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SAMPLING AND RECONSTRUCTION OF SURFACES AND HIGHER DIMENSIONAL MANIFOLDS

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Abstract. We present a new sampling theorem for surfaces and higher dimensional manifolds. The core of the proof resides in triangulation results for manifolds with boundary, not necessarily bounded. The proposed method adopts a geometric approach that is considered in the context of 2-dimensional manifolds (i.e surfaces). Further, our approach and formalism lend themselves too the derivation of a geometric theorem for non-uniform sampling of one-dimensional signals compatible with the classical Shannon-Whittaker theorem. The new approach is also considered in the context of image processing.

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1. Introduction

Sampling is an essential preliminary step in processing of any continuous signal by a digital computer. This step is the heart of any digital processing of any (presumably continues) data/signal. Undersampling causes distortions due to the aliasing of the post processed sampled data. Oversampling, on the other hand, results in time and memory consuming computational processes which, at the very least, slows down the analysis process. It is therefore important to have a measure which is instrumental in determining what is the optimal sampling rate. For 1-dimensional signals such a measure exists, and, consequently, the optimal sampling rate is given by the fundamental sampling theorem of Shannon, theorem that yielded the foundation of information theory and led technology into the digital era. Shannon's theorem asserts that a signal can be perfectly reconstructed from its samples, given that the signal is band limited within some bound on its highest frequency. Ever since the proof of Shannon's theorem in the late 1940's, deducing a similar sampling theorem for higher dimensional signals is an essential problem related to various aspects of signal processing. This is further emphasized by the vast interest and applications of image processing and the growing need of fast yet accurate techniques for processing high dimensional data such as medical and satellite images.

In this report we wish to present new sampling theorems for manifolds of dimensions ≥ 2. These theorems are derived form fundamental studies in three areas of mathematics: differential topology, differential geometry and quasi-regular maps. Both classical and recent results in these areas are combined to yield a rigorous and comprehensive sampling theory for such manifolds.

We first present sampling theorems for surfaces (dimension 2) and then for higher dimensional manifolds. In the case of surfaces, we account for those surfaces that are at least $\mathcal{C}^2$, with bounded principal curvatures. This condition is, in a way, analogous to band limited signals in the case of one dimension (the classical Shannon sampling theorem). We then present sampling theorem for surfaces that are not $\mathcal{C}^2$. Afterwards we proceed to present sampling theorems for manifolds of dimension ≥ 3. The main reasons for such a differentiated treatment of surfaces and of higher dimensional manifolds is that the geometry of surfaces is much more intuitive than that of manifolds of dimension ≥ 2. Therefore, the main ideas behind the given theorems, are more accessible in this case. Apart from this, there is also a more deeper reason to distinguish between surfaces and higher dimensional manifolds: it is rooted in the geometrical richness of manifolds of dimensions ≥ 3, as opposed to surfaces. This richness reflects on the present work through the variety of curvature measures applicable to manifolds of dimensions > 2, as compared with surfaces. In higher dimensions we can consider scalar, sectional and Ricci curvatures, each of which with its specific geometrical meaning and computational considerations. As a result, and due to the crucial role curvature plays in this whole work, when setting sampling theorems for high dimensional manifolds we first need to have a good understanding of which of the possible curvatures we would like to use.

Recently a surge in the study of fat triangulations and manifold sampling in Computational Geometry, Computer Graphics and their related fields has generated a vast amount of publications (see [AB], [BCER], [CDR1], [CDR2], [CDRR], [E], [LT], [Mee], [PA], to name just a few). However, our results are vastly extending the class of manifolds for which fat meshes and "good" samplings exist. Moreover, the method proposed herein, appertains to the vast corpus of Differential Topology and Geometry and hence inherits the mathematical correctness and conciseness of the classical apparatus.

The report is organized as follows: In Section 2 we review some preliminary results relevant to the theory. We first review aspects of sampling theory. Afterwards we present the most relevant results from differential topology that play a central role in the theoretical background of our theory. More precisely, we focus on PL-approximation of smooth manifolds and its counterpart of smoothing of PL-manifolds. These results are directly adopted in order to show that our proposed reconstruction method is accurate and also to overcome the problem of non-smoothness. In Section 3 we will provide some additional background results, combining both differential geometry and the theory of quasi-regular mappings. These results, both classical such as those of S.S Cairns, starting from the early 1930‘s, and new, due to K. Peltonen from the 1990’s and to E. Saucan from 2000’s will be later adopted to give the existence of sampling for manifolds. In Section 4 we show how to apply the surfaces/manifolds sampling results to obtain a new, geometric proof of the classical Shannon sampling theorem, and also to the analysis of images. In the final section we examine some delicate aspects of our study, and discuss extensions of this work, relating both to geometric aspects of sampling, as well as to its relationship with classical sampling theory.

2. Preliminaries

2.1. **Shannon's Theorem and Sampling Theory.** We do not present here in detail the classical Whittaker-Kotelnikov-Nyquist-Shannon theorem (or Shannon's theorem, for short), but restrict ourselves to bringing the following version:

Theorem 2.1. *Let $f \in L^2(\mathbb{R})$, such that supp$(\hat{f}) \subseteq [-\pi,\pi]$, where $\hat{f}$ denotes the Fourier transform of f. Then*

$$f(x) = \sum_{t\in\mathbb{Z}} f(t)\,\mathrm{sinc}(x-t)\,;$$

where $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$.

The classical Shannon theorem pertains to *band limited signals.* Various generalizations of it were proposed (see [PA], [SZ], [AG], [BD], [AF], [UZ], amongst others).

We conclude this brief overview of Shannon's theorem with a few remarks relevant to the sequel:

(1) Mathematically, Shannon's theorem belongs to the field of interpolation (see, e.g. [BD], [SZ]). The main – and surprising – fact is that linear interpolation (the *secant approximation*, to be more precise – see Sections 2.2, 2.3 below) basically suffices to faithfully reconstruct manifolds.
(2) The quest for reproducing kernels is natural. However, not every family of functions admits such kernels (see [Ar], pp. 380-381). Moreover, surfaces (and a fortiori higher dimensional manifolds) are geometric objects with far "wilder" smoothness properties than signals, as usually considered (see, e.g. [HSV]). Therefore, a general theory of reproducing kernels for manifolds seems difficult and remains, at this stage, yet to be developed.
(3) Shannon's theorem is equivalent to a variety of seemingly unrelated results in classical Mathematical Analysis (see [HSV]). It is plausible, and indeed probable, that precisely these variations on the given them can shed some more light on all the aspects of a sampling theory for surfaces.

2.2. **Background on PL-Topology.** We first recall a few classical definitions and notations:

Definition 2.2. *Let $a_0, \ldots, a_m \in \mathbb{R}^n$. $a_0, \ldots, a_m$ are called independent iff the vectors $v_i = a_i - a_0\, i1, ..., m$ are linearly independent.*

The set $\sigma = a_0 a_1 \ldots a_m = \{x = \alpha_i a_i \,|\, \alpha_i \geq 0, \sum \alpha_i = 1\}$ is called the m-simplex spanned by $a_0, \ldots, a_m$. The points $a_0, \ldots, a_m$ are called the vertices of σ.

The numbers α_i are called the barycentric coordinates of σ. The point $\tilde{\sigma} = \sum \frac{\alpha_i}{m+1}$ is called the barycenter of σ.

If $\{a_0, \ldots, a_k\} \subseteq \{a_0, \ldots, a_m\}$, then $\tau = a_0 \ldots a_k$ is called a face of σ, and we write $\tau < \sigma$.

