CCIT Report #546
July 2005

Strong Duality in Nonconvex Quadratic Optimization with Two Quadratic Constraints

Amir Beck* and Yonina C. Eldar†

April 12, 2005

Abstract

We consider the problem of minimizing an indefinite quadratic function subject to two quadratic inequality constraints. When the problem is defined over the complex plane we show that strong duality holds and obtain necessary and sufficient optimality conditions. Using the results in the complex case and a result on the connection between the image of the real and complex spaces under a quadratic mapping, we are able to derive a condition that ensures strong duality even in the case of real variables. Preliminary numerical simulations show that for random instances of the extended trust region subproblem, the sufficient condition hardly never fails.

1 Introduction

In this paper we consider quadratic minimization problems with two quadratic constraints both in the real and in the complex domain:

$$(Q2P_{\mathbb{C}}) \qquad \min_{\boldsymbol{z}\in\mathbb{C}^n} \{f_3(\boldsymbol{z}) : f_1(\boldsymbol{z}) \geq 0, f_2(\boldsymbol{z}) \geq 0\}, \tag{1}$$

$$(Q2P_{\mathbb{R}}) \qquad \min_{\boldsymbol{x}\in\mathbb{R}^n} \{f_3(\boldsymbol{x}) : f_1(\boldsymbol{x}) \geq 0, f_2(\boldsymbol{x}) \geq 0\}. \tag{2}$$

In the real case each function $f_j : \mathbb{R}^n \to \mathbb{R}$ is defined by $f_j(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}_j\boldsymbol{x} + 2\boldsymbol{b}_j^T\boldsymbol{x} + c_j$ with $\boldsymbol{A}_j = \boldsymbol{A}_j^T \in \mathbb{R}^{n\times n}, \boldsymbol{b}_j \in \mathbb{R}^n$ and $c_j \in \mathbb{R}$. In the complex setting, $f_j : \mathbb{C}^n \to \mathbb{R}$ is given by $f_j(\boldsymbol{z}) = \boldsymbol{z}^*\boldsymbol{A}_j\boldsymbol{z} + 2\Re(\boldsymbol{b}_j^*\boldsymbol{z}) + c_j$, where $\boldsymbol{A}_j$ are Hermitian matrices, *i.e.*, $\boldsymbol{A}_j = \boldsymbol{A}_j^*$, $\boldsymbol{b}_j \in \mathbb{C}^n$ and $c_j \in \mathbb{R}$. The condition $\boldsymbol{A}_j = \boldsymbol{A}_j^*$ can be written as $\Re(\boldsymbol{A}_j)^T = \Re(\boldsymbol{A}_j)$ and $\Im(\boldsymbol{A}_j)^T = -\Im(\boldsymbol{A}_j)$.

The problem $(Q2P_{\mathbb{R}})$ appears as a subproblem in some trust region algorithms for constrained

*MINERVA Optimization Center, Department of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel. E-mail: becka@tx.technion.ac.il.

†Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel. E-mail: yonina@ee.technion.ac.il. Fax: +972-4-8295757.

optimization [6, 8, 18] where the original problem is to minimize a general nonlinear function subject to equality constraints. The subproblem has the form

$$(TTRS) \quad \min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} + 2\boldsymbol{g}^T\boldsymbol{x} : \|\boldsymbol{x}\| \le \Delta, \|\boldsymbol{A}^T\boldsymbol{x} + \boldsymbol{c}\| \le \xi \right\}, \tag{3}$$

and is sometimes called the *two trust region problem* [1] or the *extended trust region problem* [25]. The objective function in (TTRS) is a quadratic model of the original objective in a neighborhood of the current iterate, $\boldsymbol{c} \in \mathbb{R}^m$ is a vector whose elements are the value of the constraints, $\boldsymbol{A}^T \in \mathbb{R}^{m\times n}$ is the Jacobian matrix of the constraints computed at the current iterate, and the constants δ and ξ are determined by the trust region method. More on the extended trust region subproblem and trust region algorithms can be found in [26, 27] and in the book [8].

If the original nonlinear constrained problem has complex variables, then instead of $(Q2P_{\mathbb{R}})$ one should consider the complex variant $(Q2P_{\mathbb{C}})$. Optimization problems with complex variables appear naturally in many engineering applications. For example, if the estimation problem is posed in the Fourier domain, then typically the parameters to be estimated will be complex valued, unless the original signal has strong symmetry properties [16, 19]. In the context of digital communications, many signal constellations are modelled as complex valued. Another area where complex variables naturally arise is narrowband array processing [7].

It is interesting to note that every complex quadratic problem of dimension n can be written as a *real* quadratic problem of dimension $2n$. This fact is made evident by considering the decomposition $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$, where $\boldsymbol{x} = \Re(\boldsymbol{z})$ and $\boldsymbol{y} = \Im(\boldsymbol{z})$. Then

$$f_j(\boldsymbol{z}) = (\boldsymbol{x}+i\boldsymbol{y})^*\boldsymbol{A}_j(\boldsymbol{x}+i\boldsymbol{y}) + 2\Re(\boldsymbol{b}_j^*(\boldsymbol{x}+i\boldsymbol{y})) + c_j = \boldsymbol{w}^T\mathbf{Q}_j\boldsymbol{w} + 2\boldsymbol{d}_j^T\boldsymbol{w} + c_j,$$

where

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{2n}, \mathbf{Q}_j = \begin{pmatrix} \Re(\boldsymbol{A}_j) & -\Im(\boldsymbol{A}_j) \\ \Im(\boldsymbol{A}_j) & \Re(\boldsymbol{A}_j) \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} \Re(\boldsymbol{b}_j) \\ \Im(\boldsymbol{b}_j) \end{pmatrix}.$$

However, the opposite claim is false: not every real quadratic problem of dimension $2n$ can be formulated as an n-dimensional complex quadratic problem. Evidently, the family of complex quadratic problems is a special case of real quadratic problems. Therefore, one may ask *why consider complex quadratic problems and not only real quadratic problems?*. The answer to this question is that, as we shall see, there are stronger results for complex quadratic problems than for their real counterparts (c.f. Sections 2 and 3).

A simpler (nonconvex) quadratic problem than (TTRS) is the *trust region subproblem*, which appears in trust region algorithms for *unconstrained* optimization, and has the following form:

$$(TR) \quad \min_{\boldsymbol{x}\in\mathbb{R}^n} \{\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} + 2\boldsymbol{g}^T\boldsymbol{x} : \|\boldsymbol{x}\|^2 \le \delta\}. \tag{4}$$

Problem (TR) has been studied extensively in the literature see, *e.g.,* [5, 10, 14, 15, 21, 22] and references therein. The dual problem to (TR) is given by the semidefinite problem (SDP)

$$\max_{\alpha,\lambda}\left\{\lambda\ :\ \begin{pmatrix} \boldsymbol{B}+\alpha\boldsymbol{I} & \boldsymbol{g} \\ \boldsymbol{g}^T & -\alpha\delta-\lambda \end{pmatrix}\succeq \boldsymbol{0}, \alpha\geq 0\right\} \tag{5}$$

and it can also be formulated as a maximization problem of a single variable concave problem that can be solved very efficiently (see *e.g.,* [8] and references therein). Problem (TR) enjoys many useful and attractive properties. It is known that (TR) admits no duality gap and that the semidefinite relaxation of (TR) is tight. Moreover, the solution of (TR) can be extracted from the dual solution $\bar{\alpha}$. A necessary and sufficient condition for $\bar{\boldsymbol{x}}$ to be an optimal solution of (TR) is that there exists $\bar{\alpha}\geq 0$ such that [14, 21]

$$(\boldsymbol{B}+\bar{\alpha}\boldsymbol{I})\bar{\boldsymbol{x}}+\boldsymbol{g} = \boldsymbol{0}, \tag{6}$$

$$\|\bar{\boldsymbol{x}}\|^2 \leq \delta, \tag{7}$$

$$\bar{\alpha}(\|\bar{\boldsymbol{x}}\|^2-\delta) = 0, \tag{8}$$

$$\boldsymbol{B}+\bar{\alpha}\boldsymbol{I} \succeq \boldsymbol{0}. \tag{9}$$

Unfortunately, in general these results cannot be extended to the (TTRS) problem, or to the more general problem ($Q2P_{\mathbb{R}}$). Indeed, it is known that the semidefinite relaxation of ($Q2P_{\mathbb{R}}$) is not necessarily tight [25, 26]. An exception is the case in which the functions f_1, f_2 and f_3 are all homogenous quadratic functions and there exists a positive definite linear combination of the matrices $\boldsymbol{A}_j$. In this case, Ye and Zhang [25] proved that the semidefinite relaxation of ($Q2P_{\mathbb{R}}$) is tight. Another interesting result obtained in [25] is that if f_1 is concave (thus the constraint is convex) and f_2 is linear then, although the semidefinite relaxation is *not necessarily tight*, there is an efficient algorithm for solving ($Q2P_{\mathbb{R}}$). The latter result is based on the dual cone representation approach of Sturm and Zhang [24].

In this paper we discuss both the complex and real case. The real case is analyzed using results derived for the complex case. In Section 2, we show that under some mild conditions strong duality holds for the complex valued problem ($Q2P_{\mathbb{C}}$) and that the semidefinite relaxation is tight. Our result is based on the extended version of the S-lemma derived by Fradkov and Yakubovich [11]. We also develop optimality conditions similar to those known for the TR problem (4). In Section 3, we present a method for calculating the optimal solution of ($Q2P_{\mathbb{C}}$) from the dual optimal solution based on the optimality conditions derived in Section 2 and a method for solving quadratic feasibility problems. Thus, all the results known for (TR) can essentially be extended to ($Q2P_{\mathbb{C}}$). Section 4 deals with the real case. After a brief discussion of the complex relaxation of ($Q2P_{\mathbb{R}}$), we present a sufficient condition that ensures zero duality gap (and tightness of the semidefinite relaxation) for ($Q2P_{\mathbb{R}}$). This condition is expressed via the dual optimal solution and therefore can be validated in polynomial-time. Our result is based on the connection between the image of the real and complex spaces under a quadratic mapping. Preliminary numerical experiments show that for random instances of the TTRS problem (3), this condition *hardly never fails*. The sufficient condition will also enable us to identify a family of problems with tight semidefinite relaxation. We end this paper with some appendices including some useful mathematical results that are used throughout the paper.

Notation: Instead of using an inf/sup notation, we use a min/max notation throughout the paper[1]. Vectors are denoted by boldface lowercase letters, *e.g.,* $\boldsymbol{y}$, and matrices are denoted by boldface uppercase letters *e.g.,* $\boldsymbol{A}$. For two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \succ \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is positive definite and $\boldsymbol{A} \succeq \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is positive semidefinite. $\mathcal{S}_n^+ = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} : \boldsymbol{A} \succeq \boldsymbol{0}\}$ is the set all real valued $n \times n$ positive semidefinite matrices[2] and $\mathcal{H}_n^+ = \{\boldsymbol{A} \in \mathbb{C}^{n\times n} : \boldsymbol{A} \succeq \boldsymbol{0}\}$ is the set of all complex valued $n \times n$ positive semidefinite matrices.[3] The real and imaginary part of scalars, vectors or matrices are denoted by $\Re(\cdot)$ and $\Im(\cdot)$. For an optimization problem

$$(\mathrm{P}) : \min / \max\{f(\boldsymbol{x}) : \boldsymbol{x} \in C\},$$

we denote the value of the optimal objective function by val(P). We use some standard abbreviations such as SDP (semidefinite programming), SDR (semidefinite relaxation) and LMI (linear matrix inequalities). We follow the MATLAB convention and use ";" for adjoining scalars, vectors or matrices in a column.

