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Smoothing Method of Multipliers for Sum-Max Problems

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Abstract

We study a class of nonsmooth unconstrained optimization problems, which includes the problem of minimizing the sum of pairwise maxima of smooth convex functions. Minimum l_1-norm approximation is a particular case of this problem. Combining the ideas of Lagrange multipliers and of smooth approximation of max-type function, we obtain an extended notion of nonquadratic augmented Lagrangian. Our approach does not require artificial variables, and preserves sparse structure of Hessian in many practical cases. We present the corresponding method of multipliers, and its convergence analysis for a dual counterpart, resulting in a proximal point maximization algorithm. The practical efficiency of the algorithm is supported by computational results for large-scale problems, arising in structural optimization.

1 Introduction

We consider the non-smooth convex unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n}\left\{F(x)=f(x)+\sum_{i=1}^{m}\max\left[\alpha_i h_i(x),\ \beta_i h_i(x)\right]\right\} \tag{1}$$

where $f(x)$ and $h_i(x),\ i=1,\dots,m$ are smooth convex functions, defined over entire space $\mathbb{R}^n$; $\alpha_i<\beta_i$ are certain constants. In order to guarantee convexity, we assume, that for any particular index i, the values α_i and β_i are non-negative, if $h_i(x)$ is nonlinear[1]. Problem in this setting arises for example in Truss Topology Design [1, 5]. This is a generalization of least l_1 norm approximation (regularization) problem, when the coefficients $\alpha_i=-1,\ \beta_i=1$,

[1]Without loss of generality, one could set $\alpha_i=0$ for all i by changing f to $f+\sum_i\alpha_i h_i$ and β_i to $\beta_i-\alpha_i$. This simlification would not change significantly our derivation, nor the computational burden of the method, however it can be useful in various cases.

and the functions $h_i(x)$ are affine:

$$F(x) = f(x) + \sum_{i=1}^{m} |h_i(x)| = f(x) + ||h(x)||_1 \ , \qquad (2)$$

where $h(x) =: [h_1(x), h_2(x), .., h_m(x)]^T$. This kind of problem arises in many important areas of modern signal/image processing, in the context of sparse representations, blind source separation, minimal total variation solutions, *etc.*(see for example [8, 21, 22]).

Further, for simplicity of notation, we consider the case where $\alpha_i \equiv \alpha$, $\beta_i \equiv \beta$ are independent of i; the extension onto the general case is quite straightforward.

Extended augmented Lagrangian approach Methods of multipliers, involving *nonquadratic augmented Lagrangians* [10, 6, 15, 3, 19, 20, 4, 7, 11] successfully compete with the interior-point and other methods in non-linear and semidefinite programming. They are especially efficient when a very high accuracy of solution is required. This success is explained by the fact, that due to iterative update of multipliers, the penalty parameter does not need to become extremely small in the neighborhood of solution.

Direct application of the augmented Lagrangian approach to the sum-max problem requires the introduction of artificial variables, one per element of the sum, which significantly increases the problem size and the computational burden. Alternatively, one can use smooth approximation of the max-type function [2, 9], which will keep the size of the problem unchanged, but will require the smoothing parameter to become extremely small in order to obtain an accurate solution.

In this article we propose an extended notion of augmented Lagrangian, which combines the idea of multipliers with smooth approximation of the max-type function. It allows us to keep the size of the sum-max problem unchanged, and achieves a very accurate solution under a moderate value of the smoothing parameter. We show, that the powerful tool for analysis of augmented Lagrangian algorithms, based on their correspondence to the proximal point methods in dual space [18, 13, 14, 4], can be extended to our case. The practical efficiency of the algorithm is supported by computational results for large-scale problems, arising in structural optimization.

2 Smoothing the max-function

Let us introduce a parameterized smooth approximation of the maximum function

$$r(t) = \max(\alpha t, \beta t) \ ,$$

shown in Figure 1. The smoothing function $\varphi(t; \mu, c)$, $\alpha < \mu < \beta$, $c > 0$ has two parameters: μ and c. Parameter c defines accuracy of approximation of the max-function $r(\cdot)$, the approximation becomes perfect as $c \to \infty$. Parameter μ