Definition 2.3. *Let $A, B \subset \mathbb{R}^n$. We define the join $A * B$ of A and B as $A * B = \{\alpha a + \beta b \,|\, a \in A, b \in B\,;\, \alpha, \beta \geq 0,\ \alpha + \beta = 1\}$. If $A = \{a\}$, then $A * B$ is called the cone with vertex a and base B.*

Definition 2.4. *A collection K of simplices is called a simplicial complex if*

(1) *If* $\tau < \sigma$, *then* $\tau \in K$.
(2) *Let* $\sigma_1, \sigma_2 \in K$ *and let* $\tau = \sigma_1 \cap \sigma_2$. *Then* $\tau < \sigma_1, \tau < \sigma_1$.
(3) K *is locally finite.*

$|K| = \bigcup_{\sigma \in K} \sigma$ *is called the underlying polyhedron (or polytope) of* K.

Definition 2.5. *A complex* K' *is called a subdivision of* K *iff*

(1) $K' \subset K$;
(2) *if* $\tau \in K'$, *then there exists* $\sigma \in K$ *such that* $\tau \subseteq \sigma$.

If K' *is a subdivision of* K *we denote it by* $K' \lhd K$.

Let K *be a simplicial complex and let* $L \subset K$. *If* L *is a simplicial complex, then it is called a subcomplex of* K.

Definition 2.6. *Let* $a \in |K|$. *Then*

$$St(a, K) = \bigcup_{\substack{a \in \sigma \\ \sigma \in K}} \sigma$$

is called the star of $a \in K$.

If $S \subset K$, *then we define:* $St(S, K) = \bigcup_{a \in S} St(a, K)$.

Definition 2.7. *Let* $\sigma = a_0 a_1 \dots a_m$ *and let* $f : \sigma \to \mathbb{R}^p$. *The map* f *is called linear iff for any* $x = \sum \alpha_i a_i \in \sigma$, *it holds that* $f(x) = \sum \alpha_i f(a_i)$.

Let K, L *be complexes, and let* $f : |K| \to |L|$. *Then* f *is called linear (relative to* K *and* L*) iff for any* $\sigma \in K$, $\tau = f(\sigma) \in L$.

The map $f : K \to L$ *is called piecewise linear (PL) iff there exists a subdivision* K' *of* K *such that* $f : K' \to L$ *is linear.*

If (i) $f : K \to L$ *is a homeomorphism of* $|K|$ *onto* $|L|$, *(ii)* $f|_\sigma$ *is linear and (iii)* $\tau = f|_\sigma \in L$, *for any* $\sigma \in K$, *then* f *is called a linear homeomorphism.*

Definition 2.8. *A cell* γ *is a bounded subset of* $\mathbb{R}^n$ *defined by:*

$$\gamma = \{x \in \mathbb{R}^n \mid \sum_j \alpha_{ij} x_j \geq \beta_i;\ i = 1, \dots, p\},$$

for some constants $\alpha_{i,j}$ *and* β_i.

The dimension m *of* γ *is defined as* $\min\{dim\,\Pi \mid \gamma \subset \Pi, \Pi$ *a hyperplane in* $\mathbb{R}^n\}$.

Let γ *be an* m*-dimensional cell. The* $(m-1)$*-cells* β_j *of* $\partial\gamma$ *are called its* $(m-1)$*-faces, the* $(m-2)$*-faces of each* β_j *are called the* $(m-2)$*-faces of* γ, *etc. By convention* $\emptyset$ *and* γ *are also faces of* γ.

A cell complex is defined in the same manner as a simplicial complex, more exactly, a cell complex K is a collection of cells that satisfy conditions 1.–3. of Definition 2.4.

Subcomplexes are also defined analogous to the simplicial case. In particular, the q skeleton K^q of K, $K^q = \{\gamma \,|\, \gamma \in K,\ dim\,\gamma \leq q\}$ is a subcomplex of K.

Lemma 2.9. *Let K be cell complex. Then K has a simplicial subdivision.*

Proof. See [Mun], Lemma 7.8. □

We next define the concept of embedding for complexes, but first we need some basic definitions:

Definition 2.10. *Let K be a simplicial complex.*

(1) *$f : |K| \rightarrow M^n$ is $\mathcal{C}^r$ differentiable (relative to $|K|$) iff $f|_\sigma \in \mathcal{C}^r(\sigma)$, for any simplex $\sigma \in K$.*
(2) *$f : |K| \rightarrow M^n$ is non-degenerate iff $rank(f|_\sigma) = dim(\sigma)$, for any simplex $\sigma \in K$.*

Definition 2.11. *Let σ be a simplex, and let $f : \sigma \rightarrow \mathbb{R}^n$, $f \in \mathcal{C}^r$. For $a \in \sigma$ we define $df_a : \sigma \rightarrow \mathbb{R}^n$ as follows: $df_a(x) = Df(a) \cdot (x - a)$, where $Df(a)$ denotes the formal derivative $Df(a) = (\partial f_i / \partial x^j)_{1 \leq i,j \leq n}$, computed with respect to some orthogonal coordinate system contained in $\Pi(\sigma)$, where $\Pi(\sigma)$ is the hyperplane determined by σ. The map $df_a : \sigma \rightarrow \mathbb{R}^n$ does not depend upon the choice of this coordinate system.*

Note that $df_a|_{\sigma \cap \tau}$ is well defined, for any $\sigma, \tau \in \overline{St}(a, K)$. Therefore, the map $df_a : \overline{St}(a, K) \rightarrow \mathbb{R}^n$ is well-defined and continuous, and it is called – in analogy to the case of differentiable manifolds – the differential of f.

Remark 2.12. *In contrast to the differential case, the tangent space $T_{f(p)}(M^n)$ is a union of of polyhedral tangent cones, It, therefore, does not possess a natural vector space structure (see* [Th]*, p. 196).*

Definition 2.13. *Let K be a simplicial complex, let M^n be a $\mathcal{C}^r$ submanifold of $\mathbb{R}^N$, and let $f : K \rightarrow M^n$ be a $\mathcal{C}^r$ map. Then, f is called*

(1) *an immersion, iff $df_\sigma : \overline{St}(\sigma, K) \rightarrow \mathbb{R}^n$ is injective for each and every $\sigma \in K$;*

(2) *an embedding, iff it is an immersion and a homeomorphism on the image f(K);*
(3) *a* $\mathcal{C}^r$ *triangulation, iff it is an embedding such that* $f(K) = M^n$.

Remark 2.14. *If the class of the map* f *is not relevant,* f *will be called simply a triangulation.*

Definition 2.15. *Let* $f : K \to \mathbb{R}^n$ *be a* $\mathcal{C}^r$ *map, and let* $\delta : K \to \mathbb{R}^*_+$ *be a continuous function. Then* $g : |K| \to \mathbb{R}^n$ *is called a* δ*-approximation to* f *iff:*
(i) There exists a subdivision K' *of* K *such that* $g \in \mathcal{C}^r(K', \mathbb{R}^n)$ *;*
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$ *, for any* $x \in |K|$ *;*
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{eucl}(x, a)$ *, for any* $a \in |K|$ *and for all* $x \in \overline{St}(a, K')$.

Definition 2.16. *Let* K' *be a subdivision of* K*,* $U = \overset{\circ}{U}$*, and let* $f \in \mathcal{C}^r(K, \mathbb{R}^n)$*,* $g \in \mathcal{C}^r(K', \mathbb{R}^n)$*.* g *is called a* δ*-approximation of* f *(on* U*) iff conditions (ii) and (iii) of Definition 2.6. hold for any* $a \in U$.

The most natural and intuitive δ-approximation to a given mapping f is the *secant map induced by* f:

Definition 2.17. *Let* $f \in \mathcal{C}^r(K)$ *and let* s *be a simplex,* $s < \sigma \in K$*. Then the linear map:* $L_s : s \to \mathbb{R}^n$*, defined by* $L_s(v) = f(v)$ *where* v *is a vertex of* s*, is called the* secant map induced by f.

2.3. **PL-Approximation of Smooth Manifolds.** We show in this section that the apparent "naive" secant approximation of surfaces (and higher dimensional manifolds) represents a good approximation, both in distances and in angles, provided that the secant approximation induced by a triangulations that satisfies a certain un-degeneracy condition called "fatness" (or "thickness").