2 Strong Duality and Optimality Conditions for $(Q2P_{\mathbb{C}})$

In this section we use an extended version of the S-lemma in order to prove a strong duality result for problem $(Q2P_{\mathbb{C}})$ and to show that the semidefinite relaxation of $(Q2P_{\mathbb{C}})$ is tight. Necessary and sufficient optimality conditions, similar to those known for the TR problem (4) (conditions (6)-(9)), are also derived. All of the above mentioned results are true under extremely mild regularity conditions.

2.1 Strong duality for $(Q2P_{\mathbb{C}})$

In this section we use a result of [11], which extends the usual S-lemma to the complex case. Recall that for the real case [4, 11], the nonhomogenous S-lemma states that if there exists $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that $\bar{\boldsymbol{x}}^T \boldsymbol{A}_1 \bar{\boldsymbol{x}} + 2\boldsymbol{b}_1^T \bar{\boldsymbol{x}} + c_1 > 0$, then the following two conditions are equivalent:

1. $\boldsymbol{x}^T \boldsymbol{A}_1 \boldsymbol{x} + 2\boldsymbol{b}_1^T \boldsymbol{x} + c_1 \geq 0$ for every $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x}^T \boldsymbol{A}_2 \boldsymbol{x} + 2\boldsymbol{b}_2^T \boldsymbol{x} + c_2 \geq 0$.

[1] The fact that we use this notation does not mean that we assume that the optimum is attained and/or finite.
[2] A real valued positive semidefinite matrix is also symmetric.
[3] A complex valued positive semidefinite is also Hermitian.

2. There exists $\lambda \geq 0$ such that $\begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1^T & c_1 \end{pmatrix} \succeq \lambda \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{b}_2 \\ \boldsymbol{b}_2^T & c_2 \end{pmatrix}$.

Extensions of the S-lemma in the real case are in general not true. For example, the natural extension for the case of two quadratic inequalities that imply a third quadratic inequality is not true in general (see the example in [4]). However, the following result of Fradkov and Yakubuvich [11, Theorem 2.2] extends the S-lemma to the complex case. This result will be the key ingredient in proving strong duality.

Theorem 2.1 (Extended S-Lemma [11]). *Let*

$$f_j(\boldsymbol{z}) = \boldsymbol{z}^* \boldsymbol{A}_j \boldsymbol{z} + 2\Re(\boldsymbol{b}_j^* \boldsymbol{z}) + c_j, \quad \boldsymbol{z} \in \mathbb{C}^n, i = 1, 2, 3$$

where $\boldsymbol{A}_j$ are $n \times n$ Hermitian matrices, $\boldsymbol{b}_j \in \mathbb{C}^n$ and $c_j \in \mathbb{R}$. Suppose that there exists $\tilde{\boldsymbol{z}} \in \mathbb{C}^n$ such that $f_1(\tilde{\boldsymbol{z}}) > 0, f_2(\tilde{\boldsymbol{z}}) > 0$. Then the following two claims are equivalent:

1. *$f_3(\boldsymbol{z}) \geq 0$ for every $\boldsymbol{z} \in \mathbb{C}^n$ such that $f_1(\boldsymbol{z}) \geq 0$ and $f_2(\boldsymbol{z}) \geq 0$.*

2. *There exists $\alpha, \beta \geq 0$ such that*

$$\begin{pmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3 \\ \boldsymbol{b}_3^* & c_3 \end{pmatrix} \succeq \alpha \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1^* & c_1 \end{pmatrix} + \beta \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{b}_2 \\ \boldsymbol{b}_2^* & c_2 \end{pmatrix}.$$

The standard Lagrangian dual of $(Q2P_{\mathbb{C}})$ is given by

$$(D_{\mathbb{C}}) \qquad \max_{\alpha \geq 0, \beta \geq 0, \lambda} \left\{ \lambda \;\middle|\; \begin{pmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3 \\ \boldsymbol{b}_3^* & c_3 - \lambda \end{pmatrix} \succeq \alpha \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1^* & c_1 \end{pmatrix} + \beta \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{b}_2 \\ \boldsymbol{b}_2^* & c_2 \end{pmatrix} \right\}. \tag{10}$$

Problem $(D_{\mathbb{C}})$ is sometimes called Shor's relaxation [20]. Theorem 2.2 below states that if problem $(Q2P_{\mathbb{C}})$ is strictly feasible then $\text{val}(Q2P_{\mathbb{C}}) = \text{val}(D_{\mathbb{C}})$ *even* in the case where the value is equal to $-\infty$.

Theorem 2.2 (Strong duality for complex valued quadratic problems). *Suppose that problem $(Q2P_{\mathbb{C}})$ is strictly feasible,* i.e., *there exists $\tilde{\boldsymbol{z}} \in \mathbb{C}^n$ such that $f_1(\tilde{\boldsymbol{z}}) > 0, f_2(\tilde{\boldsymbol{z}}) > 0$. Then,*

1. *if val$(Q2P_{\mathbb{C}})$ is finite then the maximum of problem $(D_{\mathbb{C}})$ is attained and val$(Q2P_{\mathbb{C}})$ = val$(D_{\mathbb{C}})$.*

2. *val$(Q2P_{\mathbb{C}}) = -\infty$ if and only if $(D_{\mathbb{C}})$ is not feasible.*

Proof: 1. Suppose first that $\text{val}(Q2P_{\mathbb{C}})$ is finite. Then clearly

$$\text{val}(Q2P_{\mathbb{C}}) = \max_{\lambda} \{\lambda : \text{val}(Q2P_{\mathbb{C}}) \geq \lambda\}. \tag{11}$$

Now, the statement $\mathrm{val}(Q2P_{\mathbb{C}}) \geq \lambda$ holds true if and only if the implication

$$f_1(\boldsymbol{z}) \geq 0, f_2(\boldsymbol{z}) \geq 0 \Rightarrow f_3(\boldsymbol{z}) \geq \lambda$$

is valid. By Theorem 2.1 this is equivalent to

$$\exists \alpha, \beta \geq 0 \quad \begin{pmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3 \\ \boldsymbol{b}_3^* & c_3 - \lambda \end{pmatrix} \succeq \alpha \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1^* & c_1 \end{pmatrix} + \beta \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{b}_2 \\ \boldsymbol{b}_2^* & c_2 \end{pmatrix} \tag{12}$$

Therefore, by replacing the constraint in the maximization problem (11) with the LMI (12), we obtain that $\mathrm{val}(Q2P_{\mathbb{C}}) = \mathrm{val}(D_{\mathbb{C}})$. The maximum of $(D_{\mathbb{C}})$ is attained at $(\bar{\lambda}, \bar{\alpha}, \bar{\beta})$, where $\bar{\lambda}$ is the (finite) value $\mathrm{val}(Q2P_{\mathbb{C}})$ and $\bar{\alpha}, \bar{\beta}$ are the corresponding nonnegative constants that satisfy the LMI (12) for $\lambda = \bar{\lambda} = \mathrm{val}(Q2P_{\mathbb{C}})$.

2. We will prove that $\mathrm{val}(Q2P_{\mathbb{C}}) > -\infty$ if and only if problem $(D_{\mathbb{C}})$ is feasible. Indeed, $\mathrm{val}(Q2P_{\mathbb{C}}) > -\infty$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\mathrm{val}(Q2P_{\mathbb{C}}) \geq \lambda$, which is equivalent to the implication

$$f_1(\boldsymbol{z}) \geq 0, f_2(\boldsymbol{z}) \geq 0 \Rightarrow f_3(\boldsymbol{z}) \geq \lambda.$$

By Theorem 2.1 the latter implication is the same as the LMI (12), which is the same as feasibility of $(D_{\mathbb{C}})$. $\square$

It is interesting to note that the dual problem to $(D_{\mathbb{C}})$ is the so-called semidefinite relaxation of $(Q2P_{\mathbb{C}})$:

$$(SDR_{\mathbb{C}}) \quad \min_{\boldsymbol{Z}} \{\mathrm{Tr}(\boldsymbol{Z}\boldsymbol{M}_3) : \mathrm{Tr}(\boldsymbol{Z}\boldsymbol{M}_1) \geq 0, \mathrm{Tr}(\boldsymbol{Z}\boldsymbol{M}_2) \geq 0, Z_{n+1,n+1} = 1, \boldsymbol{Z} \in \mathcal{H}_{n+1}^{+}\}, \tag{13}$$

where

$$\boldsymbol{M}_j = \begin{pmatrix} \boldsymbol{A}_j & \boldsymbol{b}_j \\ \boldsymbol{b}_j^* & c_j \end{pmatrix}.$$

A result on the tightness of the semidefinite relaxation $(SDR_{\mathbb{C}})$ for problem $(Q2P_{\mathbb{C}})$ is given in the following theorem.

Theorem 2.3 (Tightness of the semidefinite relaxation). *Suppose that both problems $(Q2P_{\mathbb{C}})$ and $(D_{\mathbb{C}})$ are strictly feasible and let $(\bar{\lambda}, \bar{\alpha}, \bar{\beta})$ be an optimal solution of $(D_{\mathbb{C}})$. Then problems $(Q2P_{\mathbb{C}})$,$(D_{\mathbb{C}})$ and $(SDR_{\mathbb{C}})$ (problems (1),(10) and (13) respectively) attain their solutions and*

$$val(Q2P_{\mathbb{C}}) = val(D_{\mathbb{C}}) = val(SDR_{\mathbb{C}}).$$

Proof: Since $(Q2P_{\mathbb{C}})$ is strictly feasible, by Theorem 2.2 the maximum of problem $(D_{\mathbb{C}})$ is attained and $\mathrm{val}(Q2P_{\mathbb{C}}) = \mathrm{val}(D_{\mathbb{C}})$. Assume to the contrary that the minimum of problem $(Q2P_{\mathbb{C}})$ is not attained. Then there exists a sequence $\{\boldsymbol{z}^n\}$ such that $\|\boldsymbol{z}^n\| \to \infty, f_1(\boldsymbol{z}^n) \geq 0, f_2(\boldsymbol{z}^n) \geq 0$ and

$$f_3(\boldsymbol{z}^n) \to \text{val}(Q2P_{\mathbb{C}}). \tag{14}$$

Since problem $(D_{\mathbb{C}})$ is strictly feasible

$$\exists \tilde{\alpha} \geq 0, \tilde{\beta} \geq 0 \text{ such that } \boldsymbol{A}_3 \succ \tilde{\alpha}\boldsymbol{A}_1 + \tilde{\beta}\boldsymbol{A}_2. \tag{15}$$

Now,

$$f_3(\boldsymbol{z}^n) \geq f_3(\boldsymbol{z}^n) - \tilde{\alpha} f_1(\boldsymbol{z}^n) - \tilde{\beta} f_2(\boldsymbol{z}^n), \tag{16}$$

where the latter inequality is true since $f_1(\boldsymbol{z}^n), f_2(\boldsymbol{z}^n), \tilde{\alpha}$ and $\tilde{\beta}$ are all nonnegative. Consider the function $q(\boldsymbol{z}) \equiv f_3(\boldsymbol{z}) - \tilde{\alpha} f_1(\boldsymbol{z}) - \tilde{\beta} f_2(\boldsymbol{z})$. Then by (15), $q(\boldsymbol{z})$ is a coercive function and thus, since $\|\boldsymbol{z}^n\| \to \infty$ it follows that $\lim_{n\to\infty} q(\boldsymbol{z}^n) = \infty$, which is a contradiction to (14). All that is left to prove is that the minimum of the SDR problem (13) is attained and that val($D_{\mathbb{C}}$)=val($SDR_{\mathbb{C}}$). By Lemma B.1 (see Appendix B), the strict feasibility of $(Q2P_{\mathbb{C}})$ implies that the SDR problem (13) is also strictly feasible. Therefore, since $(D_{\mathbb{C}})$ and $(SDR_{\mathbb{C}})$ are strictly feasible conic problems which are dual to each other, we have by the conic duality theorem (see *e.g.,* [4]) that both problems $(D_{\mathbb{C}})$ and $(SDR_{\mathbb{C}})$ attain their solutions and val($SDR_{\mathbb{C}}$)=val($D_{\mathbb{C}}$). □

Strict feasibility of the dual problem $(D_{\mathbb{C}})$ is equivalent to saying that there exist $\tilde{\alpha} \geq 0, \tilde{\beta} \geq 0$ such that $\boldsymbol{A}_3 \succ \tilde{\alpha}\boldsymbol{A}_1 + \tilde{\beta}\boldsymbol{A}_2$. This condition is automatically satisfied if at least one of the constraints is strongly convex or if the objective function is strongly convex (see also [25, Proposition 2.1]), an assumption that is true in many practical scenarios, for example in the TTRS problem (3).