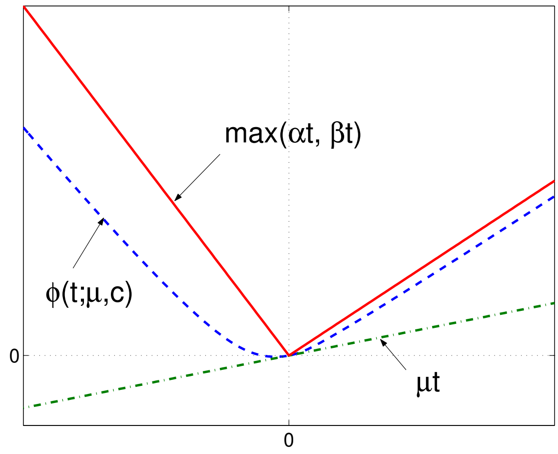

Figure 1: Max$(\alpha t, \beta t)$ – solid line; $\varphi(t;\mu,c)$ – dashed line; μt – dot-dashed line

determines the derivative of φ at $t = 0$; it will serve as a Lagrange multiplier. The graph of the linear function μt is tangent to the plot of $\varphi(\cdot;\mu,c)$ at the origin.

The function φ possesses the following properties:

(φ1) $\varphi(t;\mu,c)$ is convex in t;

(φ2) $\varphi(0;\mu,c) = 0$;

(φ3) $\varphi'_t(0;\mu,c) = \mu$;

(φ4) $\lim_{t\to-\infty} \varphi'_t(t;\mu,c) = \alpha$;

(φ5) $\lim_{t\to+\infty} \varphi'_t(t;\mu,c) = \beta$;

(φ6) $\lim_{c\to+\infty} \varphi(t;\mu,c) = r(t)$.

The particular form of function φ we introduce and prefer to use in our computations consists of three smoothly connected branches:

$$\varphi(t,\mu,c) = \begin{cases} \alpha t - p_1 \log \dfrac{t}{\tau_1} + s_1, & t < \tau_1 \le 0 \\ \dfrac{ct^2}{2} + \mu t, & \tau_1 \le t \le \tau_2 \\ \beta t - p_2 \log \dfrac{t}{\tau_2} + s_2, & t > \tau_2 \ge 0\,. \end{cases} \tag{3}$$

The coefficients $p_1, p_2, s_1, s_2, \tau_1, \tau_2$ are chosen to make the function φ continuous and twice differentiable at the joint points τ_1 and τ_2:

$$\begin{aligned}
\tau_1 &= \frac{\alpha-\mu}{2c}; \quad \tau_2 = \frac{\beta-\mu}{2c}; \\
p_1 &= c\tau_1^2; \qquad p_2 = c\tau_2^2; \\
s_1 &= \frac{c}{2}\tau_1^2 + (\mu-\alpha)\tau_1; \\
s_2 &= \frac{c}{2}\tau_2^2 + (\mu-\beta)\tau_2.
\end{aligned}$$

When $\tau_i = 0$, we put $p_i \log \frac{t}{\tau_i} = c\tau_i^2 \log \frac{t}{\tau_i} = 0$. One can note, that when $\alpha \to 0$ and $\beta \to \infty$, $\varphi(\cdot)$ becomes a *quadratic-logarithmic* penalty for inequality constraint in nonquadratic augmented Lagrangian [4]. On the other hand, when $\alpha \to -\infty$ and $\beta \to \infty$, $\varphi(\cdot)$ becomes a quadratic penalty for equality constraint in a standard quadratic augmented Lagrangian [16]. In this way our approach generalizes known augmented Lagrangian techniques.

3 Extended Notion of Lagrangian, Augmented Lagrangian and Duality

A standard way to introduce augmented Lagrangian for the sum-max problem would be to reformulate it as a smooth constrained optimization problem using artificial variables, which will increase the problem size significantly. Instead we introduce extended notions of Lagrangian and augmented Lagrangian, which keep the problem size unchanged, preserving at the same time many important classical properties of these functions. We propose the following extended notion of Lagrangian

$$L(x,u) = f(x) + \sum_{i=1}^{m} u_i h_i(x), \quad \alpha \le u_i \le \beta. \tag{4}$$

and corresponding extended notion of augmented Lagrangian

$$M(x,u,c) = f(x) + \sum_{i=1}^{m} \varphi(h_i(x), u_i, c), \quad \alpha \le u_i \le \beta. \tag{5}$$

As we will show in this section, *there exists a vector of multipliers u^*, such that an unconstrained minimizer x^* of the augmented Lagrangian provides an optimal solution to Problem (1)*, *i.e.* we do not need to force the smoothing parameter c towards zero in order to solve the problem. This property serves as a basis for the method of multipliers presented in the next section.