2.3.1. *Fat Triangulations.* We begin this section with the following informal, intuitive definition:

Definition 2.18. *A triangle in* $\mathbb{R}^2$ *is called "fat" (or* φ*-fat, to be more precise) iff all its angles are larger than a prescribed value* φ.

In other words, fat triangles are those that do not "deviate" to much from being equiangular (regular), hence fat triangles are not too "slim". This can be defined more formally by requiring that the the ratio $\frac{r}{R} \geq \varphi$, for some $\varphi > 0$, where r denotes the radius of the inscribed circle of τ (*inradius*) and R denotes the radius of the circumscribed circle of τ (*circumradius*). (See Fig. 2.1 below.)

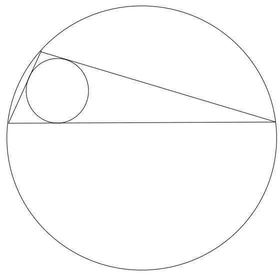

FIGURE 1. Thin triangle - Peltonen's definition.

One can easily check, by elementary methods, that the angle-condition and the radii condition are equivalent. Even if, perhaps, more intuitive, the angle condition is more difficult to properly formulate in higher dimension, therefore we opt for the following formal definition of fatness:

Definition 2.19. *A k-simplex $\tau \subset \mathbb{R}^n$, $2 \leq k \leq n$, is φ-fat if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$. A triangulation of a submanifold of $\mathbb{R}^n$, $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is φ-fat if all its simplices are φ-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is fat if there exists $\varphi \geq 0$ such that all its simplices are φ-fat; for any $i \in \mathbf{I}$.*

Proposition 2.20 ([CMS])**.** *There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma < \tau} \measuredangle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{2.1}$$

and

$$\varphi(\tau) \leq \frac{Vol_j(\sigma)}{diam^j\, \sigma} \leq c(k) \cdot \varphi(\tau)\,, \tag{2.2}$$

where φ denotes the fatness of the simplex τ, $\measuredangle(\tau, \sigma)$ denotes the (internal) dihedral angle of the face $\sigma < \tau$ and $Vol_j(\sigma)$; $diam\, \sigma$ stand for the Euclidian

j-volume and the diameter of σ respectively. (If $dim\,\sigma = 0$, then $Vol_j(\sigma) = 1$, by convention.)

Condition 2.1 is just the expression of fatness as a function of dihedral angles in all dimensions, while Condition 2.2 expresses fatness as given by "large area/diameter". Diameter is important since fatness is independent of scale.

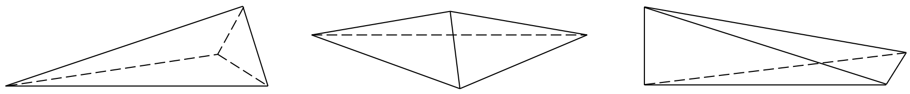

FIGURE 2. Slivers

One can gain some insight into the equivalence of all the definitions above, by analyzing the 3-dimensional examples below. (See [E] for a complete classification of "slim" triangles in dimensions 2 and 3.)

Remark 2.21. *The above definition is the one introduced in* [Pe] *and we employ it, as already noted, mainly for briefness. For other, equivalent definitions of fatness see* [Ca1], [Ca2], [CMS], *(based upon angles),* [Mun] *(the most similar to the one given above – see below) and* [Tu] *(based upon area/diameter).*

2.3.2. *The Main Result.* While, by Proposition 2.20, we could have employed any of the equivalent definitions of fatness, the computations in the proposition below are performed for

$$\varphi(\sigma) = \frac{r(\sigma)}{diam(\sigma)}\,;$$

(where the notations are as above).

Proposition 2.22 ([Mun], Lemma 9.3)**.** *Let $f : \sigma \to \mathbb{R}^n$ be of class $\mathcal{C}^k$. Then, for $\delta, f_0 > 0$, there exists $\varepsilon > 0$, such that, for any $\tau < \sigma$, such that $diam(\tau) < \varepsilon$ and such that $\varphi(\tau) > \varphi_0$, the secant map L_τ is a δ-approximation of $f|\tau$.*

Proof We first show that *(i)* $F_b(x) = f(b) + Df(b)\cdot(x-b)$, where b is the barycenter of σ, is a $\delta/2$-approximation to f on a sufficient small neigbourhood of b; then we prove that *(ii)* if $\tau < \sigma$ satisfies the conditions from the statement of the theorem, then L_σ is a $\delta/2$-approximation to F_b. This two assertions suffice to prove the theorem.

Proof of (i) Immediate from the definition of Df. We impose the additional requirement $||f(x) - F_b(x)||/||x-b|| < \delta\varphi_0/4$, for $||x-b|| < \varepsilon$, (Here $||\cdot||$ denotes the Euclidean norm.)

Before we proceed further we need the following result: Let $L, F : \tau \to \mathbb{R}^n$ be linear maps, s.t. $||L(x) - F(x)|| < c\,, \forall x \in \tau$. Then, $||DL(x)\cdot\mathbf{u} - DF(x)\cdot\mathbf{u}|| \leq c/r(\tau)$, ll $\mathbf{u}$ in the plane of τ, $||\mathbf{u}|| = 1$. (This follows immediately from *(i)* - for details see [CMS], p. 91.)

Proof of (ii) Let $v_0, ..., v_k$ be the vertices of τ, and let $x \in \tau, x = \sum \alpha_i v_i$. Then, by the linearity of L_s and F_s it follows that $L_s(x) = \sum \alpha_i L_s(v_i) = \sum \alpha_i f(v_i)$ and $F_b(x) = \sum \alpha_i F_b(v_i)$. Hence:

$$||L_s(x) - F_b(x)|| = \Big|\Big|\sum \alpha_i \frac{||f(x) - F_b(x)||}{||x-b||}\Big|\Big| \leq \max ||f(x) - F_b(x)||\,.$$

But $||f(x) - F_b(x)|| < \delta/2$ and $||L_s(x) - F_b(x)|| < \delta/2$, for all $||x-b|| < \varepsilon$. Moreover, $||f(x) - F_b(x)||/||x-b|| < \delta\varphi_0/4$, for all $||x-b|| < \varepsilon$, and, since $\varphi_0 \leq r(\tau)/diam(\tau)$, it follows that:

$$||L_s(x) - F_b(x)|| < \max ||v_i - b||\delta\varphi_0/4 \leq diam(\tau)\delta\varphi_0/4 \leq \delta\, r(\tau)/4\,.$$

This concludes the proof of *(ii)*, and, hence, of the proposition.

□

2.4. **Smoothing of Manifolds.** In this section we focus our attention on the problem of smoothing of manifold. That is, approximating a manifold of differentiability type $\mathcal{C}^r$, $r \geq 0$, by manifolds of type $\mathcal{C}^\infty$. Of special interest is the case where $r = 0$. Later, when posing our sampling theorem we will make a use of this in two respects. One of them will be as a post-processing step where, after reproducing a PL surface out of the samples, we can smoothen it to get a smooth reproduced surface. Another aspect in which smoothing is useful is as a pre-processing step, when we wish to extend the sampling theorem to surfaces which are not necessarily smooth. Smoothing will take place followed by sampling of the smoothed surface, yielding a sampling for the non-smooth one as well. As a major reference to this we use [Mun], Chap 4. Similar results can also be found in [HM] and others.

Remark 2.23. *The question of smoothing of manifolds (and hence its solution) is much less obvious when an additional requirement of "geometric" approximation is added. s.a. as a "good" curvature (Gauss, mean, etc.) convergence is also imposed. For the proof of this in the case of surfaces refer to* [BH].

2.4.1. *Partition of Unity.* Smoothing will be done while using a smoothing convolution kernel. This kernel will be a $\mathcal{C}^\infty$ function. Before introducing this kernel we present the vary basic idea of partition of unity. At a first glance it seems unrelated, but in fact this is the core of the smoothing process.

