+Q_1)-\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_m P+Q_0+Q_1)\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}\right)}{1-\frac{\log(1+\nu_m P+Q_0)}{\log(1+\nu_m P+Q_0+Q_1)}-\frac{\log(1+\nu_\ell P+Q_0)}{\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}+\frac{\log(1+\nu_m P+Q_0)\log(1+\nu_\ell P+Q_0+Q_1)}{\log(1+\nu_m P+Q_0+Q_1)\log(1+\nu_\ell P+Q_0+Q_1+Q_2)}}.$$

Note that the conditional CDF of $\varepsilon_k$ conditioned on $\nu_m$ and $\nu_\ell$ is given by:

$$\begin{aligned}\Pr\left\{\varepsilon_k\le\alpha\;\middle|\;\nu_m=\eta,\,\nu_\ell=\xi\right\}&=\Pr\left\{\frac{R}{\log(1+\nu_k P+Q_0)}\le\alpha\;\middle|\;\nu_m=\eta,\,\nu_\ell=\xi\right\}\\&=\Pr\left\{\nu_k\ge\frac{e^{\frac{R}{\alpha}}-1-Q_0}{P}\;\middle|\;\nu_m=\eta,\,\nu_\ell=\xi\right\}\\&=\begin{cases}\sum_{j=k}^{m-1}C_{m-1}^{j}e^{(-\frac{e^{\frac{R}{\alpha}}-1}{P}+\eta)j}(1-e^{-\frac{e^{\frac{R}{\alpha}}-1}{P}+\eta})^{m-1-j}, & \\ \qquad \eta<\frac{1}{P}\left(e^{\frac{R}{\alpha}}-1-Q_0\right),\;0\le\alpha<1\\ 1, \qquad \eta\ge\frac{1}{P}\left(e^{\frac{R}{\alpha}}-1-Q_0\right)\;\text{ or }\;\alpha\ge 1\end{cases}\end{aligned}$$

so we can express the conditional probability

$$\Pr\left\{\varepsilon_k<1\middle|\nu_m=\eta,\,\nu_\ell=\xi\right\}=$$
$$=\Pr\left\{\varepsilon_k<\mu(R,P,Q_0,Q_1,Q_2,\nu_m,\nu_\ell)\middle|\nu_m=\eta,\,\nu_\ell=\xi\right\}=$$
$$=\begin{cases}0, & \mu(R,P,Q_0,Q_1,Q_2,\eta,\xi)\le 0\\ \sum_{j=k}^{m-1}C_{m-1}^{j}e^{(-\frac{e^{\frac{R}{\mu}}-1-Q_0}{P}+\eta)j}(1-e^{-\frac{e^{\frac{R}{\mu}}-1-Q_0}{P}+\eta})^{m-1-j}, & \\ & 0<\mu(R,P,Q_0,Q_1,Q_2,\eta,\xi)<1\\ 1, & \mu(R,P,Q_0,Q_1,Q_2,\eta,\xi)\ge 1\end{cases}.$$

Finally, the probability $p_\ell\,(R, P, Q_0, Q_1, Q_2, k, m)$ is expressed as

$$
\begin{aligned}
&p_\ell\,(R, P, Q_0, Q_1, Q_2, k, m) = \Pr\Big\{\varepsilon_\ell < 1\Big\} = \\
&= \int_0^\infty\!\!\int_0^\eta \Pr\left\{\varepsilon_\ell < 1 \;\Big|\; \nu_m = \eta,\ \nu_\ell = \xi\right\} p_{\nu_m,\nu_\ell}(\eta,\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \int_0^\infty\!\!\int_0^\eta \Pr\left\{\varepsilon_k < \mu(R, P, Q_0, Q_1, Q_2, \nu_m, \nu_\ell)\middle|\nu_m = \eta,\ \nu_\ell = \xi\right\} p_{\nu_m,\nu_\ell}(\eta,\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \int_0^{\frac{e^R-1-Q_0}{P}} \mathrm{d}\xi \int_\xi^\infty \mathrm{d}\eta \sum_{j=k}^{m-1} C_{m-1}^j e^{(-\frac{e^{\frac{R}{\mu}}-1-Q_0}{P}+\eta)j}(1-e^{-\frac{e^{\frac{R}{\mu}}-1-Q_0}{P}+\eta})^{m-1-j} p_{\nu_m,\nu_\ell}(\eta,\xi) + \\
&\quad + \Pr\left\{\nu_\ell > \frac{e^R-1-Q_0}{P}\right\}
\end{aligned}
$$

where $p_{\nu_m,\nu_\ell}(\eta,\xi) = \frac{K!}{(m-1)!(\ell-m-1)!(K-\ell)!} e^{-\eta m}(1-e^{-\xi})^{K-\ell}(e^{-\xi}-e^{-\eta})^{\ell-m-1}e^{-\xi}$.

By specializing the result of appendix D to $\ell = 1$ and $\ell = 2$ we get for the first two users the following decoding probabilities

$$
\begin{aligned}
p_1(R, P) &= \Pr\{\varepsilon_1 < 1\} = \Pr\left\{\nu_1 > \frac{e^R-1}{P}\right\} = 1 - \left(1 - e^{-\frac{e^R-1}{P}}\right)^K \\
p_2(R, P, Q_1) &= \int_{\max(0,\frac{e^R-1-Q}{P})}^{\frac{e^R-1}{P}} \exp\left\{-\frac{e^{\frac{R}{\mu_2(R,P,Q,\beta_2)}}-1}{P} + \beta_2\right\} p_{\nu_2}(\beta_2)\,\mathrm{d}\beta_2 + \\
&\quad + \Pr\left\{\nu_2 > \frac{e^R-1}{P}\right\}.
\end{aligned}
$$

where $p_{\nu_2}(\beta_2) = K(K-1)(1-e^{-\beta_2})^{K-2}e^{-2\beta_2}$.

As was explained in subsection 3.5 we get a lower bound by setting

$$
Q_0 = 0, \qquad Q_1 = kQ, \qquad Q_2 = (m-k)Q
$$

and an upper bound by setting

$$
Q_0 = (k-1)Q, \qquad Q_1 = (m-k)Q, \qquad Q_2 = (\ell-m)Q.
$$

Finally, after optimizing the decoding probabilities for every user over all possible selections of $k$ and $m$ and averaging over all users we obtain the desired results.

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