Duality of the Sum-Max Problem

We will show that duality properties similar to those of the standard Lagrangian and augmented Lagrangian take place in our extensions. Let us denote the following sets of indeces:

$$I_n(x) = \{i : h_i(x) < 0\}; \tag{6}$$

$$I_0(x) = \{i : h_i(x) = 0\}; \tag{7}$$

$$I_p(x) = \{i : h_i(x) > 0\}; \tag{8}$$

Lemma 1 [necessary and sufficient optimality conditions]
A vector $x^* \in \mathbb{R}^n$ is a solution of the problem (1) iff there exists a vector $u^* \in \mathbb{R}^m$ such that

$$\nabla f(x^*) + \sum_{i=1}^{m} u_i^* \nabla h_i(x^*) = 0 \ , \tag{9}$$

$$u_i^* = \alpha, \qquad i \in I_n(x^*) \ , \tag{10}$$

$$u_i^* = \beta, \qquad i \in I_p(x^*) \ , \tag{11}$$

$$\alpha \le u_i^* \le \beta, \ i \in I_0(x^*) \ . \tag{12}$$

Proof. Subdifferential set of the convex function $F(x)$ in (1) can be represented as

$$\partial F(x) = \{\nabla f(x) + \sum_{i=1}^{m} u_i \nabla h_i(x)\} \ , \tag{13}$$

where

$$u_i = \alpha, \qquad i \in I_n(x), \tag{14}$$

$$u_i = \beta, \qquad i \in I_p(x), \tag{15}$$

$$\alpha \le u_i \le \beta, \ i \in I_0(x), \tag{16}$$

so (9) - (12) represent the sufficient and necessary optimality condition

$$0 \in \partial F(x^*) \tag{17}$$

(see the subdifferential calculus developed in [16, 12]) □

Lemma 2 *For any given $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, $\alpha \le u \le \beta$, the following inequalities take place:*

$$F(x) \ge M(x, u, c) \ge L(x, u) \ . \tag{18}$$

Proof. Follows immediately from the inequality

$$\max(\alpha t, \beta t) \ge \varphi(t, \mu, c) \ge \mu t \ , \quad \alpha \le \mu \le \beta \tag{19}$$

illustrated by Fig.1. □

Theorem 1 [saddle point of the Lagrangian]
A vector x^* is a solution of the problem (1) iff there exists a vector $u^* \in \mathbb{R}^m$, $\alpha \le u^* \le \beta$, such that the pair (x^*, u^*) is a saddle point of the Lagrangian (4):

$$L(x^*, u) \le L(x^*, u^*) \le L(x, u^*) \quad \forall x \in \mathbb{R}^n, \alpha \le u \le \beta \tag{20}$$

Proof. Suppose that x^* is a solution of the problem (1). Then by Lemma 1, conditions (9) – (12) hold. By (9)

$$\nabla_x L(x^*, u^*) = 0, \tag{21}$$

so by convexity of the Lagrangian with respect to x

$$x^* \in \arg\min_x L(x, u^*). \tag{22}$$

It is easy to check, that by (10) – (12)

$$u_i^* h_i(x) = \max(\alpha h_i(x), \beta h_i(x)),$$

hence $L(x^*, u^*) = F(x^*)$. Taking into account inequality (18), we obtain

$$L(x^*, u^*) \geq L(x^*, u) \quad , \quad \alpha \leq u \leq \beta. \tag{23}$$

So by (22) and (23) the pair (x^*, u^*) is a saddle point.

Conversely, suppose that (x^*, u^*) is a saddle point of the Lagrangian. By definition (22) holds, hence (9) holds as well. By (20) and (4) we have

$$\sum_i u_i^* h_i(x^*) \geq \sum_i u_i h_i(x^*), \quad \alpha \leq u_i \leq \beta. \tag{24}$$

This implies that (10) and (11) must be satisfied, otherwise we could violate (24) putting corresponding u_i to α or β. Hence, conditions (9) – (12) hold, and by Lemma 1 the vector x^* is an optimal solution for the problem (1). □

Now we introduce the dual function

$$G(u) = \min_x L(x, u). \tag{25}$$

By Theorem 1 the *strong duality* holds:

$$F(x^*) = \min F(x) = \max_{\alpha \leq u \leq \beta} G(u) = G(u^*). \tag{26}$$

Corollary 1 *Optimal solution x^* is a minimizer of the augmented Lagrangian $M(\cdot, u^*, c)$*

Proof follows from Theorem 1 and Lemma 2. □

This corollary has an important meaning: *using optimal multipliers, one can obtain an exact solution of the original problem by minimization of the augmented Lagrangian, without forcing the smoothing parameter to zero.* This will serve as a ground for the method of multipliers presented below.