Lemma 2.24. *For every* $0 < \epsilon < 1$ *there exists a* $\mathcal{C}^\infty$ *function* $\psi_1 : \mathbb{R} \to [0,\ 1]$ *such that,* $\psi_1 \equiv 0$ *for* $|x| \geq 1$ *and* $\psi_1 = 1$ *for* $|x| \leq (1-\epsilon)$*. Such a function is called* partition of unity.

Let $c^n(\epsilon)$ be the ϵ cube around the origin in $\mathbb{R}^n$ (i.e. $X \in \mathbb{R}^n$; $-\epsilon \leq x_i \leq \epsilon\,, \forall i = 1, ..., n$). We can use the above partition of unity in order to obtain a non-negative $\mathcal{C}^\infty$ function, ψ, on $\mathbb{R}^n$ such that $\psi = 1$ on $c^n(\epsilon)$ and $\psi \equiv 0$ outside $c^n(1)$. Define $\psi(x_1, ..., x_n) = \psi_1(x_1) \cdot \psi_1(x_2) \cdots \psi_1(x_n)$.

Remark 2.25. *Let* U *be some open subset of* $\mathbb{R}^n$ *and let* $K \subset U$ *be some compact proper subset of* U*. Partition of unity enables us to define a non-negative* $\mathcal{C}^\infty$ *function which equals* 1 *on* K*, equals* 0 *outside* $\overline{U}$ *and is monotonically decreasing as we approach* ∂U*. This powerful property will enable us to build the desired smoothing kernel that will be used later.*

2.4.2. *Smoothing of Manifolds.* We now introduce the main theorem regarding smoothing of PL-manifolds.

Theorem 2.26 ([Mun])**.** *Let* M *be a* $\mathcal{C}^r$ *manifold,* $0 \leq r < \infty$*, and* $f_0 : M \to \mathbb{R}^k$ *a* $\mathcal{C}^r$ *embedding. Then, there exists a* $\mathcal{C}^\infty$ *embedding* $f_1 : M \to \mathbb{R}^k$ *which is a* δ*-approximation of* f_0*.*

The above theorem is a consequence of the following lemma concerning smoothing of maps:

Lemma 2.27. [Mun] *Let* U *be an open subset of* $\mathbb{R}^m$ *or* $\mathbb{H}^m$*. Let* A *be a compact subset of an open set* V *such that* $\overline{V} \subset U$*. Let* $f_0 : U \to \mathbb{R}^n$

be a $\mathcal{C}^r$ map, $0 \leq r$. Let δ be a positive number. Then there exists a map $f_1 : U \to \mathbb{R}^n$ such that

(1) *f_1 is $\mathcal{C}^\infty$ on A.*
(2) *$f_1 = f_0$ outside V.*
(3) *f_1 is a δ-approximation of f_0*
(4) *f_1 is $\mathcal{C}^r$-homotopic to f_0 via a homotopy f_t satisfying (2) and (3) above.*

Proof We may assume that $\overline{V}$ is compact and that U is open in $\mathbb{R}^n$, (if U is open $\mathbb{H}^n$ we can extend f_0 in a neighborhood of the boundary $\partial\mathbb{H}^n$). Let W be an open set containing A such that $\overline{W} \subset V$. Let $\psi : \mathbb{R}^m \to \mathbb{R}^+$ be a $\mathcal{C}^\infty$ map, such that $\psi = 1$ on A and $\psi \equiv 0$ outside W.

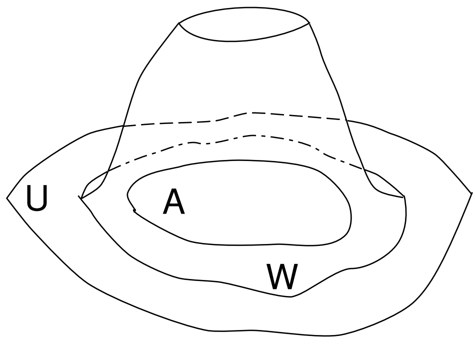

FIGURE 3. Partition of unity on A.

Define $g = \psi \cdot f$, then $g : \mathbb{R}^m \to \mathbb{R}^n$ satisfying $g = f$ on A and $g \equiv 0$ outside W. Inside A, g is of the same differentiability type as f whereas outside W it is $\mathcal{C}^\infty$.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which is positive on $int(c^m(\epsilon))$ and vanishes outside $\overline{c^m(\epsilon)}$. ϵ is some positive number yet to be defined. Further assume that $\int_{\mathbb{R}^m} \varphi = 1$. Such a function can be obtained by, say, taking a partition of unity, φ_0, supported on some open subset of $c^m(\epsilon)$ and then factorizing it by $\int_{\mathbb{R}^m} \varphi_0$. Such a function will be called a convolution kernel.

For $x \in \mathbb{R}^m$, define

$$h(x) = \int_{c^m(\epsilon)} \varphi(y) g(x+y) dy.$$

Choose ϵ so that $\sqrt{m}\epsilon < d(W, \mathcal{R}^m \backslash V)$. then $h \equiv 0$ outside V.

Let

$$f_1(x) = f_0(x) \cdot (1 - \psi(x)) + h(x).$$

Since ψ and h vanishes outside V conclusion (2) of the lemma is fulfilled. Inside A we have $f_1(x) = h(x)$. Since

$$h = \int_{c^m(\epsilon)} \varphi(y) g(x+y) dy = \int_{W + c^m(\epsilon)} \varphi(z-x) g(z) dz = \int_{\mathbb{R}^m} \varphi(z-x) g(z) dz;$$

and φ is $\mathcal{C}^\infty$, h is also $\mathcal{C}^\infty$ inside W and in particular on A, thus fulfilling conclusion (1).

By its definition $f_1 = f_0 + (h - g)$, so we have to choose ϵ small enough so that h is a δ-approximation to g. By the mean value theorem (h is actually some weighted mean of g), we have :

$$h^i(x) = g^i(x + y^i);$$

$$\frac{\partial h^i}{\partial x^j} = \frac{\partial g^i(x + y^{ij})}{\partial x^j};$$

where y^i and y^{ij} are points in $c^m(\epsilon)$. We only have to take care that ϵ is so small that

$$|g^i(x) - g^i(x')| < \delta;$$

and

$$|\frac{\partial g^i}{\partial x^j}(x) - \frac{\partial g^i}{\partial x^j}(x')| < \delta;$$

for

$$|x - x'| < \epsilon;$$

this completes part (3).

Finally, we construct the desired homotopy between f_0 and f_1. Before doing so let us stress the meaning of having such a homotopy. It is, that we can get from f_0 to f_1 by a sequence of continuous deformations.

Let $\alpha(t)$ be a monotonic $\mathcal{C}^\infty$ function such that, $\alpha = 0$ for $0 \leq t \leq 1/3$ and $\alpha = 1$ for $2/3 \leq t \leq 1$.

Put

$$f_t(x) = \alpha(t) f_1(x) + (1 - \alpha(t)) f_0(x).$$

Then, outside V $f_t \equiv f_0$ and f_t is a $\mathcal{C}^r$ homotopy between f_0 and f_1 satisfying,

$$|f_t - f_0| < \delta,$$

and whenever f_0 is differentiable,

$$|f_t - f_0| < \delta.$$

This completes the proof.

$\square$

3. Fat Triangulation

3.1. **Theorems.** In this section we review, in chronological order, existence theorems dealing with fat triangulations on manifolds. For detailed proofs see the original papers.

Theorem 3.1 (Cairns, [Ca3])**.** *Every compact $\mathcal{C}^2$ Riemannian manifold admits a fat triangulation.*

Remark 3.2. *For a somewhat more general result, the proof of which employs less elementary methods than the one we sketch below, and that does not generalize to open manifolds, see* [Ca1], [Ca2].