2.2 Optimality conditions

So far we have shown that under strict feasibility, problem $(Q2P_{\mathbb{C}})$ admits no gap with its dual problem $(D_{\mathbb{C}})$ and with the semidefinite relaxation $(SDR_{\mathbb{C}})$. In this section we derive necessary and sufficient optimality conditions. Theorem 2.4 below will be very useful in Section 3, where a method for extracting the optimal solution of $(Q2P_{\mathbb{C}})$ from the optimal solution of the dual problem $(D_{\mathbb{C}})$ will be described.

Theorem 2.4. *Suppose that both problems* $(Q2P_{\mathbb{C}})$ *and* $(D_{\mathbb{C}})$ *are strictly feasible, and let* $(\bar{\alpha}, \bar{\beta}, \bar{\lambda})$ *be an optimal solution of* $(D_{\mathbb{C}})$*. Then* $\bar{\boldsymbol{z}}$ *is an optimal solution of* $(Q2P_{\mathbb{C}})$ *if and only if*

$$(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)\bar{\boldsymbol{z}} + \boldsymbol{b}_3 - \bar{\alpha}\boldsymbol{b}_1 - \bar{\beta}\boldsymbol{b}_2 = \boldsymbol{0}, \tag{17}$$

$$f_1(\bar{\boldsymbol{z}}), f_2(\bar{\boldsymbol{z}}) \geq 0, \tag{18}$$

$$\bar{\alpha} f_1(\bar{\boldsymbol{z}}) = \bar{\beta} f_2(\bar{\boldsymbol{z}}) = 0. \tag{19}$$

Proof: From the strong duality result (Theorem 2.2) and from saddle point optimality conditions (see *e.g.,* [2, Theorem 6.2.5]) it follows that $\bar{\boldsymbol{z}}$ is an optimal solution of $(Q2P_{\mathbb{C}})$ if and only if the following conditions are satisfied

$$\begin{aligned} &\bar{\boldsymbol{z}} \in \operatorname{argmin} \mathcal{L}(\boldsymbol{z}, \bar{\alpha}, \bar{\beta}), \\ &f_1(\bar{\boldsymbol{z}}) \geq 0, f_2(\bar{\boldsymbol{z}}) \geq 0, \\ &\bar{\alpha} f_1(\bar{\boldsymbol{z}}) = \bar{\beta} f_2(\bar{\boldsymbol{z}}) = 0, \end{aligned} \tag{20}$$

where $\mathcal{L}$ is the Lagrangian function, *i.e.,*

$$\mathcal{L}(\boldsymbol{z}, \alpha, \beta) = f_3(\boldsymbol{z}) - \alpha f_1(\boldsymbol{z}) - \beta f_2(\boldsymbol{z}).$$

Note that (20) implies in particular that $\min_{\boldsymbol{z}} \mathcal{L}(\boldsymbol{z}, \bar{\alpha}, \bar{\beta}) > -\infty$ and thus by Lemma C.2, condition (20) is equivalent to condition (17).□

We now present necessary and sufficient optimality conditions for $(Q2P_{\mathbb{C}})$, which are a natural generalization of the optimality conditions (6) – (9) for the trust region subproblem. We only assume strict feasibility of $(Q2P_{\mathbb{C}})$ and $(D_{\mathbb{C}})$. Notice that for the complex version of the two trust region problem (TTRS), strict feasibility of $(D_{\mathbb{C}})$ is always satisfied since the norm constraint is strongly convex.

Theorem 2.5. *Suppose that both problems $(Q2P_{\mathbb{C}})$ and $(D_{\mathbb{C}})$ are strictly feasible. Then $\bar{\boldsymbol{z}}$ is an optimal solution of $(Q2P_{\mathbb{C}})$ if and only if there exists $\alpha, \beta \geq 0$ such that*

(i) $(\boldsymbol{A}_3 - \alpha \boldsymbol{A}_1 - \beta \boldsymbol{A}_2)\bar{\boldsymbol{z}} + \boldsymbol{b}_3 - \alpha \boldsymbol{b}_1 - \beta \boldsymbol{b}_2 = 0.$

(ii) $f_1(\bar{\boldsymbol{z}}), f_2(\bar{\boldsymbol{z}}) \geq 0.$

(iii) $\alpha f_1(\bar{\boldsymbol{z}}) = \beta f_2(\bar{\boldsymbol{z}}) = 0.$

(iv) $\boldsymbol{A}_3 - \alpha \boldsymbol{A}_1 - \beta \boldsymbol{A}_2 \succeq 0.$

Proof: The necessary part is trivial since $\bar{\boldsymbol{z}}$, $\bar{\alpha}$ and $\bar{\beta}$ of Theorem 2.4 satisfy conditions (i)-(iv). Suppose now that conditions (i)-(iv) are satisfied. Then by condition (ii), $\bar{\boldsymbol{z}}$ is feasible and therefore $f_3(\bar{\boldsymbol{z}}) \geq \operatorname{val}(Q2P_{\mathbb{C}})$. To prove the reverse inequality ($f_3(\bar{\boldsymbol{z}}) \leq \operatorname{val}(Q2P_{\mathbb{C}})$), consider the unconstrained minimization problem:

$$\min_{\boldsymbol{z} \in \mathbb{C}^n} \{f_3(\boldsymbol{z}) - \bar{\alpha} f_1(\boldsymbol{z}) - \bar{\beta} f_2(\boldsymbol{z})\}. \tag{21}$$

Moreover,

$$\begin{aligned} \operatorname{val}((21)) &\leq \min_{\boldsymbol{z} \in \mathbb{C}^n} \{f_3(\boldsymbol{z}) - \bar{\alpha} f_1(\boldsymbol{z}) - \bar{\beta} f_2(\boldsymbol{z}) : f_1(\boldsymbol{z}) \geq 0, f_2(\boldsymbol{z}) \geq 0\} \\ &\leq \min_{\boldsymbol{z} \in \mathbb{C}^n} \{f_3(\boldsymbol{z}) : f_1(\boldsymbol{z}) \geq 0, f_2(\boldsymbol{z}) \geq 0\} = \operatorname{val}(Q2P_{\mathbb{C}}) \end{aligned} \tag{22}$$

By Lemma C.3, conditions (i) and (iv) imply that $\bar{\boldsymbol{z}}$ is an optimal solution of (21) so that

$$f_3(\bar{\boldsymbol{z}}) - \bar{\alpha} f_1(\bar{\boldsymbol{z}}) - \bar{\beta} f_2(\bar{\boldsymbol{z}}) = \mathrm{val}((21)) \leq \mathrm{val}(Q2P_{\mathbb{C}}). \tag{23}$$

where the latter inequality follows from (22). By condition (iii) we have that $f_3(\bar{\boldsymbol{z}}) = f_3(\bar{\boldsymbol{z}}) - \bar{\alpha} f_1(\bar{\boldsymbol{z}}) - \bar{\beta} f_2(\bar{\boldsymbol{z}})$. Combining this with (23) we conclude that $f_3(\bar{\boldsymbol{z}}) \leq \mathrm{val}(Q2P_{\mathbb{C}})$. $\Box$

The results obtained in Section 2 are very similar to known results on quadratic optimization with a single quadratic constraint, which are summarized, for convenience, in Appendix A. Specifically, Theorems 2.2, 2.3, 2.4 and 2.5 are extensions of Theorems A.1, A.2, A.3 and A.4 respectively in the complex case. Note that the assumptions under which our theorems hold are equivalent to the conditions under which the corresponding theorems in the case of a single quadratic constraint hold. The essential difference is that our derivations are true only over the complex plane, while the results in Appendix A are true also when restricting attention to real variables.

3 Finding an Explicit Solution of $(Q2P_{\mathbb{C}})$

In this section we use the optimality conditions of Theorem 2.4 in order to describe a method for extracting the solution of problem $(Q2P_{\mathbb{C}})$ from the solution of the dual problem $(D_{\mathbb{C}})$. In Section 3.1 we show, given the optimal solution of $(D_{\mathbb{C}})$, how to reduce $(Q2P_{\mathbb{C}})$ into a quadratic feasibility problem. In Section 3.2 we develop a method for solving the quadratic feasibility problem and in Section 3.3 we give some numerical examples.

3.1 Reduction to a quadratic feasibility problem

Suppose that both $(Q2P_{\mathbb{C}})$ and $(D_{\mathbb{C}})$ are strictly feasible. From Theorem 2.4, $\bar{\boldsymbol{z}}$ is an optimal solution if it satisfies (17), (18) and (19). If $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2 \succ \boldsymbol{0}$, then the (unique) solution to the primal problem $(Q2P_{\mathbb{C}})$ is given by

$$\bar{\boldsymbol{z}} = -(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)^{-1}(\boldsymbol{b}_3 - \bar{\alpha}\boldsymbol{b}_1 - \bar{\beta}\boldsymbol{b}_2).$$

Next, suppose that $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$ is positive semidefinite but not positive definite. In this case (17) can be written as $\boldsymbol{z} = \boldsymbol{B}\boldsymbol{w} + \boldsymbol{a}$, where the columns of $\boldsymbol{B}$ form a basis for the null space of $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$ and $\boldsymbol{a} = -(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)^{\dagger}(\boldsymbol{b}_3 - \bar{\alpha}\boldsymbol{b}_1 - \bar{\beta}\boldsymbol{b}_2)$ is a solution of (17). It follows that $\bar{\boldsymbol{z}} = \boldsymbol{B}\bar{\boldsymbol{w}} + \boldsymbol{a}$ is an optimal solution to $(Q2P_{\mathbb{C}})$ if and only if conditions (18) and (19) of Theorem 2.4 are satisfied, *i.e.,*

$$g_1(\bar{\boldsymbol{w}}) \geq 0, g_2(\bar{\boldsymbol{w}}) \geq 0, \bar{\alpha} g_1(\bar{\boldsymbol{w}}) = 0, \bar{\beta} g_2(\bar{\boldsymbol{w}}) = 0, \quad (g_j(\boldsymbol{w}) \equiv f_j(\boldsymbol{B}\boldsymbol{w} + \boldsymbol{a})). \tag{24}$$

We are left with the problem of finding a vector which is a solution of a system of two quadratic equalities or inequalities as described in Table 1. This problem will be called the *Quadratic Feasibility Problem.*

No.	Case	Feasibility Problem
I	$\bar{\alpha} = 0, \bar{\beta} = 0$	$g_1(\boldsymbol{w}) \geq 0$ and $g_2(\boldsymbol{w}) \geq 0$
II	$\bar{\alpha} > 0, \bar{\beta} = 0$	$g_1(\boldsymbol{w}) = 0$ and $g_2(\boldsymbol{w}) \geq 0$
III	$\bar{\alpha} = 0, \bar{\beta} > 0$	$g_1(\boldsymbol{w}) \geq 0$ and $g_2(\boldsymbol{w}) = 0$
IV	$\bar{\alpha} > 0, \bar{\beta} > 0$	$g_1(\boldsymbol{w}) = 0$ and $g_2(\boldsymbol{w}) = 0$

Table 1: Cases of the quadratic feasibility problem.