4 Smoothing Method of Multipliers

We introduce the method of multipliers, which performs the following steps at iteration k:

1. Minimize augmented Lagrangian in x

$$x^{k+1} = \arg\min_x M(x, u^k, c_k); \tag{27}$$

2. Update the multipliers

$$u_i^{k+1} = \varphi'(h_i(x_{k+1}), u_i^k, c_k), \tag{28}$$

   where the derivative φ' is taken with respect to the first argument

3. Update the smoothing parameter (optionally)

$$c_{k+1} = \gamma c_k, \quad \gamma > 1. \tag{29}$$

In practical implementation we sometimes restrict relative change of multipliers to some bounds in order to stabilize the method:

$$\gamma_1 < \frac{u_i^{k+1} - \alpha}{u_i^k - \alpha} < \gamma_2 \tag{30}$$

$$\gamma_1 < \frac{\beta - u_i^{k+1}}{\beta_i - u_i^k} < \gamma_2 \tag{31}$$

$$\alpha + \delta < u_i^{k+1} < \beta - \delta. \tag{32}$$

We also restrict the smoothing parameter by some maximal value c_{max}. In numerical experiments presented in this work, our choice of parameters was $\gamma = \gamma_1 = \frac{1}{\gamma_2} = 2$, $\delta = 10^{-6}$, $c_{max} = 10^3$. In general, the algorithm is rather insensitive to changes in the parameters in order of magnitude or more.

5 Proximal Point Algorithm as a Dual Interpretation of the Method

We show here that the algorithm (27 – 29) generates the same sequence of multipliers $\{u^k\}$ as an appropriate (non-quadratic) proximal point algorithm applied to the maximization of the dual objective function G over the box

$$\max_{\alpha \le u \le \beta} G(u) \ , \tag{33}$$

and state the convergence of the algorithm (see Appendix A for the proofs.)

Proposition 1 *After the update of multipliers (28) the vector x^{k+1} becomes a minimizer of the Lagrangian, which gives the value of the dual function*

$$L(x^{k+1}, u^{k+1}) = \min_x L(x, u^{k+1}) = G(u^{k+1}) \ . \tag{34}$$

Proposition 2 *If $\bar{x}$ is a minimizer of the Lagrangian*

$$L(\bar{x}, \bar{u}) = \min_x L(x, \bar{u}) = G(\bar{u}) \ , \tag{35}$$

then

$$h(\bar{x}) \in \partial G(\bar{u}) \ , \tag{36}$$

where $\partial G(\bar{u})$ denotes a subdifferential set of the concave function G, and $h(x) =: (h_1(x), h_2(x), .., h_m(x))^T$

Lemma 3 [correspondence between method of multipliers and prox algorithm] *The algorithm (27),(28),(29) generates the same sequence of multipliers $\{u_k\}$ as a proximal point algorithm given by the following recurrent relation*

$$u^{k+1} = \arg\max_u \{G(u) - D_k(u, u^k)\}, \tag{37}$$

where the proximal term D_k is given by

$$D_k(u, u^k) = \sum_{i=1}^{m} \varphi^*(u_i, u_i^k, c_k) \ , \tag{38}$$

and φ^* is the conjugate function of φ with respect to the first argument, given by Legendre transformation (see e.g. [16])

$$\varphi^*(\lambda, \mu, c) = \sup_t \{\lambda t - \varphi(t, \mu, c)\}. \tag{39}$$

For example, the conjugate of the quadratic-logarithmic function (3) can be easily obtained by integrating the inverse of its derivative (see Figure 2)

$$\varphi^*(\lambda, \mu, c) = \begin{cases} -p_1 \log \dfrac{\lambda - \alpha}{l_1 - \alpha} + \dfrac{1}{2c}(l_1 - \mu)^2, & \alpha < \lambda < l_1 \\ \dfrac{1}{2c}(\lambda - \mu)^2, & l_1 \le \lambda \le l_2 \\ -p_2 \log \dfrac{\lambda - \beta}{l_2 - \beta} + \dfrac{1}{2c}(l_2 - \mu)^2, & l_2 < \lambda < \beta \end{cases} \tag{40}$$