Theorem 3.3 (Peltonen, [Pe])**.** *Every open (unbounded) $\mathcal{C}^\infty$ Riemannian manifold admits a fat triangulation.*

Theorem 3.4 (Saucan, [S2])**.** *Let M^n be an n-dimensional $\mathcal{C}^1$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then, any uniformly fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Remark 3.5. *Theorem above holds also when the compactness condition of the boundary components is replaced by the condition that ∂M^n is endowed with a fat triangulation $\mathcal{T}$ such that $\inf\limits_{\sigma \in \mathcal{T}} diam\, \sigma > 0$.*

Corollary 3.6. *M^n be as above admits a fat triangulation.*

Corollary 3.7. *Let M^n be an n-dimensional, $n \leq 4$ (resp. $n \leq 3$), PL (resp. topological) connected manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

3.2. **Methods.**

3.2.1. *Background.* Let us establish first some notations: Let M^n denote an n-dimensional complete Riemannian manifold, and let M^n be isometrically embedded into $\mathbb{R}^\nu$ ("ν"-s existence is guaranteed by Nash's Theorem (see, e.g. [Pe], [Spi5]).

Let $\mathbb{B}^\nu(x,r) = \{y \in \mathbb{R}^\nu \,|\, d_{eucl} < r\}$; $\partial\mathbb{B}^\nu(x,r) = \mathbb{S}^{\nu-1}(x,r)$. If $x \in M^n$, let $\sigma^n(x,r) = M^n \cap \mathbb{B}^\nu(x,r)$, $\beta^n(x,r) = exp_x\big(\mathbb{B}^n(0,r)\big)$, where: exp_x denotes the exponential map: $exp_x : T_x(M^n) \to M^n$ and where $\mathbb{B}^n(0,r) \subset T_x\big(M^n\big)$, $\mathbb{B}^n(0,r) = \{y \in \mathbb{R}^n \,|\, d_{eucl}(y,0) < r\}$.

Remark 3.8. *Neither of the following (homeomorphisms) is guaranteed:*

(1) $\sigma^n(x,r) \simeq \mathbb{B}^n(0,r)$
(2) $\beta^n(x,r) \simeq \mathbb{B}^n(0,r)$.

Also, let us give the following definitions, which generalize in a straightforward manner the classical ones used for surfaces in $\mathbb{R}^3$:

Definition 3.9. (1) $\mathbb{S}^{\nu-1}(x,r)$ *is* tangent *to* M^n *at* $x \in M^n$ *iff there exists* $\mathbb{S}^n(x,r) \subset \mathbb{S}^{\nu-1}(x,r)$, *s.t.* $T_x(\mathbb{S}^n(x,r)) \equiv T_x(M^n)$.

(2) *Let* $l \subset \mathbb{R}^\nu$ *be a line, then* l *is* secant *to* $X \subset M^n$ *iff* $|\, l \cap X| \geq 2$.

Definition 3.10. (1) $\mathbb{S}^{\nu-1}(x,\rho)$ *is an* osculatory sphere *at* $x \in M^n$ *iff:*

(a) $\mathbb{S}^{\nu-1}(x,\rho)$ *is tangential at* x;
and
(b) $\mathbb{B}^n(x,\rho) \cap M^n = \emptyset$.

(2) *Let* $X \subset M^n$. *Then*

$$\omega = \omega_X = \sup\{\rho > 0 \,|\, \mathbb{S}^{\nu-1}(x,\rho) \text{ osculatory at any } x \in X\}$$

is called the maximal osculatory radius *at* X.

Remark 3.11. (1) *There exists an osculatory sphere at any point of* M^n *(see* [Ca3] *).*

(2) *If* X *is compact, then* $\omega_X > 0$.

3.3. **The Classical Case.** In the compact case the method is to produce a point set $A \subseteq M^n$, that is maximal with respect to the following density condition:

$$d(a_1, a_2) \geq \eta\,, \tag{3.1}$$

where

$$\eta < \omega_M\,. \tag{3.2}$$

Remark 3.12. *In this special case* $\omega_M > 0$.

One makes use of the fact that for a compact manifold M^n we have $|A| < \aleph_0$, to construct the finite cell complex "cut out of M" by the ν-dimensional Dirichlet complex, whose (closed) cells are given by:

$$\bar{c}_k = \bar{c}^{\nu}_k = \{x \in \mathbb{R}^{\nu} \,|\, d_{eucl}(a_k, x) \leq d_{eucl}(a_i, x),\ a_i \in A\,,\ a_i \neq a_k\}, \tag{3.3}$$

i.e. the (closed) cell complex $\{\bar{\gamma}^n_k\}$, where:

$$\{\bar{\gamma}^n_k\} = \bar{\gamma}_k = \bar{c}_k \cap M^n \tag{3.4}$$

(see [Ca3], [Pe] (for details)).

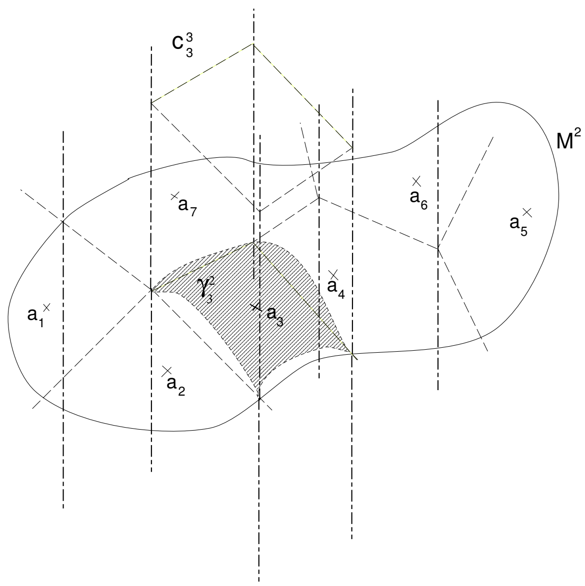

FIGURE 4. Dirichlet (Voronoi) cells – the compact surfaces case.

Remark 3.13. *A result equivalent to 3.1 is proven in* [AB]*, using basically the same method as Cairns' original one. However, the proof given in* [AB] *is more technical. Moreover, the seminal papers of Cairns are not referenced wherein.*

Remark 3.14. *Voronoi cell partitioning is also employed in "classical" sampling theory (see* [SZ]*).*

3.4. **Open Riemannian Manifolds.** In adapting Cairns' method to the non-compact case, one has to allow for some (obviously-required) modifications. We present below the main steps of Peltonen's proof:

The construction devised by Peltonen consists of two parts:

Part 1 This Proceeds in two steps:

Step A Comstruct an exhaustive set $\{E_i\}$ of M^n, generated by the pair (U_i, η_i), where:

(1) U_i is the relatively compact set $E_i \setminus \bar{E}_{i-1}$ and
(2) η_i is a number that controls the fatness of the simplices of the triangulation of E_i , constructed in Part 2, such that it will not differ to much on adjacent simplices, i.e.:
   (i) The sequence $(\eta_i)_{i\geq 1}$ descends to 0 ;
   (ii) $2\eta_i \geq \eta_{i-1}$.

The geometric feature that controls the sets E_i, U_i and the numbers η_i is the *maximal connectivity radius*:

Definition 3.15. *Let $U \subset M^n, U \neq \emptyset$, be a relatively compact set, and let $T = \bigcup_{x\in\bar{U}} \sigma(x,\omega_U)$. $\kappa_U = \max\{r \,|\, \sigma^n(x,r), connected\ for\ all\ s \leq \omega_U,\ x \in \bar{T}\}$, is called the* maximal connectivity radius *at U.*

Lemma 3.16.

$$\omega_U \leq \frac{\sqrt{3}}{3}\kappa_U \, . \tag{3.5}$$

The maximal connectivity radius and the maximal osculatory radius are interconnected by the following inequality:

Proof See Lemma 3.1, [Pe].