Remark: Since $(\bar{\lambda}, \bar{\alpha}, \bar{\beta})$ is an optimal solution of the dual problem $(D_{\mathbb{C}})$, we must have $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2 \succeq \boldsymbol{0}$. Thus, the first case is possible only when $\boldsymbol{A}_3 \succeq \boldsymbol{0}$.

We summarize the above discussion in the following theorem:

Theorem 3.1. *Suppose that both problems* $(Q2P_{\mathbb{C}})$ *and* $(D_{\mathbb{C}})$ *are strictly feasible and let* $(\bar{\alpha}, \bar{\beta}, \bar{\lambda})$ *be an optimal solution of problem* $(D_{\mathbb{C}})$*. Then*

1. *if* $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2 \succ \boldsymbol{0}$*, then the (unique) optimal solution of* $(Q2P_{\mathbb{C}})$ *is given by*

$$\bar{\boldsymbol{z}} = -(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)^{-1}(\boldsymbol{b}_3 - \bar{\alpha}\boldsymbol{b}_1 - \bar{\beta}\boldsymbol{b}_2).$$

2. *if* $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2 \succeq \boldsymbol{0}$ *but not positive definite then the solutions of* $(Q2P_{\mathbb{C}})$ *are given by* $\boldsymbol{z} = \boldsymbol{B}\boldsymbol{w} + \boldsymbol{a}$*, where* $\boldsymbol{B} \in \mathbb{C}^{n \times d}$ *is a matrix whose columns form a basis for* $\mathcal{N}(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)$*,* $\boldsymbol{a}$ *is a solution of system (17), and* $\boldsymbol{w} \in \mathbb{C}^d$ *(d is the dimension of* $\mathcal{N}(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)$*) is any solution of the quadratic feasibility problem (24).*

3.2 Solving the quadratic feasibility problem

We now develop a method for solving all possible instances of the quadratic feasibility problem described in Table 1, under the condition that f_1 is a strongly concave quadratic function, *i.e.,* $\boldsymbol{A}_1 \prec \boldsymbol{0}$ (and thus the corresponding constraint is strongly convex)[4]. The strong concavity of $g_1(\boldsymbol{w}) = f_1(\boldsymbol{B}\boldsymbol{w} + \boldsymbol{a})$ follows immediately. All of these feasibility problems have at least one solution, a fact that follows from their construction. By applying an appropriate linear transformation on g_1, we can assume without loss of generality that $g_1(\boldsymbol{w}) = \gamma - \|\boldsymbol{w}\|^2$ $(\gamma \geq 0)$.

[4]Note that this assumption readily implies that problem $(D_{\mathbb{C}})$ is strictly feasible.

Our approach for solving the quadratic feasibility problem will be to use the solutions of optimization problems of one of the following types:

$$\min_{\boldsymbol{z}\in\mathbb{C}^n}\{\boldsymbol{z}^*\mathbf{Q}\boldsymbol{z}+2\Re(\boldsymbol{f}^*\boldsymbol{z}):\|\boldsymbol{z}\|^2\leq\gamma\}, \tag{25}$$

$$\min_{\boldsymbol{z}\in\mathbb{C}^n}\{\boldsymbol{z}^*\mathbf{Q}\boldsymbol{z}+2\Re(\boldsymbol{f}^*\boldsymbol{z}):\|\boldsymbol{z}\|^2=\gamma\}, \tag{26}$$

where $\mathbf{Q}=\mathbf{Q}^*\in\mathbb{C}^n$ and $\boldsymbol{f}\in\mathbb{C}^n$. Problem (25) is just the trust region subproblem. The fact that this is a complex valued problem does not change matters since it can be converted to a real valued trust region subproblem (4) (see the Introduction). Problem (26) can be solved similarly to (25); a complete description of the solution of (26) can be found in [12, p. 825].

We split our analysis according to the different cases:

Case I: In this case we need to solve the quadratic feasibility problem

$$g_2(\boldsymbol{w})\geq 0,\quad \|\boldsymbol{w}\|^2\leq\gamma. \tag{27}$$

Since the system (27) has at least one solution, it follows that the value of the optimization problem

$$\max\{g_2(\boldsymbol{w}):\|\boldsymbol{w}\|^2\leq\gamma\} \tag{28}$$

is nonnegative and therefore any optimal solution of (28) is a solution to (27).

Case II. Similarly to case I, we solve the optimization problem

$$\max\{g_2(\boldsymbol{w}):\|\boldsymbol{w}\|^2=\gamma\}. \tag{29}$$

Any optimal solution of (29) will be a solution to the feasibility problem

$$g_2(\boldsymbol{w})\geq 0,\quad \|\boldsymbol{w}\|^2=\gamma.$$

Case III. In this case, the feasibility problem takes on the following form:

$$g_2(\boldsymbol{w})=0,\quad \|\boldsymbol{w}\|^2\leq\gamma. \tag{30}$$

To find a solution to (30), we first solve the following optimization problems:

$$\min\{g_2(\boldsymbol{w}):\|\boldsymbol{w}\|^2\leq\gamma\}, \tag{31}$$

$$\max\{g_2(\boldsymbol{w}):\|\boldsymbol{w}\|^2\leq\gamma\}. \tag{32}$$

Let $\boldsymbol{w}^0$ be a solution to (31) and $\boldsymbol{w}^1$ a solution to (32). Then $\|\boldsymbol{w}^0\|^2\leq\gamma,\|\boldsymbol{w}^1\|^2\leq\gamma$. Since the system (30) has at least one solution, it follows that

$$g_2(\boldsymbol{w}^0) \leq 0 \leq g_2(\boldsymbol{w}^1). \tag{33}$$

To find $\bar{\boldsymbol{w}}$ that solves (30), consider the function $\varphi(\eta) = g_2(\boldsymbol{w}^0 + \eta(\boldsymbol{w}^1 - \boldsymbol{w}^0))$. By (33) we have

$$\varphi(0) \leq 0 \leq \varphi(1).$$

Since φ is a continuous function we conclude that there exists $\bar{\eta} \in [0, 1]$ such that

$$\varphi(\bar{\eta}) = 0. \tag{34}$$

Eq. (34) is a scalar quadratic equation. Let $\bar{\eta} \in [0, 1]$ be a solution of (34). Finally, $\bar{\boldsymbol{w}} = \boldsymbol{w}^0 + \bar{\eta}(\boldsymbol{w}^1 - \boldsymbol{w}^0)$ is a solution to (30) since (34) can be written as $g_2(\bar{\boldsymbol{w}}) = 0$ and the inequality $\|\bar{\boldsymbol{w}}\|^2 \leq \gamma$ holds true since $\bar{\boldsymbol{w}}$ is a convex combination of $\boldsymbol{w}^0$ and $\boldsymbol{w}^1$.

Case IV. The feasibility problem in this case is the following:

$$g_2(\boldsymbol{w}) = 0, \quad \|\boldsymbol{w}\|^2 = \gamma. \tag{35}$$

To find a solution to (35), we first solve the following optimization problems:

$$\min\{g_2(\boldsymbol{w}) : \|\boldsymbol{w}\|^2 = \gamma\}, \tag{36}$$

$$\max\{g_2(\boldsymbol{w}) : \|\boldsymbol{w}\|^2 = \gamma\}. \tag{37}$$

Let $\boldsymbol{w}^0$ and $\boldsymbol{w}^1$ be optimal solutions to (36) and (37) respectively. Then $\|\boldsymbol{w}^0\|^2 = \|\boldsymbol{w}^1\|^2 = \gamma$. Since system (35) must have at least one solution, we conclude that $g_2(\boldsymbol{w}^0) \leq 0 \leq g_2(\boldsymbol{w}^1)$. The case in which $\boldsymbol{w}^0$ and $\boldsymbol{w}^1$ are linearly dependent can be analyzed in the same way as Case III. If $\boldsymbol{w}^0$ and $\boldsymbol{w}_1$ are linearly independent we can define

$$\boldsymbol{u}(\eta) = \boldsymbol{w}^0 + \eta(\boldsymbol{w}^1 - \boldsymbol{w}^0), \quad \boldsymbol{w}(\eta) = \sqrt{\gamma}\frac{\boldsymbol{u}(\eta)}{\|\boldsymbol{u}(\eta)\|}, \quad \eta \in [0, 1].$$

By the definition of $\boldsymbol{w}(\eta)$, we have that $\|\boldsymbol{w}(\eta)\|^2 = \gamma$ for every $\eta \in [0, 1]$ and that $g_2(\boldsymbol{w}(0)) \leq 0 \leq g_2(\boldsymbol{w}(1))$. All that is left is to find a root to the scalar equation

$$g_2(\boldsymbol{w}(\eta)) = 0, \quad \eta \in [0, 1],$$

which can be written as

$$\gamma\frac{\boldsymbol{u}(\eta)^*\boldsymbol{A}_2\boldsymbol{u}(\eta)}{\|\boldsymbol{u}(\eta)\|^2} + \frac{2\sqrt{\gamma}}{\|\boldsymbol{u}(\eta)\|}\Re(\boldsymbol{b}_2^*\boldsymbol{u}(\eta)) + c_2 = 0, \quad \eta \in [0, 1]. \tag{38}$$

It is elementary to see that all the solutions of (38) also satisfy the following *quartic* scalar equation.

$$\left(\gamma \boldsymbol{u}(\eta)^* \boldsymbol{A}_2 \boldsymbol{u}(\eta) + c_2 \|\boldsymbol{u}(\eta)\|^2\right)^2 = 4\gamma \|\boldsymbol{u}(\eta)\|^2 (\Re(\boldsymbol{b}_2^* \boldsymbol{u}(\eta)))^2. \tag{39}$$

The solution of the quadratic feasibility problem is given by $w(\eta)$, where η is one of the solutions (39). Notice that (39) has at most four solutions, which have explicit algebraic expressions.

3.3 Numerical examples

In this section we demonstrate the previous results through some numerical examples. All the SDP problems in this section were solved using SeDuMi [23].

3.3.1 Example 1

Consider the quadratic optimization problem

$$\min\{\boldsymbol{z}^* \boldsymbol{A}_3 \boldsymbol{z} + 2\Re(\boldsymbol{b}_3^* \boldsymbol{z}) + c_3 : \|\boldsymbol{z}\|^2 \leq 4, \|\boldsymbol{z} - \boldsymbol{e}\|^2 \leq 4\},$$

where

$$n = 4, \boldsymbol{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{A}_3 = \begin{pmatrix} 6 & 1+i & 1+i & 4+2i \\ 1-i & 2 & 1+2i & 1-3i \\ 1-i & 1-2i & 6 & 3-3i \\ 4-2i & 1+3i & 3+3i & 0 \end{pmatrix}, \boldsymbol{b}_3 = \begin{pmatrix} 3+i \\ 2+2i \\ 1 \\ i \end{pmatrix}, c_3 = 4.$$

The minimum eigenvalue of $\boldsymbol{A}_3$ is equal to -4.278 and therefore the objective function is not convex. Since obviously all the conditions of Theorem 2.3 are satisfied, the strong duality result holds and we can solve this problem via the dual problem $(D_{\mathbb{C}})$ by using Theorem 2.4. The solution to $(D_{\mathbb{C}})$ is given by

$$\bar{\lambda} = -10.5424, \quad \bar{\alpha} = 0.7399, \quad \bar{\beta} = 5.804.$$

Note that $\bar{\lambda} = -10.5424$ is the optimal value and the fact that both $\bar{\alpha}$ and $\bar{\beta}$ are positive implies that both constraints are active. In this example we have $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2 \succ \boldsymbol{0}$ and thus by Theorem 3.1 the optimal solution is given by

$$\bar{\boldsymbol{z}} = -(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)^{-1}(\boldsymbol{b}_3 - \bar{\alpha}\boldsymbol{b}_1 - \bar{\beta}\boldsymbol{b}_2) = \begin{pmatrix} -0.3660 - 0.1046i \\ 0.8336 + 0.1578i \\ 0.2740 + 0.6465i \\ 1.2583 - 1.0290i \end{pmatrix}.$$

The next example was especially "tailored" so that $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$ is not positive definite but rather just positive semidefinite.