where

$$l_1 = c\tau_1 + \mu; \quad l_2 = c\tau_2 + \mu;$$

To simplify the notation we will omit the third argument in φ and denote

$$\psi(\lambda, \mu) \equiv \varphi^*(\lambda, \mu) \tag{41}$$

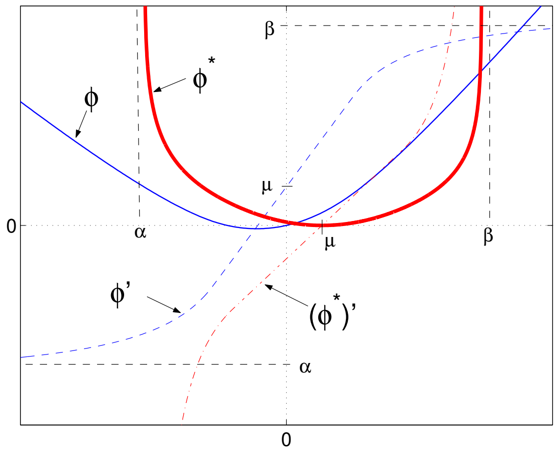

Figure 2: Smoothing function φ (solid line) and its conjugate ψ (bold solid line). Corresponding derivatives (dashed and dot-dashed lines) are mutually inverse functions, hence their plots are symmetric with respect to the 45^o line.

Let $\alpha < \mu < \beta$. The conjugate function ψ has the following properties:

(ψ1.) ψ_1' increases monotonically
[since φ_1' increases monotonically and $\psi_1' = (\varphi_1')^{-1}$]

(ψ2.) $\psi(\cdot, \mu)$ is convex [by ψ1]

(ψ3.) $\psi_1'(\mu, \mu) = 0$ [since $\varphi_1'(0, \mu) = \mu$]

(ψ4.) $\psi(\lambda, \mu) = \infty$ for $\lambda < \alpha$ or $\lambda > \beta$ (barrier property)
$\lim_{\lambda \searrow \alpha} \psi_1'(\lambda, \mu) = -\infty, \quad \lim_{\lambda \nearrow \beta} \psi_1'(\lambda, \mu) = \infty$
[since $\lim_{t \to -\infty} \varphi'(t, \mu) = \alpha; \quad \lim_{t \to \infty} \varphi'(t, \mu) = \beta$]

(ψ5.) $\psi(\lambda, \lambda) = 0$
[$\psi(\lambda, \lambda) = \varphi^*(\lambda, \lambda) =: \sup(t\lambda - \varphi(t, \lambda)) = -\varphi(0, \lambda) = 0 \quad$ since $\varphi\prime(0, \lambda) = \lambda$]

(ψ6.) $\psi(\lambda, \mu) \geq 0$ [by ψ5, ψ3 and ψ2].

Denote

$$\mathrm{Box}(\alpha, \beta) = \{u \in \mathbb{R}^m : \alpha \leq u \leq \beta\} \ . \tag{42}$$

Due to ($\psi 1 - \psi 6$), the proximal term (38) is a convex function with respect to the first argument and the following properties are satisfied:

(D1.) $D(u, v) \geq 0 \ , u, v \in \mathrm{Box}(\alpha, \beta)$

(D2.) $D(u, u) = 0 \ , u \in \mathrm{interior}(\mathrm{Box}(\alpha, \beta))$

(D3.) $D(u, v) = \infty$ if $v \in \mathrm{interior}(\mathrm{Box}(\alpha, \beta))$, $u \notin \mathrm{Box}(\alpha, \beta)$

Theorem 2 [monotonicity and convergence of the proximal point algorithm] *Let the proximal term (38) of the the algorithm (37) be based on the conjugate of the quadratic-logarithmic function (3). Then*

(a) the sequence $\{u^k\}$ generated by the prox-algorithm (37) belongs to the $\mathrm{Box}(\alpha, \beta)$ *and the sequence of the function values $G(u^k)$ is nondecreasing;*

(b) the set of accumulation points of the method (34) is nonempty and belongs to the solution set of the dual problem (33).

6 Computational Example

In this section we demonstrate effectiveness of our method as applied to the optimal truss topology design.