□

The numbers η_i are chosen such that they will satisfy the following condition:

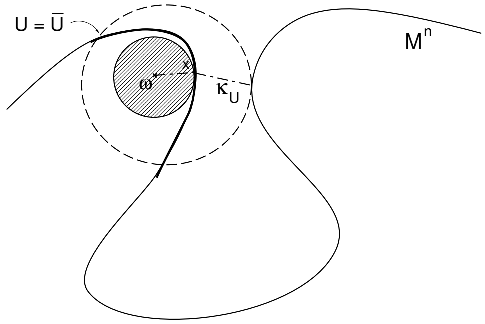

FIGURE 5. Maximal connectivity radius at U.

$$\eta_i \leq \frac{1}{4}\min_{i\geq 1}\{\omega_{\bar{U}_{i-1}}, \omega_{\bar{U}_i}, \omega_{\bar{U}_{i+1}}\}\,.$$

We now proceed with

Step B

(1) Produce a maximal set A, $|A| \leq \aleph_0$, s.t. $A \cap U_i$ satisfies:

(i) a density condition, namely:

$$d(a,b) \geq \eta_i/2\,, \forall i\,;$$

and

(ii) a "gluing" condition for U_i, U_{i+1}, i.e. their intersection is large enough.

Note that according to the density condition (i), the following holds:

For any i and for any $x \in \bar{U}_i$, $\exists a \in A$ s.t. $d(x,a) \leq \eta_i/2$.

(2) Prove that the Dirichlet complex $\{\bar{\gamma}_i\}$ defined by the sets A_i is a cell complex and every cell has a finite number of faces (so it can be triangulated in a standard manner).

Part 2 Consider first the dual complex Γ, and prove that it is a Euclidian simplicial complex with a "good" (i.e. proper) density. Project then Γ on M^n (using the normal map). Finally, prove that the resulting complex $\widetilde{\Gamma}$ can be triangulated by fat simplices. Indeed, the fatness of any n-dimensional simplex $\gamma \in \widetilde{\Gamma}$, contained in the set U_i is given by the following bound:

$$\frac{r_\gamma}{R_\gamma} \geq \frac{1}{2^{5n+1}} \frac{(n+2)^{\frac{n+1}{2}}}{(n+1)^{n+1}}\,. \tag{3.6}$$

Remark 3.17. *In the course of Peltonen's construction M^n is presumed to be isometrically embedded in some $\mathbb{R}^{N_1}$, where the existence of N_1 is guaranteed by Nash's Theorem (see* [Pe], [Spi5]*).*

3.5. **Manifolds With Boundary of Low-Differentiability.** The idea of the proof of Theorem 3.5 is to build first two fat triangulations: $\mathcal{T}_1$ of a product neighbourhood N of ∂M^n in M^n and $\mathcal{T}_2$ of $int\, M^n$ (its existence follows from Peltonen's result), and then to "mash" the two triangulations into a new triangulation $\mathcal{T}$, while retaining their fatness. While the mashing procedure of the two triangulations is basically the one developed in the original proof of Munkres' theorem, the triangulation of $\mathcal{T}_1$ has been modified, in order to ensure the fatness of the simplices of $\mathcal{T}_1$. More precisely we prove the following Theorem (see [S2]):

Theorem 3.18. *Let M^n be a $\mathcal{C}^r$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any fat $\mathcal{C}^r$-triangulation of ∂M^n can be extended to a $\mathcal{C}^r$-triangulation $\mathcal{T}$ of M^n, $1 \leq r \leq \infty$, the restriction of which to a product neighbourhood $\widetilde{K}_0 = \partial M^n \times I_0$ of ∂M^n in M^n is fat.*

In the general case we employ a method for fattening triangulations developed in [CMS]. The core of this methods resides in the following result:

Lemma 3.19. *(*[CMS]*, Lemma 6.3.) Let $\mathcal{T}_1, \mathcal{T}_2$ be two fat triangulations of open sets $U_1, U_2 \subset \mathbb{R}^n$, $B_r(0) \subseteq U_1 \cap U_2$, having common fatness $\geq \varphi_0$ and such that $d_1 = \inf_{\sigma_1 \in \mathcal{T}_1} diam\, \sigma_1 \leq d_2 = \inf_{\sigma_2 \in \mathcal{T}_2} diam\, \sigma_2$. Then there exist φ_0^*-fat triangulations $\mathcal{T}_1', \mathcal{T}_2'$, $\varphi_0^* = \varphi_0^*(\varphi_0)$, of open sets $V_1, V_2 \subseteq B_r(0)$, such that*

(1) $\mathcal{T}_i'\big|_{B_{r-8d_2}(0)} = \mathcal{T}_i\big|_{B_{r-8d_2}(0)}$, $i = 1, 2$;
(2) *$\mathcal{T}_1'$ and $\mathcal{T}_2'$ agree near their common boundary.*

Moreover:

(3) $\inf_{\sigma_1' \in \mathcal{T}_1'} diam\, \sigma_1' \leq 3d_1/2$, $\inf_{\sigma_2' \in \mathcal{T}_2'} diam\, \sigma_2' \leq d_2$.

Remark 3.20. *A more elementary, geometric approach in two and three dimensions was developed in* [S1].

Remark 3.21. *For the treatment of the same problem in the context of Computational Geometry, see e.g* [E], [PA].

Classical smoothing results are applied to derive Corollary 3.7 (see Section 2.2.3 and [Th]).

4. Sampling Theorems

4.1. **Surfaces.**

4.1.1. *Smooth Surfaces.*

Theorem 4.1. *Let Σ be a connected, non-necessarily compact smooth surface (i.e. of class $\mathcal{C}^k, k \geq 2$), with finitely many boundary components. Then, there exists a sampling scheme of Σ, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\Big(\frac{1}{k(p)}\Big)$, where $k(p) = \max\{|k_1|, |k_2|\}$, and k_1, k_2 are the principal curvatures of Σ, at the point $p \in \Sigma$.*

Proof The existence of the sampling scheme follows immediately from Corollary 3.6, where the sampling points are the vertices of the triangulation. The fact that the density is a function solely of $k = max\{|k_1|, |k_2|\}$ follows from the proof of Theorem 3.3 and from the fact that the osculatory radius $\omega_\gamma(p)$ at a point p of a curve γ equals $1/k_\gamma(p)$, where $k_\gamma(p)$ is the curvature of γ at p; hence that the maximal osculatory radius (of Σ) at p is: $\omega(p) = \max\{|k_1|, |k_2|\} = \max\{\frac{1}{\omega_1}, \frac{1}{\omega_2}\}$. (Here ω_1, ω_2 denote the minimal, respective maximal sectional osculatory radii at p.)

$\square$

Remark 4.2. *Since for unbounded surfaces it may well be that $\kappa \to \infty$, it follows that an infinite density of the sampling is possible. However, for practical implementations, where such cases are excluded, we have the following corollary:*

Corollary 4.3. *Let $\Sigma, \mathcal{D}$ be as above. Assume that there exists $k_0 > 0$, such that $k_0 \geq k(p)$, for all $p \in \Sigma$. Then there exists a sampling of Σ having uniformly bounded density.*

Proof The proof is deduced immediately from Theorem 4.1 above.