3.3.2 Example 2

Consider problem $(Q2P_{\mathbb{C}})$ with

$$n=3, \boldsymbol{A}_3 = \begin{pmatrix} 3 & 1 & -3i \\ 1 & 6 & 3-2i \\ 3i & 3+2i & 1 \end{pmatrix}, \boldsymbol{b}_3 = \begin{pmatrix} -4.9184+4.3374i \\ -7.9923+2.8916i \\ -5.5549-3.3531i \end{pmatrix}, c_3 = 0, \boldsymbol{A}_1 = -\boldsymbol{I}_3, \boldsymbol{b}_1 = \boldsymbol{0}, c_1 = 3$$

$$\boldsymbol{A}_2 = \begin{pmatrix} 4 & 6+i & 4-3i \\ 6-i & 4 & 3 \\ 4+3i & 3 & 6 \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} 2+i \\ 1+i \\ 2 \end{pmatrix}, c_2 = -39.$$

This is obviously not a convex optimization problem since $\boldsymbol{A}_3$ is not positive semidefinite and $\boldsymbol{A}_2$ is not negative semidefinite. Since all the conditions of Theorem 2.3 are satisfied, strong duality holds and we can use Theorem 3.1 in order to extract the solution of $(Q2P_{\mathbb{C}})$ from the optimal solution of $(D_{\mathbb{C}})$. The solution of the dual problem $(D_{\mathbb{C}})$ is given by

$$\bar{\lambda} = -21.5416, \quad \bar{\alpha} = 2.1374, \quad \bar{\beta} = 0.$$

The eigenvalues of the matrix $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$ are 0,5.5819 and 10.8303. Thus, $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$ is not invertible and in order to find an optimal solution we will transform the problem into a quadratic feasibility problem. The null space of $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$ is spanned by the vector $\boldsymbol{u} = (0.0604 + 0.4512i; -0.3103 + 0.1465i; 0.8216)$ and a solution to the system (17) is given by $\boldsymbol{a} = (0.6305 - 0.4769i; 0.6226 - 0.2963i; 0.5303 + 0.3523i)$. Thus, the set of solutions to (17) is given by $\{\boldsymbol{z} = \boldsymbol{a} + w\boldsymbol{u} : w \in \mathbb{C}\}$. Since in this example $\bar{\alpha} > 0$ and $\bar{\beta} = 0$, we need to solve the quadratic feasibility problem

$$\begin{aligned} f_1(\boldsymbol{a} + w\boldsymbol{u}) &= 0, \\ f_2(\boldsymbol{a} + w\boldsymbol{u}) &\geq 0, \end{aligned}$$

which is the same as

$$\begin{aligned} 0 &= -x^2 - y^2 - 0.0439x + 0.0463y + 1.4943, \\ 0 &\leq 2.2634x^2 + 2.2634y^2 + 11.4971x - 9.7040y - 16.1033, \end{aligned}$$

where $x = \Re(w)$ and $y = \Im(w)$. The solution to the later feasibility problem is the optimal solution of the minimization problem

$$\begin{array}{ll} \max\limits_{x,y} & 2.2634x^2 + 2.2634y^2 + 11.4971x - 9.7040y - 16.1033 \\ \text{s.t.} & x^2 + y^2 + 0.0439x - 0.0463y - 1.4943 = 0 \end{array}$$

whose solution is given by $(x, y) = (0.9127, -0.7640)$. Hence $w = 0.9127 - 0.764i$ and the solution to the primal problem $(Q2P_{\mathbb{C}})$ is given by $\boldsymbol{z} = \boldsymbol{a} + w\boldsymbol{u} = (1.0304 - 0.1112i; 0.4513 + 0.0745i; 1.2802 - 0.2754i)$.

4 The Real Case

In this section we consider the real valued problem $(Q2P_{\mathbb{R}})$ in which the data and variables are assumed to be real valued. The dual problem to $(Q2P_{\mathbb{R}})$ is given by

$$(D_{\mathbb{R}}) \quad \max_{\alpha \geq 0, \beta \geq 0, \lambda} \left\{ \lambda \;\middle|\; \begin{pmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3 \\ \boldsymbol{b}_3^T & c_3 - \lambda \end{pmatrix} \succeq \alpha \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1^T & c_1 \end{pmatrix} + \beta \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{b}_2 \\ \boldsymbol{b}_2^T & c_2 \end{pmatrix} \right\}. \qquad (40)$$

Note that this is *exactly the same* as problem $(D_{\mathbb{C}})$ (problem (10)), where here we use fact that the data is real and therefore $\boldsymbol{b}_j^* = \boldsymbol{b}_j^T$. The semidefinite relaxation in this case is given by

$$(SDR_{\mathbb{R}}) \quad \min_{\boldsymbol{X}} \{\mathrm{Tr}(\boldsymbol{X}\boldsymbol{M}_3) : \mathrm{Tr}(\boldsymbol{X}\boldsymbol{M}_1) \geq 0, \mathrm{Tr}(\boldsymbol{X}\boldsymbol{M}_2) \geq 0, X_{n+1,n+1} = 1, \boldsymbol{X} \in \mathcal{S}_{n+1}^+\}. \qquad (41)$$

In contrast to the complex case, strong duality results are, generally speaking, not true for $(Q2P_{\mathbb{R}})$. It is not known weather $(Q2P_{\mathbb{R}})$ is a tractable problem or not and in that respect, if there is an efficient algorithm for finding its solution. In this section we use the results obtained for $(Q2P_{\mathbb{C}})$ in order to establish several results on the real valued problem $(Q2P_{\mathbb{R}})$. In Section 4.1 we show that if the constraints of $(Q2P_{\mathbb{R}})$ are convex then the complex valued problem $(Q2P_{\mathbb{C}})$, considered as a relaxation of $(Q2P_{\mathbb{R}})$, can produce an approximate solution. A result on the tightness of the approximation is given. In Section 4.2 we derive a result on the image of the real and complex space under a quadratic mapping. This result will enable us to bridge between the real and complex case. Using the latter result, a sufficient condition for zero duality gap is proved in Section 4.3. The sufficient condition is expressed via the optimal dual variables and thus can be verified in polynomial time. Preliminary numerical results show that for the TTRS problem (3) this condition hardly never fails. Using the condition, we identify a family of problems with zero duality gap.

4.1 The complex relaxation

As already mentioned, val$(Q2P_{\mathbb{R}})$ is not necessarily equal to val$(D_{\mathbb{R}})$. However, the *complex counterpart* $(Q2P_{\mathbb{C}})$ does satisfy val$(Q2P_{\mathbb{C}})$ = val$(D_{\mathbb{R}})$ and we can always find a *complex valued* solution to $(Q2P_{\mathbb{C}})$ that attains the bound val$(D_{\mathbb{R}})$. Therefore, we can consider $(Q2P_{\mathbb{C}})$ as a tractable relaxation (*the complex relaxation*) of the real valued problem $(Q2P_{\mathbb{R}})$. The following example, whose data is taken from Yuan [26, p.59], illustrates this fact:

Example: Consider the following real valued quadratic optimization problem

$$\min_{x_1,x_2\in\mathbb{R}}\{-2x_1^2+2x_2^2+4x_1 : x_1^2+x_2^2-4\leq 0, x_1^2+x_2^2-4x_1+3\leq 0\}, \tag{42}$$

which is a special case of $(Q2P_{\mathbb{R}})$ with:

$$\boldsymbol{A}_1=\boldsymbol{A}_2=-\boldsymbol{I}, \boldsymbol{A}_3=\begin{pmatrix}-2 & 0\\ 0 & 2\end{pmatrix}, \boldsymbol{b}_1=\boldsymbol{0}, \boldsymbol{b}_2=(2;0), \boldsymbol{b}_3=(2;0), c_1=4, c_2=-3, c_3=0.$$

The solution to the dual problem is given by $\bar{\alpha}=1, \bar{\beta}=1$ and $\bar{\lambda}=-1$. It is easy to see that the optimal solution to $(Q2P_{\mathbb{R}})$ is given by $x_1=2, x_2=0$ and its corresponding optimal solution is 0. The duality gap is thus 1. By the strong duality result of Theorem 2.2, We can find a complex valued solution with value equal to the value of the dual problem -1 to the complex counterpart:

$$\min_{z_1,z_2\in\mathbb{C}}\{-2|z_1|^2+2|z_2|^2+4\Re(z_1) : |z_1|^2+|z_2|^2-4\leq 0, |z_1|^2+|z_2|^2-4\Re(z_1)+3\leq 0\}. \tag{43}$$

Using the techniques described in Section 3 we obtain that the solution of problem (43) is $z_1=7/4+\sqrt{15/16}i, z_2=0$ with function value -1. □

The following Theorem states that if the constraints of $(Q2P_{\mathbb{C}})$ are convex (as in the two trust region problem), then we can extract an *approximate real solution* from the optimal complex solution $\bar{\boldsymbol{z}}$ by just taking $\bar{\boldsymbol{x}}=\Re(\bar{\boldsymbol{z}})$. In the following theorem we show that the approximate real solution is a feasible solution of $(Q2P_{\mathbb{R}})$ and we bound the distance of its function value from the optimal function value in terms of the imaginary part of the complex valued solution $\bar{\boldsymbol{z}}$.

Theorem 4.1. *Suppose that both $(Q2P_{\mathbb{R}})$ and $(D_{\mathbb{R}})$ are strictly feasible. Let $\boldsymbol{A}_1,\boldsymbol{A}_2\in\mathbb{R}^{n\times n}$ be negative definite matrices, $\boldsymbol{A}_3=\boldsymbol{A}_3^T\in\mathbb{R}^{n\times n}$, $\boldsymbol{b}_j\in\mathbb{R}^n$ and $c_j\in\mathbb{R}$. Let $\bar{\boldsymbol{z}}$ be an optimal complex valued solution of $(Q2P_{\mathbb{C}})$ and let $\bar{\boldsymbol{x}}=\Re(\bar{\boldsymbol{z}})$. Then $\bar{\boldsymbol{x}}$ is a feasible solution of $(Q2P_{\mathbb{R}})$ and*

$$f_3(\bar{\boldsymbol{x}})-val(Q2P_{\mathbb{R}})\leq\Im(\bar{\boldsymbol{z}})^T\boldsymbol{A}_3\Im(\bar{\boldsymbol{z}}).$$

Proof: To show that $\bar{\boldsymbol{x}}$ is a feasible solution of $(Q2P_{\mathbb{R}})$ note that for $\boldsymbol{z} \in \mathbb{C}^n, j = 1, 2$ one has

$$\begin{array}{rcl} 0 & \leq & f_j(\boldsymbol{z}) = \boldsymbol{z}^* \boldsymbol{A}_j \boldsymbol{z} + 2\Re(\boldsymbol{b}_j^* \boldsymbol{z}) + c_j \\ & = & \Re(\boldsymbol{z})^T \boldsymbol{A}_j \Re(\boldsymbol{z}) + \Im(\boldsymbol{z})^T \boldsymbol{A}_j \Im(\boldsymbol{z}) + 2\boldsymbol{b}_j^T \Re(\boldsymbol{z}) + c_j \\ & \overset{\boldsymbol{A}_j \prec \boldsymbol{0}}{\leq} & \Re(\boldsymbol{z})^T \boldsymbol{A}_j \Re(\boldsymbol{z}) + 2\boldsymbol{b}_j^T \Re(\boldsymbol{z}) + c_j = f_j(\Re(\boldsymbol{z})). \end{array}$$

Thus, since $\bar{\boldsymbol{z}}$ is feasible, so is $\Re(\bar{\boldsymbol{z}})$. Finally,

$$f_3(\Re(\bar{\boldsymbol{z}})) - \mathrm{val}(Q2P_{\mathbb{R}}) \leq f_3(\Re(\bar{\boldsymbol{z}})) - \mathrm{val}(Q2P_{\mathbb{C}}) = f_3(\Re(\bar{\boldsymbol{z}})) - f_3(\bar{\boldsymbol{z}}) = \Im(\bar{\boldsymbol{z}})^T \boldsymbol{A}_3 \Im(\bar{\boldsymbol{z}}).$$

□

In our example the approximate solution is $(7/4, 0)$ and its function value is equal to 0.875.