6.1 Truss Topology Design problem formulation

The problem is to find the stiffest truss which carries a given load and which consists of bars of a given total volume. The bars of the truss are a subset of the bars connecting all of a set of a priori chosen nodal points. The volume t_i of each bar is within prescribed upper and lower bounds U_i and L_i. The original formulation of the problem is the following (see [1] and [5] for details):

$$\min_{x,t} f^T x \tag{43}$$

subject to

$$A(t)x = f \tag{44}$$

$$\sum_{i=1}^{m} t_i = v \tag{45}$$

$$0 \le L_i \le t_i \le U_i\,, \quad i = 1, \ldots, m \tag{46}$$

where

N	–	number of nodes in the truss
$t = (t_i)$	–	m-dimensional vector of the bars' volumes ("design variables")
m	–	maximum number of bars ($m = \frac{1}{2}N(N-1)$)
$x = (x_j)$	–	n-dimensional vector of the displacements of the nodes ("analysis variables")
n	–	number of analysis variables $n = 2N$ (2D-trusses) or $n = 3N$ (3D-trusses)
f	–	n-dimensional vector of the loads on the nodes
v	–	given total volume of the truss
$L = (L_i)$	–	m-dimensional vector of lower bounds on the bars' volumes
$U = (U_i)$	–	m-dimensional vector of upper bounds on the bars' volumes
$A(t)$	–	symmetric positive semidefinite $n \times n$ matrix, the stiffness matrix.

The matrix $A(t)$ is given in terms of matrices A_i, which are symmetric positive semidefinite $n \times n$ matrices. A_i $(i = 1, 2, \dots, m)$ determine the geometry of the connection of node i to the other nodes. Typical case:

$$A(t) = \sum_{i=1}^{m} t_i A_i \ ,$$

where A_i are matrices of rank 1.

It was proved in [1] that this problem is equivalent to the following one:

$$\min_{x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}} \left\{ \lambda v - f^T x + \sum_{i=1}^{m} \max \left\{ \left(\frac{1}{2} x^T A_i x - \lambda \right) U_i, \left(\frac{1}{2} x^T A_i x - \lambda \right) L_i \right\} \right\}. \tag{47}$$

Note that (47) is a sum-max problem, and thus can be solved by our algorithm discussed above.

6.2 Results

The major computational load of the algorithm is connected to the assembling of the Hessian matrices and solution of the Newton systems at the internal unconstrained optimization phase. Typically it takes the order of n^3 operations for each Newton step, while the overall number of steps grows moderately with the dimension of the problem (see Table 1.) The number of external iterations (multiplier updates) varies from 7 to 13 in all the experiments. We also produced comparison with the ordinary smoothing method by freezing the multipliers at a constant level. This increased typical solution time by about 50%, which demonstrates the usefulness of the multiplier approach.

In the case of the standard method of multipliers, which should use a slack variable for each max term, the total number of the variables would increase up to 30 times in our examples, *i.e.* computational burden would be about 30^3 times higher!

7 Conclusions

In this article we have developed a Lagrangian duality scheme for sum-max problems, and obtained a new kind of augmented Lagrangian, which uses smooth approximation of max-type functions, and does not require artificial variables.

We have demonstrated the correspondence of the suggested method of multipliers to a proximal point algorithm in dual space, which provides foundation for the convergence proof. Numerical experiments with large scale problems show practical efficiency of the method.

Acknowledgments

I am grateful to Aharon Ben-Tal from the Technion for attracting my attention to the sum-max formulation of the truss topology design problem, which origi-

# of variables	# of max terms	Upper bound on bar volume	Lower bound on bar volume	# of Newton steps	# of grad. evaluations
98	150	10	0	23	72
		0.1	0	19	80
		0.1	0.01	18	77
126	1234	10	0	46	147
		0.1	0	30	124
		0.1	0.001	30	91
		10	0.001	46	136
242	4492	10	0	44	136
342	8958	10	0	45	147
		0.1	0	36	124
		0.1	0.0005	32	121
450	15556	10	0	102	795
		0.01	0	160	1376
		0.001	0.0001	64	386

Table 1: Numerical Results for the Truss Topology Design Problems. Accuracy in optimal value of the objective function is 6 decimal digits.

nally stimulated this work. I am also grateful to Michal Kocvara and Jochem Zowe from the University of Erlangen/Nuremberg for useful discussions and collaboration in studies of truss design using the presented method. We used the truss topology design subroutines developed by Martin Bendsoe from the Technical University of Denmark. This research has been supported in part by the Minerva Optimization Center, by the Ollendorff Minerva Center and the HASSIP Research Network Program HPRN-CT-2002-00285, sponsored by the European Commission.

Appendix A.