$\square$

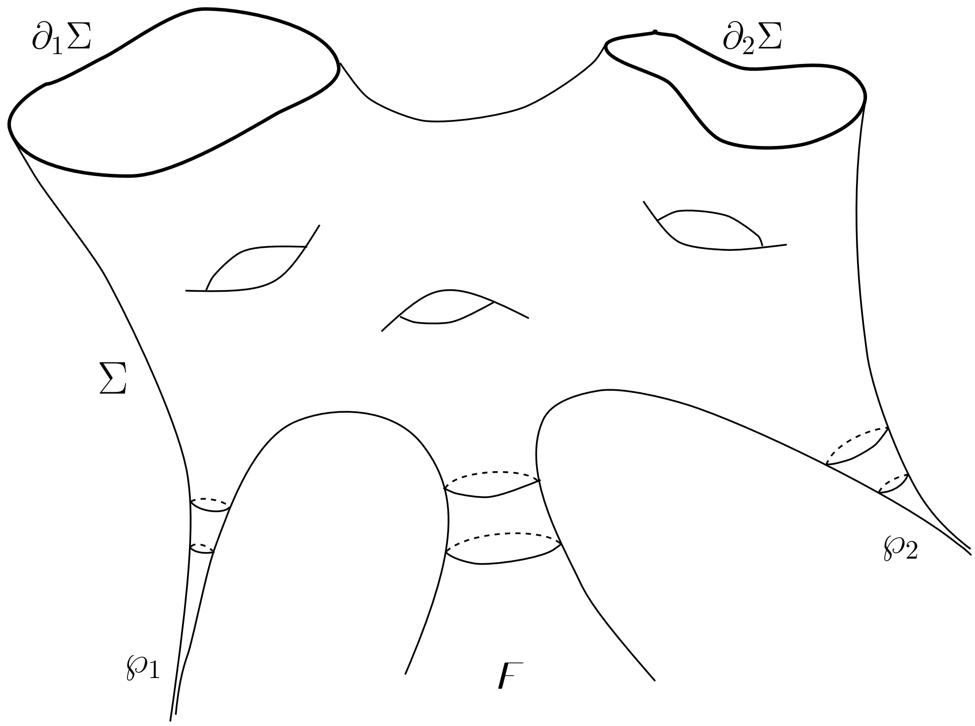

FIGURE 6. A non-compact surface Σ, with two boundary components $\partial_1\Sigma$ and $\partial_2\Sigma$. Observe the cusps $\wp_1$ and $\wp_2$ and the funnel F.

.

Corollary 4.4. *In the following cases there exist k_0 as in Corollary 4.3 above:*

(1) Σ *is compact.*
(2) *There exist H_1, H_2, K_1, K_2, such that $H_1 \leq H(p) \leq H_2$ and $K_1 \leq K(p) \leq K_2$, for any $p \in \Sigma$, where H, K denote the mean, respective Gauss curvature. (That is both mean and Gauss curvatures are* pinched.*)*
(3) *The Willmore integrand $W(p) = H^2(p) - K(p)$ and K (or H) are pinched.*

Proof

(1) It follows immediately from a compactness argument and from the continuity of the principal curvature functions.
(2) Since $K = k_1k_2$, $H = \frac{1}{2}(k_1+k_2)$, the bounds for K and H imply the desired one for k.
(3) Reasoning analogous to that of (ii), applies in the case of $W = \frac{1}{4}(k_1 - k_2)^2$.

This concludes the proof of the theorem.

□

Remark 4.5. *Condition (iii) on W is not only compact, it has the additional advantage that the Willmore energy $\int_\Sigma W dA$ (where dA represents the area element of Σ) is a conformal invariant of Σ. See* [ASZ] *for its importance in quasi-conformal mappings and their applications to imaging.*

4.1.2. *Non-Smooth Surfaces.* We begin by proposing the following definition:

Definition 4.6. *Let Σ be a (connected) surface of class $\mathcal{C}^0$, and let Σ_δ be a δ-approximation of Σ. A sampling of Σ_δ is called a δ-*sampling of Σ*.*

Theorem 4.7. *Let Σ be a connected, non-necessarily compact surface of class $\mathcal{C}^0$. Then, for any $\delta > 0$, there exists a δ-sampling of Σ, such that if $\Sigma_\delta \to \Sigma$, then $\mathcal{D}_\delta \to \mathcal{D}$, where $\mathcal{D}_\delta$ and $\mathcal{D}$ denote the densities of Σ_δ and Σ, respectively.*

Proof The proof is an immediate consequence of Theorem 3.5 and its proof and the methods exposed in Section 2.4. We adopt the sampling of some smooth δ-approximation of Σ.

□

Corollary 4.8. *Let Σ be a $\mathcal{C}^0$ surface with finitely many points at which Σ fails to be smooth. Then every δ-sampling of a smooth δ-approximation of Σ is in fact, a sampling of Σ apart of finitely many small neighborhoods of the points where Σ is not smooth.*

Proof From Lemma 2.27 and theorem 2.26 we have that any such δ-approximation, Σ_δ, coincides with Σ outside of finitely many such small neighborhoods.

□

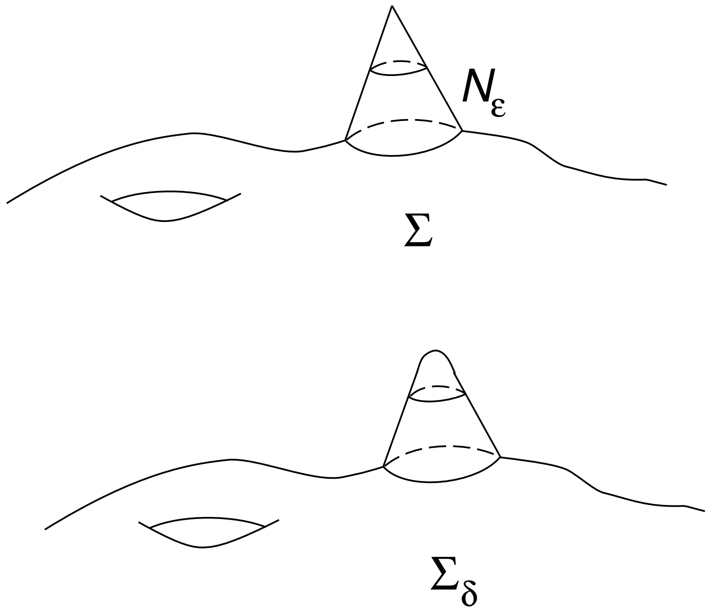

FIGURE 7. A neighbourhood $\mathcal{N}_\varepsilon$ such that $\Sigma \equiv \Sigma_\delta$ outside $\mathcal{N}_\varepsilon$

.

Remark 4.9. *Even in the case where $\Sigma_\delta \in \mathcal{C}^2$, and curvature measures exist for Σ (e.g. if Σ is a PL-surface), it does not follow that the curvature measures converge punctually to the curvatures of Σ (see* [BH] *and the discussion in Section 2.3.1). However, if Σ is compact and with empty boundary, the desired convergence property holds (*[BH]*).*

4.1.3. *Reconstruction.* We use the secant map as defined in Definition 2.17 in order to reproduce a PL-surface as a δ-approximation for the sampled surface. As said in the beginning of Section 2.3 we may now use smoothing in order to obtain a $\mathcal{C}^\infty$ approximation.

4.2. **Higher Dimensional Manifolds.** Theorem 4.1 and Corollary 4.3 have straightforward generalizations to any dimension:

Theorem 4.10. *Let $\Sigma^n, n \geq 3$ be a connected, not necessarily compact, smooth manifold, with finitely many compact boundary components. Then, there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) =$*

$\mathcal{D}\left(\frac{1}{k(p)}\right)$, *where* $k(p) = \max\{|k_1|, ..., |k_{2n}|\}$, *and where* $k_1, ..., k_{2n}$ *are the principal (normal) curvatures of* Σ^n, *at the point* $p \in \Sigma^n$.

Corollary 4.11. *Let* $\Sigma^n, \mathcal{D}$ *be as above. If there exists* $k_0 > 0$, *such that* $k(p) \leq k_0$, *for all* $p \in \Sigma^n$, *then there exists a sampling of* Σ^n *of finite density everywhere.*

Some of the conclusions of Corollary 4.4 also generalize. In particular we have:

Corollary 4.12. *If* Σ^n *is compact, then there exists a sampling of* Σ^n *having uniformly bounded density.*

However, more geometric conditions, such as those given in Corollary 4.4 are hard to impose in higher dimension, hence the study of such precise geometric constraints is left for further study.