4.2 The Image of the Complex and Real Space under a Quadratic Mapping

We now prove a result (Theorem 4.3) on the image of the spaces $\mathbb{C}^n$ and $\mathbb{R}^n$ under a quadratic mapping, which is composed from two nonhomogeneous quadratic functions. This result will be one of the key ingredients in proving the sufficient condition in Section 4.3. Results on the image of quadratic mappings play an important role in nonconvex quadratic optimization (see *e.g.,* [13, 17] and references therein). One important result, which is very relevant to our analysis, is due to Polyak [17, Theorem 2.2]:

Theorem 4.2 ([17]). *Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n \times n}, (n \geq 2)$ be symmetric matrices for which the following condition is satisfied:*

$$\exists \alpha, \beta \in \mathbb{R} \text{ such that } \alpha \boldsymbol{A}_1 + \beta \boldsymbol{A}_2 \succ \boldsymbol{0}. \tag{44}$$

Let $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$ and define $f_j(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A}_j \boldsymbol{x} + 2\Re(\boldsymbol{b}_j^T \boldsymbol{x}) + c_j$. Then the set

$$W = \{(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^n\}$$

is closed and convex.

The main result in this section is Theorem 4.3 below that states that the image of $\mathbb{C}^n$ and $\mathbb{R}^n$ under the quadratic mapping defined in Theorem 4.2 are the same. Later on, in Section 4.3, this theorem will enable us to make the connection between the complex and real cases.

Theorem 4.3. *Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n \times n}, (n \geq 2)$ be symmetric matrices for which the following condition is satisfied:*

$$\exists \alpha, \beta \in \mathbb{R} \text{ such that } \alpha \boldsymbol{A}_1 + \beta \boldsymbol{A}_2 \succ \boldsymbol{0}. \tag{45}$$

Let $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$ and define $f_j(\boldsymbol{z}) = \boldsymbol{z}^* \boldsymbol{A}_j \boldsymbol{z} + 2\Re(\boldsymbol{b}_j^* \boldsymbol{z}) + c_j$. Then the sets

$$F = \{(f_1(\boldsymbol{z}), f_2(\boldsymbol{z})) : \boldsymbol{z} \in \mathbb{C}^n\}, \quad W = \{(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^n\}$$

are equal.

The proof of Theorem 4.3 relies on the following Lemma:

Lemma 4.1. *Let $\boldsymbol{A}$ be a real $n \times n$ symmetric matrix, $\boldsymbol{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Consider the following real and complex minimization problems:*

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + 2\boldsymbol{b}^T \boldsymbol{x} + c : \|\boldsymbol{x}\|^2 = \beta\}, \tag{46}$$

$$\min_{\boldsymbol{z} \in \mathbb{C}^n} \{\boldsymbol{z}^* \boldsymbol{A} \boldsymbol{z} + 2\Re(\boldsymbol{b}^* \boldsymbol{z}) + c : \|\boldsymbol{z}\|^2 = \beta\}. \tag{47}$$

Then val((46)) = val((47)).

Proof: The values of problems (46) and (47) are equal to

$$\max_{\mu} \{\mu : \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + 2\boldsymbol{b}^T \boldsymbol{x} + c \geq \mu \text{ for every } \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \|\boldsymbol{x}\|^2 = \beta\} \tag{48}$$

and

$$\max_{\mu} \{\mu : \boldsymbol{z}^* \boldsymbol{A} \boldsymbol{z} + 2\Re(\boldsymbol{b}^* \boldsymbol{z}) + c \geq \mu \text{ for every } \boldsymbol{z} \in \mathbb{C}^n \text{ such that } \|\boldsymbol{z}\|^2 = \beta\} \tag{49}$$

respectively. By Theorem B.3, the value of both problems is equal to the value of the optimization problem

$$\begin{array}{ll} \max_{\mu,\lambda} & \mu \\ \text{s.t.} & \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^T & c - \mu \end{pmatrix} \succeq \lambda \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\beta \end{pmatrix}. \end{array}$$

Thus, val((46))=val((47)). □

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3: First notice that by Theorem 4.2 we have that both W and F are convex. Obviously $W \subseteq F$. To prove the opposite, we first assume without loss of generality that $f_1(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$. The latter assumption is possible since condition (45) is satisfied. Suppose that $(a, b) \in F$, *i.e.*, $a = \|\boldsymbol{z}\|^2, b = f_2(\boldsymbol{z})$ for some $\boldsymbol{z} \in \mathbb{C}^n$. Let

$$b_{min} = \min\{f_2(\boldsymbol{z}) : \|\boldsymbol{z}\|^2 = a\} \text{ and } b_{max} = \max\{f_2(\boldsymbol{z}) : \|\boldsymbol{z}\|^2 = a\}.$$

By Theorem 4.1, there must be two *real* vectors $\boldsymbol{x}^0, \boldsymbol{x}^1 \in \mathbb{R}^n$ such that $\|\boldsymbol{x}^0\|^2 = \|\boldsymbol{x}^1\|^2 = a$ and $f_2(\boldsymbol{x}^0) = b_{min} \leq b \leq b_{max} = f_2(\boldsymbol{x}^1)$. Therefore, $(a, b_{min}), (a, b_{max}) \in W$. Since W is convex we conclude that (a, b), being a convex combination of (a, b_{min}) and (a, b_{max}), also belongs to W. □

4.3 A sufficient condition for zero duality gap of $(Q2P_{\mathbb{R}})$

We now use the results on the complex valued problem $(Q2P_{\mathbb{C}})$ in order to find a sufficient condition for zero duality gap and tightness of the semidefinite relaxation of the real valued problem $(Q2P_{\mathbb{R}})$.

4.3.1 The Condition

We first develop a sufficient condition for zero duality gap in problem $(Q2P_{\mathbb{R}})$. Our derivation is based on the fact that if an optimal solution of the complex valued problem $(Q2P_{\mathbb{C}})$ is *real valued* then the real valued problem $(Q2P_{\mathbb{R}})$ admits no gap with its dual problem $(D_{\mathbb{R}})$.

Theorem 4.4. *Suppose that both problems $(Q2P_{\mathbb{R}})$ and $(D_{\mathbb{R}})$ are strictly feasible and that the following condition is satisfied:*

$$\exists \hat{\alpha}, \hat{\beta} \in \mathbb{R} \text{ such that } \hat{\alpha}\boldsymbol{A}_1 + \hat{\beta}\boldsymbol{A}_2 \succ \boldsymbol{0}. \tag{50}$$

Let $(\bar{\lambda}, \bar{\alpha}, \bar{\beta})$ be an optimal solution of the dual problem $(D_{\mathbb{R}})$. If

$$\dim\left(\mathcal{N}(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)\right) \neq 1 \tag{51}$$

then $val(Q2P_{\mathbb{R}}) = val(D_{\mathbb{R}}) = val(SDR_{\mathbb{R}})$ and there exists a real valued solution to the complex valued problem $(Q2P_{\mathbb{C}})$.

Proof: First, the fact that $\text{val}(D_{\mathbb{R}}) = \text{val}(SDR_{\mathbb{R}})$ follows from the fact that both $(SDR_{\mathbb{R}})$ and $(D_{\mathbb{R}})$ are strictly feasible. Now, suppose that the dimension of $\mathcal{N}(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)$ is 0. Then by (17), a solution to $(Q2P_{\mathbb{C}})$ is given by the real valued vector

$$\bar{\boldsymbol{x}} = -(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)^{-1}(\boldsymbol{b}_3 - \bar{\alpha}\boldsymbol{b}_1 - \bar{\beta}\boldsymbol{b}_2).$$

Since $(Q2P_{\mathbb{C}})$ has a real valued solution it follows that

$$\text{val}(Q2P_{\mathbb{R}}) = \text{val}(Q2P_{\mathbb{C}}) = \text{val}(D_{\mathbb{R}}),$$

where the last equality follows from Theorem 2.2.

Next, suppose that the dimension of $\mathcal{N}(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)$ is greater or equal to 2. By Theorem 3.1, any optimal solution $\bar{\boldsymbol{z}}$ of $(Q2P_{\mathbb{C}})$ is given by $\bar{\boldsymbol{z}} = \boldsymbol{B}\bar{\boldsymbol{w}} + \boldsymbol{a}$, where $\bar{\boldsymbol{w}} \in \mathbb{C}^d$ is a solution of the system

$$g_1(\boldsymbol{w}) \geq 0, g_2(\boldsymbol{w}) \geq 0, \bar{\alpha}g_1(\boldsymbol{w}) = 0, \bar{\beta}g_2(\boldsymbol{w}) = 0, \quad (g_j(\boldsymbol{w}) \equiv f_j(\boldsymbol{B}\boldsymbol{w} + \boldsymbol{a})). \tag{52}$$

Here $d = dim(\mathcal{N}(\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2)) \geq 2$. Now, obviously $(g_1(\bar{\boldsymbol{w}}), g_2(\bar{\boldsymbol{w}})) \in S_1$ where

$$S_1 = \{(g_1(\boldsymbol{w}), g_2(\boldsymbol{w})) : \boldsymbol{w} \in \mathbb{C}^d\}.$$

The matrix $\boldsymbol{B}$ has full column rank since its columns are linearly independent. Hence, (50) implies that

$$\hat{\alpha}\boldsymbol{B}^T\boldsymbol{A}_1\boldsymbol{B} + \hat{\beta}\boldsymbol{B}^T\boldsymbol{A}_2\boldsymbol{B} \succ \boldsymbol{0}. \tag{53}$$

The LMI (53) together with the fact that $d \geq 2$ implies that the conditions of Theorem 4.3 are satisfied and thus $S_1 = S_2$, where

$$S_2 = \{(g_1(\boldsymbol{x}), g_2(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^d\}.$$

Therefore, there exists $\bar{\boldsymbol{x}} \in \mathbb{R}^d$ that satisfies $g_j(\bar{\boldsymbol{w}}) = g_j(\bar{\boldsymbol{x}})$ and as a result, (52) has a real valued solution. To conclude, $\bar{\boldsymbol{z}} = \boldsymbol{B}\bar{\boldsymbol{x}} + \boldsymbol{a} \in \mathbb{R}^n$ is a real valued vector which is an optimal solution to $(Q2P_{\mathbb{C}})$. □

4.3.2 Numerical Experiments

To demonstrate the fact that for the TTRS problem (3), the sufficient condition of Theorem 4.4 hardly never fails for random problems, we considered different values of m and n (the number of constraints and the number of variables in the original nonlinear problem) and randomly generated 1000 instances of $\boldsymbol{B}, \boldsymbol{g}, \boldsymbol{A}$ and $\boldsymbol{c}$. We chose $\Delta = 0.1$ and

$$\xi = \|\boldsymbol{A}^T(-\alpha\boldsymbol{A}\boldsymbol{c}) + \boldsymbol{c}\|,$$

where

$$\alpha = \min\left\{\frac{\Delta}{\|\boldsymbol{A}\boldsymbol{c}}, \frac{\boldsymbol{c}^T(\boldsymbol{A}^T\boldsymbol{A})\boldsymbol{c}}{\boldsymbol{c}^T(\boldsymbol{A}^T\boldsymbol{A})^2\boldsymbol{c}}\right\},$$

as was suggested in the trust region algorithm of Celis, Dennis and Tapia [6]. The results are given in Table 2.

n	m	distribution	N_{suf}	mean	sd
10	1	Normal	997	5.50	2.34
10	1	Uniform	1000	1.61	0.62
10	10	Normal	1000	5.04	2.31
10	10	Uniform	1000	1.60	0.61
100	1	Normal	1000	13.15	2.65
100	1	Uniform	1000	3.75	0.64
100	100	Normal	1000	12.54	2.31
100	100	Uniform	1000	3.71	0.65

Table 2: Results for TTRS.