Proof of Proposition 1 By optimality condition for (27)

$$0 = f'(x^{k+1}) + \sum_{i=1}^{m} \varphi'[g_i(x^{k+1}), u_i^k, c_k] h_i'(x^{k+1}) \ . \tag{48}$$

Taking into account the updating formula (28), we obtain

$$0 = f'(x^{k+1}) + \sum_{i=1}^{m} u_i^{k+1} h_i'(x^{k+1}) \ , \tag{49}$$

which is exactly the optimality condition for the Lagrangian $L(x^{k+1}, u^{k+1})$ with respect to the first argument. □

Proof of Proposition 2 We will show that the vector h can be used as a gradient in Gradient Inequality for the concave function G(u)

$$\begin{aligned} G(u) &= \min_x \{f(x) + \sum u_i h_i(x)\} \le f(\bar{x}) + \sum u_i h_i(\bar{x}) \\ &= f(\bar{x}) + \sum \bar{u}_i h_i(\bar{x}) + \sum (u_i - \bar{u}_i) h_i(\bar{x}) \\ &= G(\bar{u}) + (u - \bar{u})^T h(\bar{x}) \end{aligned}$$

□

Proof of Lemma 3 By the property of conjugate function

$$(\varphi^*)' = (\varphi')^{-1} \tag{50}$$

and updating formula (28), we obtain

$$(\varphi^*)'(u_i^{k+1}, u_i^k, c_k) = h_i(x^{k+1}). \tag{51}$$

So the gradient of the proximal term with respect to the first argument will be

$$\nabla_1 D_k(u^{k+1}, u^k) = h(x^{k+1}); \tag{52}$$

Taking into account (34) and (36), we obtain

$$\nabla_1 D_k(u^{k+1}, u^k) \in \partial G(u^{k+1}) \ , \tag{53}$$

which is precisely the necessary and sufficient condition for u^{k+1} to attain the maximum in (37). □

Proof of Theorem 2

(a) The sequence $\{u^k\}$ generated by the method (34) belongs to the Box(α, β) (42) by barrier property $(D3)$. Now:

$$\begin{aligned} G(u^{k+1}) - D(u^{k+1}, u^k) &\ge G(u^k) - D(u^k, u^k) \\ &= G(u^k) \ , \ \text{ by } \ (D2) \ . \end{aligned}$$

Hence

$$G(u^{k+1}) - G(u^k) \ge D(u^{k+1}, u^k) \ge 0 \ \text{ by } \ (D1) \ . \tag{54}$$

(b) Box(α, β) is a compact set, hence the sequence $\{u^k\}$ has accumulation points. Arguing by contradiction, suppose that there is a non-optimal accumulation point $\bar{u}$. Denote an "active" set of indeces

$$I_a(\bar{u}) = \{i : \alpha < \bar{u}_i < \beta\}. \tag{55}$$

Consider a feasible neighborhood of the point $\bar{u}$, which is separated from the solution set U^* and from the boundaries of the feasible box in those coordinates, which are not at the boundary for $\bar{u}$:

$$N_\gamma(\bar{u}) = B_\gamma(\bar{u}) \bigcap \text{Box}(\alpha, \beta) \ , \tag{56}$$

where $B_\gamma(\bar{u})$ is a ball centered in $\bar{u}$ with radius

$$\gamma = \frac{1}{2} \min \left\{ dist(\bar{u}, U^*), \ \min_{i \in I_a(\bar{u}_i)} \{\bar{u}_i - \alpha, \ \beta - \bar{u}_i\}, \ \beta - \alpha \right\} . \tag{57}$$

For any point from $N_\gamma(\bar{u})$ the necessary optimality conditions are not satisfied, i.e. there exists a small positive number ϵ such that for any subgradient

$$g \in \bigcup_{u \in N_\gamma(\bar{u})} \partial G(u) \ , \tag{58}$$

at least for one value of the index i one of the following take place:

$$g_i < -\epsilon \ , \quad \bar{u}_i > \alpha \tag{59}$$

$$g_i > \epsilon \ , \quad \bar{u}_i < \beta \tag{60}$$

Let $u^k \in N_\gamma(\bar{u})$ be a point generated by the prox-algorithm. By (53)

$$\nabla_1 D_k(u^k, u^{k-1}) \in \partial G(u^k) \ , \tag{61}$$

hence taking into account (59), (60), (61) and (38), we conclude that there exists such an index i that either

$$\psi_1'(u_i^k, u_i^{k-1}) > \epsilon \ , \quad \alpha \le u_i^k \le \beta - \gamma \tag{62}$$

or

$$\psi_1'(u_i^k, u_i^{k-1}) < -\epsilon \ , \quad \alpha + \gamma \le u_i^k \le \beta \tag{63}$$

It is shown in Proposition 3 in Appendix B, that conditions (62) and (63) entail

$$\psi(u_i^k, u_i^{k-1}) > \delta > 0 \ , \tag{64}$$

where δ is a constant depending on ϵ, γ, and c. Recall that $\bar{u}$ is an accumulation point, so by (54) and (64) there exists an infinite set of indeces $\{k_j\}$ such that

$$G(u^{k_j}) - G(u^{k_j - 1}) > \delta > 0 \ .$$

Thus, the infinite sequence $G(u^{k_j})$ increases each step by at least a positive constant, as shown above. Hence, it cannot be bounded, contrary to assumption $G^* < \infty$. □

Appendix B

Proposition 3 *Let $\psi(\cdot,\cdot)$ be the conjugate of the quadratic-logarithmic function (3) and one of the following relations takes place*

$$\psi_1'(\bar{\lambda}, \mu) > \epsilon \ , \quad \alpha \le \mu < \bar{\lambda} \le \beta - \gamma \tag{65}$$

or

$$\psi_1'(\bar{\lambda},\mu) < -\epsilon \ , \quad \alpha+\gamma \leq \bar{\lambda} < \mu \leq \beta \tag{66}$$

Then

$$\psi(\bar{\lambda},\mu) > \delta > 0 \ , \tag{67}$$

where δ is a constant depending on ϵ, γ, and c.

Proof Consider the condition (65) (proof under condition (66) is similar). Suppose that we know (see Proposition 4) that

$$\psi''(\lambda,\mu) < M = \text{const}(\gamma), \quad \mu \leq \lambda \leq \bar{\lambda} \ . \tag{68}$$

Then taking into account that $\min_\lambda \psi(\lambda,\mu) = \psi(\mu,\mu) = 0$ (by properties ($\psi 5$) and ($\psi 6$)), we conclude that $\psi(\lambda,\mu)$ majorates the quadratic function $q(\lambda)$ with the following properties

$$q'(\bar{\lambda}) = \psi'(\bar{\lambda},\mu); \quad q''(\lambda) = M; \quad \min q(\lambda) = q(\mu) = 0 \ .$$

This gives us a lower bound

$$\psi(\bar{\lambda},\mu) \geq q(\bar{\lambda}) = \frac{\left(q'(\bar{\lambda})\right)^2}{2M} > \frac{\epsilon}{2M} \ .$$

This proof is similar to that of Lemma 2 in [4], so the reader is refereed there for more details. □

Proposition 4 *Under condition (65) of Proposition 3, the following upper bound on second derivative takes place*

$$\psi''(\lambda,\mu) \leq \frac{1}{\min\{c, \gamma^2/p_2\}} \tag{69}$$

Proof Consider the first and the second derivatives of the function (3) with respect to t

$$\varphi'(t,\mu,c) = \begin{cases} \alpha - p_1/t, & t < \tau_1 \leq 0 \\ ct+\mu, & \tau_1 \leq t \leq \tau_2 \\ \beta - p_2/t, & t > \tau_2 \geq 0 \ , \end{cases} \tag{70}$$

$$\varphi''(t,\mu,c) = \begin{cases} p_1/t^2, & t < \tau_1 \leq 0 \\ c, & \tau_1 \leq t \leq \tau_2 \\ p_2/t^2, & t > \tau_2 \geq 0 \ . \end{cases} \tag{71}$$

As we already mentioned, if follows directly from the definition of the conjugate function, that the derivative of $\varphi(\cdot,\cdot)$ with respect to the first argument is an inverse function of the derivative of $\psi(\cdot,\cdot)$, *i.e.* if for some t, $\lambda = \varphi'(t,\mu)$ then $t = \psi'(\lambda,\mu)$.

Now let $\mu < \lambda \leq \beta - \gamma$ and $\varphi'(t,\mu) = \lambda$. Then taking into account (70), we get

$$\beta - p_2/t \leq \beta - \gamma$$

which gives us

$$t \leq \frac{p_2}{\gamma} \ . \tag{72}$$

On other hand, from $\varphi'(t,\mu) = \lambda > \mu$, convexity of $\varphi(\cdot,\mu)$ and the fact that $\varphi'(0,\mu) = \mu$, we conclude that $t > 0$, *i.e.* only the second and the third branches of $\varphi''(t,\mu)$ in (71) are relevant to our case. Therefore substituting (72) into (71), we obtain

$$\varphi''(t,\mu,c) \geq \min\{c, \gamma^2/p_2\} \ .$$

Taking into account that by property of mutually inverse functions φ' and ψ'

$$\psi''(\lambda,\mu) = \frac{1}{\varphi''(t,\mu)}$$

we complete the proof. □

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