In the table, *distribution* is the distribution from which the coefficients of $\boldsymbol{B}, \boldsymbol{g}, \boldsymbol{A}$ and $\boldsymbol{c}$ are generated. There are two possibilities: uniform distribution ($U[0,1]$) or standard normal distribution ($N(0,1)$). N_{suf} is the number of problems satisfying the sufficient condition (51) out of 1000. *mean* and *sd* are the mean and standard deviation of the minimal eigenvalue of the matrix $\boldsymbol{A}_3 - \bar{\alpha}\boldsymbol{A}_1 - \bar{\beta}\boldsymbol{A}_2$. It is interesting to note that *almost all* the instances satisfied condition (51) except for only 3 instances in the case $n = 10, m = 1$ with data generated from the normal distribution. Of course, these experiments reflect the situation in random problems and the results might be different (for better or for worse) if the data will be generated differently.

4.3.3 A family of problems with zero duality gap

Consider the problem of minimizing an indefinite quadratic function subject to a norm constraint and a linear inequality constraint:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \{\boldsymbol{x}^T\mathbf{Q}\boldsymbol{x} + 2\boldsymbol{b}^T\boldsymbol{x} : \|\boldsymbol{x}\|^2 \leq \delta, \boldsymbol{a}^T\boldsymbol{x} \leq \xi\}. \tag{54}$$

This problem was considered in [24, 25], where it was shown that the semidefinite relaxation is not always tight, although a polynomial-time algorithm for solving this problem was presented. We will find a condition on the data $(\mathbf{Q}, \boldsymbol{a}, \boldsymbol{b})$ that will be sufficient for zero duality gap.

Theorem 4.5. *Suppose that problem (54) is strictly feasible and that $n \geq 2$. If the dimension of $\mathcal{N}(\mathbf{Q} - \lambda_{min}(\mathbf{Q})\boldsymbol{I})$ is at least 2, then strong duality holds for problem (54).*

Proof: Let $(\bar{\lambda}, \bar{\alpha}, \bar{\beta})$ be an optimal solution of the dual problem to (54). From the feasibility of the dual problem it follows that $\mathbf{Q} - \bar{\alpha}\boldsymbol{I} \succeq \mathbf{0}$. Now, either $\bar{\alpha} > \lambda_{min}(\mathbf{Q})$ and in that case $\mathbf{Q} - \bar{\alpha}\boldsymbol{I}$ is nonsingular and thus the dimension of $\mathcal{N}(\mathbf{Q} - \bar{\alpha}\boldsymbol{I})$ is 0 or $\bar{\alpha} = \lambda_{min}(\mathbf{Q})$ and in this case $\mathcal{N}(\mathbf{Q} - \bar{\alpha}\boldsymbol{I})$ is of dimension at least 2 by the assumptions. The result follows now from Theorem 4.4. □

A Quadratic Optimization With a Single Quadratic Constraint

For the sake of completeness we state here some known results [5, 10, 22] on the problem of minimizing an indefinite quadratic function subject to a *single* quadratic inequality constraint. The results can also be derived by using the techniques of Section 2. In our presentation we consider the optimization problem over the complex domain; note, however, that the results are also true for the real counterpart.

Consider the problem

$$\min_{\boldsymbol{z}\in\mathbb{C}^n} \{f_2(\boldsymbol{z}) : f_1(\boldsymbol{z}) \geq 0\}, \tag{55}$$

where $f_j(\boldsymbol{z}) = \boldsymbol{z}^*\boldsymbol{A}_j\boldsymbol{z} + 2\Re(\boldsymbol{b}_j^*\boldsymbol{z}) + c_j, \boldsymbol{A}_j = \boldsymbol{A}_j^* \in \mathbb{C}^{n\times n}, \boldsymbol{b}_j \in \mathbb{C}^n, c_j \in \mathbb{R}, j = 1, 2$. The dual problem to (55) is given by

$$\max_{\alpha\geq 0,\lambda}\left\{\lambda:\begin{pmatrix}\boldsymbol{A}_2 & \boldsymbol{b}_2\\ \boldsymbol{b}_2^* & c_2-\lambda\end{pmatrix}\succeq\alpha\begin{pmatrix}\boldsymbol{A}_1 & \boldsymbol{b}_1\\ \boldsymbol{b}_1^* & c_1\end{pmatrix}\right\} \tag{56}$$

and the SDR of (55) is

$$\min_{\boldsymbol{Z}}\{\mathrm{Tr}(\boldsymbol{M}_2\boldsymbol{Z}):\mathrm{Tr}(\boldsymbol{M}_1\boldsymbol{Z})\geq 0, \boldsymbol{Z}_{n+1,n+1}=1, \boldsymbol{Z}\in\mathcal{H}_{n+1}^+\}, \tag{57}$$

where $\boldsymbol{M}_j=\begin{pmatrix}\boldsymbol{A}_j & \boldsymbol{b}_j\\ \boldsymbol{b}_j^* & c_j\end{pmatrix}$.

Theorem A.1. *Suppose that problem* (55) *is strictly feasible,* i.e., *there exists* $\tilde{\boldsymbol{z}}\in\mathbb{C}^n$ *such that* $f_1(\tilde{\boldsymbol{z}})>0$. *Then,*

1. *if val*((55)) *is finite then the maximum of problem (56)) is attained and val*(55) = *val*(56)*.*
2. *val*(55) $=-\infty$ *if and only if (56) is not feasible.*

Theorem A.2. *Suppose that both problems* (55) *and (56) are strictly feasible and let* $(\bar{\lambda},\bar{\alpha},\bar{\beta})$ *be an optimal solution of* (56). *Then problems* (55),(56) *and* (57) *attain their solutions and*

$$val(55)=val(56)=val(57).$$

Theorem A.3. *Suppose that both problems* (55) *and* (56) *are strictly feasible, and let* $(\bar{\lambda},\bar{\alpha})$ *be an optimal solution of* (56). *Then* $\bar{\boldsymbol{z}}$ *is an optimal solution of* (55) *if and only if*

$$\begin{aligned}&(\boldsymbol{A}_2-\bar{\alpha}\boldsymbol{A}_1)\bar{\boldsymbol{z}}+\boldsymbol{b}_2-\bar{\alpha}\boldsymbol{b}_1=\boldsymbol{0},\\ &f_1(\bar{\boldsymbol{z}})\geq 0,\\ &\bar{\alpha}f_1(\bar{\boldsymbol{z}})=0.\end{aligned}$$

Theorem A.4. *Suppose that both problems* (55) *and* (56) *are strictly feasible. Then* $\bar{\boldsymbol{z}}$ *is an optimal solution of* (55) *if and only if there exists* $\alpha\geq 0$ *such that*

(i) $(\boldsymbol{A}_2-\alpha\boldsymbol{A}_1)\bar{\boldsymbol{z}}+\boldsymbol{b}_2-\alpha\boldsymbol{b}_1=0.$

(ii) $f_1(\bar{\boldsymbol{z}})\geq 0.$

(iii) $\alpha f_1(\bar{\boldsymbol{z}})=0.$

(iv) $\boldsymbol{A}_2-\alpha\boldsymbol{A}_1\succeq 0.$

B Technical Results

B.1 Strict feasibility of the semidefinite relaxation

Lemma B.1 below states that if the primal problem $(Q2P_{\mathbb{C}})$ is strictly feasible then its semidefinite relaxation must also be strictly feasible.

Lemma B.1. *Suppose that $(Q2P_{\mathbb{C}})$ is strictly feasible. Then the SDR problem (13) is strictly feasible as well.*

Proof: Since $(Q2P_{\mathbb{C}})$ is strictly feasible, there exists $\boldsymbol{v} \in \mathbb{C}^n$ such that $f_1(\boldsymbol{v}) > 0$ and $f_2(\boldsymbol{v}) > 0$. Therefore, $\text{Tr}(\boldsymbol{M}_1\boldsymbol{V}) > 0$ and $\text{Tr}(\boldsymbol{M}_2\boldsymbol{V}) > 0$, where $\boldsymbol{V} = \tilde{\boldsymbol{v}}\tilde{\boldsymbol{v}}^*, \quad \tilde{\boldsymbol{v}} = (\boldsymbol{v}; 1)$. Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n \in \mathbb{C}^{n+1}$ be n vectors with $(n+1)$th component equal to 0 such that the set $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \tilde{\boldsymbol{v}}\}$ is linearly independent. Consider the matrix

$$\boldsymbol{W} = \boldsymbol{V} + \sum_{i=1}^{n} \alpha_j \boldsymbol{w}_j \boldsymbol{w}_j^*,$$

where α_j are positive numbers. $W_{n+1,n+1} = 1$ since the last component of $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ is equal to zero. Moreover, for small enough α_j we have $\text{Tr}(\boldsymbol{M}_1\boldsymbol{W}) > 0$ and $\text{Tr}(\boldsymbol{M}_2\boldsymbol{W}) > 0$. From its definition, $\boldsymbol{W}$ is positive semidefinite. To prove that $\boldsymbol{W}$ is *positive definite* all that remains to show is that $\boldsymbol{a}^*\boldsymbol{W}\boldsymbol{a} = 0$ if and only if $\boldsymbol{a} = 0$. Suppose indeed that $\boldsymbol{a}^*\boldsymbol{W}\boldsymbol{a} = 0$ for $\boldsymbol{a} \in \mathbb{C}^{n+1}$. Then,

$$0 = \boldsymbol{a}^*\boldsymbol{W}\boldsymbol{a} = |\boldsymbol{a}^*\tilde{\boldsymbol{v}}|^2 + \sum_{i=1}^{n} \alpha_j |\boldsymbol{a}^*\boldsymbol{w}_j|^2$$

and thus $\boldsymbol{a}^*\tilde{\boldsymbol{v}} = 0$ and $\boldsymbol{a}^*\boldsymbol{w}_j = 0$ for all i. Thus, $\boldsymbol{a}^*\boldsymbol{X} = \boldsymbol{0}$ where $\boldsymbol{X}$ is the $(n+1) \times (n+1)$ matrix whose columns are the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \tilde{\boldsymbol{v}}$. Since $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \tilde{\boldsymbol{v}}$ are linearly independent we conclude that $\boldsymbol{X}$ is nonsingular and thus $\boldsymbol{a} = \boldsymbol{0}$. □

B.2 Finsler's theorem and its variants

In this section several variants of Finsler's Theorem are proven. These results are used in Section 4.2. The "classical" Finsler's Theorem is concerned with homogenous quadratic functions.

Theorem B.1 (Finsler's Theorem [9]). *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ be symmetric matrices. Then*

$$\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} \geq 0 \textit{ for every } \boldsymbol{x} \in \mathbb{R}^n \textit{ such that } \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$$

if and only if there exists $\alpha \in \mathbb{R}$ such that $\boldsymbol{B} - \alpha\boldsymbol{A} \succeq \boldsymbol{0}$.

Theorem B.2 below is a simple extension of Finsler's Theorem to the complex case.

Theorem B.2 (Finsler's Theorem, complex version). *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ be symmetric matrices. Then the following two statements are equivalent:*

(i) $\boldsymbol{z}^*\boldsymbol{B}\boldsymbol{z} \geq 0$ *for every* $\boldsymbol{z} \in \mathbb{C}^n$ *such that* $\boldsymbol{z}^*\boldsymbol{A}\boldsymbol{z} = 0$.

(ii) There exists $\alpha \in \mathbb{R}$ *such that* $\boldsymbol{B} - \alpha\boldsymbol{A} \succeq \boldsymbol{0}$.

Proof: Writing $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$, where $\boldsymbol{x} = \Re(\boldsymbol{z})$ and $\boldsymbol{y} = \Im(\boldsymbol{z})$, the first statement can be written as

$$\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} + \boldsymbol{y}^T\boldsymbol{B}\boldsymbol{y} \geq 0 \text{ for every } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \text{ such that } \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y} = 0,$$

which, by the real version of Finsler's Theorem (Theorem B.1), holds true if and only if there exists $\alpha \in \mathbb{R}$ such that

$$\begin{pmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \alpha \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \succeq \boldsymbol{0}.$$

The latter LMI is the same as $\boldsymbol{B} - \alpha\boldsymbol{A} \succeq \boldsymbol{0}$. □

While Finsler's theorem deals with *homogenous* quadratic forms, the extended version considers *nonhomogeneous* quadratic functions.

Theorem B.3 (Extended Finsler's Theorem). *Let* $\mathbb{F}$ *be one of the fields* $\mathbb{R}$ *or* $\mathbb{C}$ *and let* $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n\times n}$ *be symmetric matrices such that*

$$\boldsymbol{A}_2 \succeq \eta\boldsymbol{A}_1 \text{ for some } \eta \in \mathbb{R}. \tag{58}$$

Let $f_j : \mathbb{F}^n \to \mathbb{R}, f_j(\boldsymbol{x}) = \boldsymbol{x}^*\boldsymbol{A}_j\boldsymbol{x} + 2\Re(\boldsymbol{b}_j^T\boldsymbol{x}) + c_j$, *where* $\mathbf{b}_j \in \mathbb{R}^n$ *and* c_j *is a real scalar*[5]. *Then the following two statements are equivalent*

(i) $f_2(\boldsymbol{x}) \geq 0$ *for every* $\boldsymbol{x} \in \mathbb{F}^n$ *such that* $f_1(\boldsymbol{x}) = 0$.

(ii) There exists $\lambda \in \mathbb{R}$ *such that* $\begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{b}_2 \\ \boldsymbol{b}_2^T & c_2 \end{pmatrix} \succeq \lambda \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1^T & c_1 \end{pmatrix}$.

Proof: $(ii) \Rightarrow (i)$ is a trivial implication. Now, suppose that (i) is satisfied. Making the change of variables $\boldsymbol{x} = (1/t)\boldsymbol{y}$, where $\boldsymbol{y} \in \mathbb{F}^n$ and $t \neq 0$, (2) becomes

$$f_2\left((1/t)\boldsymbol{y}\right) \geq 0 \text{ for every } \boldsymbol{y} \in \mathbb{R}^n, t \neq 0 \text{ such that } f_1\left((1/t)\boldsymbol{y}\right) = 0,$$

which is equivalent to

$$g_2\left(\boldsymbol{y}, t\right) \geq 0 \text{ for every } \boldsymbol{y} \in \mathbb{R}^n, t \neq 0 \text{ such that } g_1\left(\boldsymbol{y}, t\right) = 0, \tag{59}$$

[5] *In the case* $\mathbb{F} = \mathbb{R}$, f_j *can be written as* $f_j(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}_j\boldsymbol{x} + 2\boldsymbol{b}_j^T\boldsymbol{x} + c_j$.

where $g_j(\boldsymbol{y}, t) = \boldsymbol{y}^T \boldsymbol{A}_j \boldsymbol{y} + 2\boldsymbol{b}_j^T \boldsymbol{y} t + c_j t^2$. Notice that if t would not be restricted to be nonzero, then by Theorems B.1 and B.2), statement (*ii*) is true (g_1 and g_2 are homogenous quadratic functions). Thus, all is left to prove is that (59) is true for $t = 0$. However, when $t = 0$, (59) reduces to

$$\boldsymbol{y}^T \boldsymbol{A}_2 \boldsymbol{y} \geq 0 \text{ for every } \boldsymbol{y} \in \mathbb{F}^n \text{ such that } \boldsymbol{y}^T \boldsymbol{A}_1 \boldsymbol{y} = 0,$$

which, by Theorems B.1 and B.2, is equivalent to condition (58). □

Theorem B.3 is true under condition (58). This condition holds true, for instance, if $\boldsymbol{A}_2$ is positive definite or if $\boldsymbol{A}_1$ is definite (*i.e.,* positive or negative definite). The case in which $\boldsymbol{A}_1$ is definite was already proven for the real case in [24, Corollary 6].

C Known Results

Lemma C.1 below was proven in [4] for the real case. The proof for the complex is case is essentially the same and is thus omitted.

Lemma C.1 ([4], p. 163). *Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be an Hermitian matrix and let $\boldsymbol{b} \in \mathbb{C}^n$ and $c \in \mathbb{R}$. Then,*

$$\forall \boldsymbol{z} \in \mathbb{C}^n, \quad \boldsymbol{z}^* \boldsymbol{A} \boldsymbol{z} + 2\Re(\boldsymbol{b}^* \boldsymbol{z}) + c \geq 0 \tag{60}$$

if and only if

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^* & c \end{pmatrix} \succeq \mathbf{0}. \tag{61}$$

Lemma C.2. *Let $f : \mathbb{C}^n \to \mathbb{R}$ be the quadratic function $f(\boldsymbol{z}) = \boldsymbol{z}^* \boldsymbol{A} \boldsymbol{z} + 2\Re(b^* \boldsymbol{z}) + c$, where $\boldsymbol{A} = \boldsymbol{A}^* \in \mathbb{C}^{n\times n}, \boldsymbol{b} \in \mathbb{C}^n$ and $c \in \mathbb{R}$. Suppose that $\min_{\boldsymbol{z}\in\mathbb{C}^n} f(\boldsymbol{z}) > -\infty$. Then the set of optimal solutions is given by $\{\boldsymbol{z} : \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b} = 0\}$.*

Lemma C.3 ([3]). *Let $f : \mathbb{C}^n \to \mathbb{R}$ be the quadratic function $f(\boldsymbol{z}) = \boldsymbol{z}^* \boldsymbol{A} \boldsymbol{z} + 2\Re(b^* \boldsymbol{z}) + c$, where $\boldsymbol{A} = \boldsymbol{A}^* \in \mathbb{C}^{n\times n}, \boldsymbol{b} \in \mathbb{C}^n$ and $c \in \mathbb{R}$. Then, $\min_{\boldsymbol{z}\in\mathbb{C}^n} f(\boldsymbol{z}) > -\infty$ if and only if the following two conditions hold:*
(i) $\boldsymbol{b} \in \mathcal{R}(\boldsymbol{A})$.
(ii) $\boldsymbol{A} \succeq \mathbf{0}$.
and in that case the set of optimal solutions is given by $\{\boldsymbol{z} : \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b} = 0\}$.

References

[1] K. Anstreicher and H. Wolkowicz. On Lagrangian relaxation of quadratic matrix constraints. *SIAM J. Matrix Anal. Appl.*, 22(1):41–55 (electronic), 2000.

[2] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming: Theory and Algorithms.* John Wiley & Sons, Inc., second edition, 1993.

[3] A. Beck and M. Teboulle. Global optimality conditions for quadratic optimization problems with binary constraints. *SIAM J. Optim.*, 11:179–188, 2000.

[4] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization.* MPS-SIAM Series on Optimization, 2001.

[5] A. Ben-Tal and M. Teboulle. Hidden convexity in some nonconvex quadratically constrained quadratic programming. *Mathematical Programming*, 72(1):51–63, 1996.

[6] M. R. Celis, J. E. Dennis, and R. A. Tapia. A trust region strategy for nonlinear equality constrained optimization. In *Numerical optimization, 1984 (Boulder, Colo., 1984)*, pages 71–82. SIAM, Philadelphia, PA, 1985.

[7] P. Chevalier and B. Picinbono. Complex linear-quadratic systems for detection and array processing. *IEEE Trans. Signal Processing*, 44(10):2631–2634, Oct. 1996.

[8] A. R. Conn, N. I. M. Gold, and P. L. Toint. *Trust-Region Methods.* MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

[9] P. Finsler. Uber das vorkommen definiter und semi-definiter formen in scharen quadratische formen. *Commentarii Matematici Helvetici*, 9:188–192, 1937.

[10] C. Fortin and H. Wolkowicz. The trust region subproblem and semidefinite programming. *Optim. Methods Softw.*, 19(1):41–67, 2004.

[11] A. L. Fradkov and V. A. Yakubovich. The S-procedure and the duality relation in convex quadratic programming problems. *Vestnik Leningrad. Univ.*, (1 Mat. Meh. Astronom. Vyp. 1):81–87, 155, 1973.

[12] W. Gander, G. H. Golub, and U. von Matt. A constrained eigenvalue problem. *Linear Algebra Appl.*, 114-115:815–839, 1989.

[13] J. N. Hiriart-Urruty and M. Torki. Permanently going back and forth between the "quadratic world" and the "convexity world" in optimization. *Appl. Math. Optim.*, 45(2):169–184, 2002.

[14] J. M. Martínez. Local minimizers of quadratic functions on Euclidean balls and spheres. *SIAM J. Optim.*, 4(1):159–176, 1994.

[15] J. J. Moré and D. C. Sorensen. Computing a trust region step. *SIAM J. Sci. Statist. Comput.*, 4(3):553–572, 1983.

[16] B. Picinbono and P. Chevalier. Widely linear estimation with complex data. *IEEE Trans. Signal Processing*, 43(8):2030–2033, Aug. 1995.

[17] B. T. Polyak. Convexity of quadratic transformations and its use in control and optimization. *J. Optim. Th. Applicns.*, 99(3):553–583, 1998.

[18] M. J. D. Powell and Y. Yuan. A trust region algorithm for equality constrained optimization. *Math. Programming*, 49(2, (Ser. A)):189–211, 1990/91.

[19] P. J. Schreier, L. L. Scharf, and C. T. Mullis. Detection and estimation of improper complex random signals. *IEEE Trans. Inform. Theory*, 51(1):306–312, Jan. 2005.

[20] N. Z. Shor. Quadratic optimization problems. *Izv. Akad. Nauk SSSR Tekhn. Kibernet.*, (1):128–139, 222, 1987.

[21] D. C. Sorensen. Newton's method with a model trust region modification. *SIAM J. Numer. Anal.*, 19(2):409–426, 1982.

[22] R. J. Stern and H. Wolkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM J. Optim.*, 5(2):286–313, 1995.

[23] J. F. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999.

[24] J. F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Math. Operations Research*, 28(2):246–267, 2003.

[25] Y. Ye and S. Zhang. New results on quadratic minimization. *SIAM J. Optim.*, 14(1):245–267, 2003.

[26] Y. Yuan. On a subproblem of trust region algorithms for constrained optimization. *Mathematical Programming*, 47:53–63, 1990.

[27] Y. Zhang. Computing a Celis-Dennis-Tapia trust-region step for equality constrained optimization. *Math. Programming*, 55(1, Ser. A):109–124, 1992.