5. Applications to Classical Sampling Theory

5.1. **1-Dimension: The Classical Shannon Sampling Theorem.**

Our approach and formalism lend themselves to the derivation of a geometric sampling theorem for 1-dimensional signals. We further show that band-limited signals considered in the context of the classical Shannon-Whittaker theorem require, indeed, a finite sampling.

Definition 5.1. *Let* $S(t)$ *be a* $\mathcal{C}^2$ *planar curve. Let* $\rho(S)$ *be its maximal absolute curvature. Without getting into the details of arc-length parameterizing one can think of the maximal absolute value of the second derivative as a sampling rate criterion. We will call* $\eta(S) = \rho(S)/2$ *the* curvature rate *of* S.

Theorem 5.2. *Let* S *is a* $\mathcal{C}^2$ *planar curve. Then it can be sampled in sampling rate* $\eta(S)$ *namely, the distance between each consecutive samples is* $\leq 1/\eta(S)$. *If* S *satisfies the condition that* $\rho(S)$ *is bounded, then the required sampling rate is finite.*

Corollary 5.3. *If* $S(t)$ *is a band-limited signal, then it necessitates a finite sampling rate (in any finite time interval) according to* $\eta(S)$.

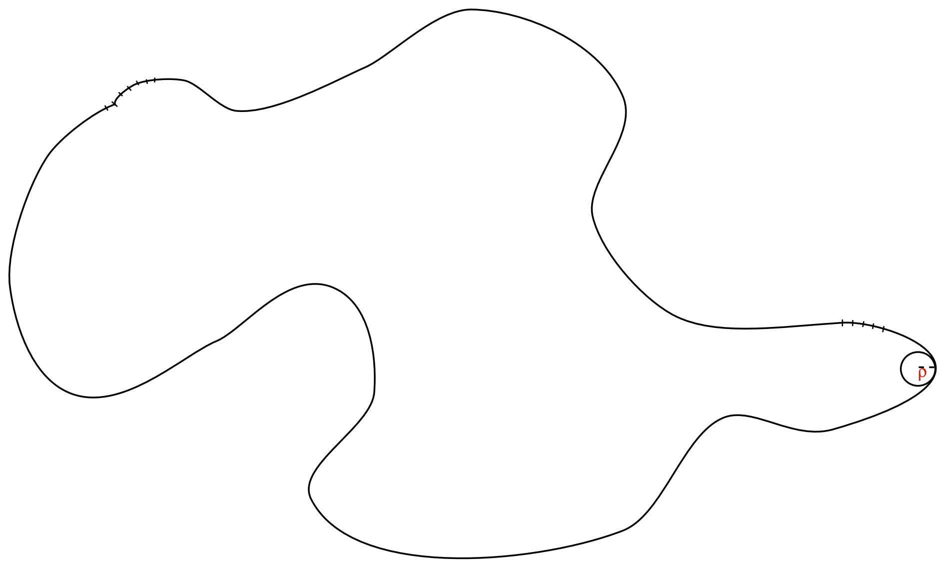

FIGURE 8. Sampling of a $\mathcal{C}^2$ curve: the sampling rate is $\eta = 1/r$, where r is the minimal radius of curvature.

Proof We view the signal's graph as a planar curve and use the theorem above. Again, omitting some technicalities of re-parameterizing, the proof amounts to showing that the second derivative of band-limited signals is everywhere bounded.

A Taylor expansion of such signals is given for instance in [MH]. In particular, for a band-limited signal $S(t)$, we have by Shannon-Whittaker:

$$S(t) = \sum_{-\infty}^{\infty} S(t_n) sinc(2W(t - t_n))\,,$$

and it is shown that its p-th derivative is given by:

$$S^p(t) = (2W)^p \sum_{-\infty}^{\infty} S(t_n)(\frac{d}{dt})^p sinc(2W(t - t_n))\,.$$

Calculations followed up by Marks and Hall ([MH]) also show that

$$(\frac{d}{dt})^p sinc(t) = \int_{-1/2}^{1/2} (2\pi i f)^p e^{2\pi i f} df = \frac{(-1)^p p!}{\pi t^{p+1}} [sin(\pi t) cos_{p/2}(\pi t) - cos(\pi t) sin_{(p-1)/2}(\pi t)]\,,$$

where,

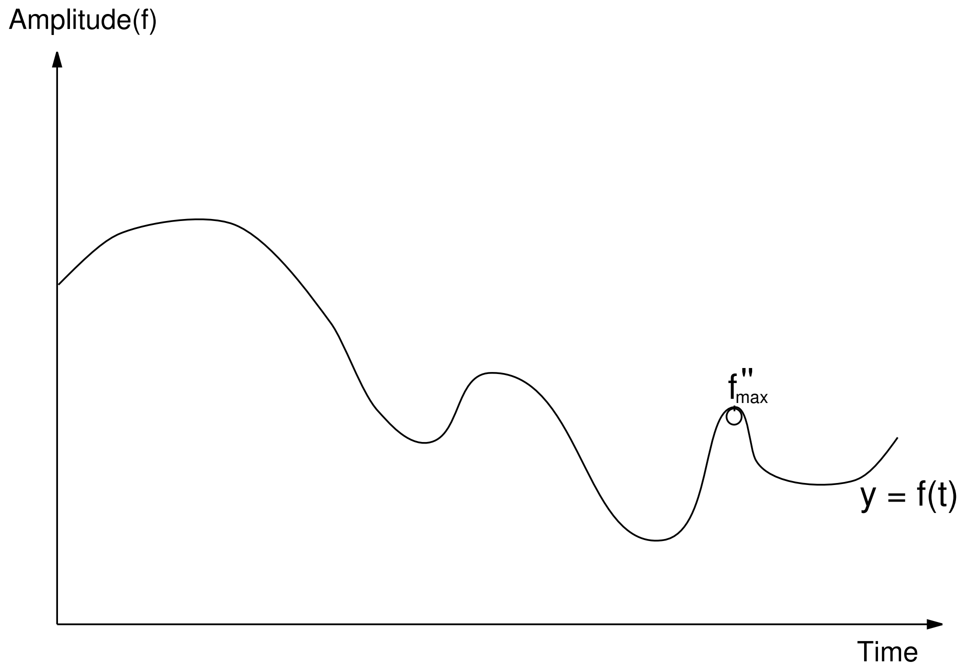

FIGURE 9. A band-limited signal $y = f(t)$.

$$cos_r(t) = \sum_{n=0}^{[r]} \frac{(-1)^n t^{2n}}{(2n)!}\,.$$

$$sin_r(t) = \sum_{n=0}^{[r]} \frac{(-1)^n t^{2n+1}}{(2n+1)!}\,.$$

The above terms have the following asymptotic behavior from which the boundedness of the second derivative (even for very large values of t), is evident.

$$(\frac{d}{dt})^p sinc(t) \to \begin{cases} (-1)^{p/2}\pi^p(sinc(t)); & p \text{ even} \\ (-1)^{(p-1)/2}\pi^p(\frac{cos(\pi t)}{\pi t}); & p \text{ odd}\,. \end{cases}$$

From the presentation above we conclude that a band-limited signal possesses a "geometric" sampling of finite rate.

□

Remark 5.4. *An approximation approach was already employed for "classical" sampling theory – see* [UZ].

5.2. **2-Dimensions: Images.** Perhaps the most direct application of the sampling theorem for surfaces is to the field of images, via "inpainting" (see, e.g. [SZ], p. 280). In this approach, images are viewed as parametrized surfaces $S = (u, v, f(u, v))$, where $(u, v) \in R$ – a rectangle of pixels, and $f(u, v) \in [0, 1]$ represents the shade of grey associated to the pixel (u, v).

Of course, if more attributes of the image are added, such as colors, luminosity, etc., then a higher dimensional manifold is obtained, and we may make again a recourse to the fitting sampling theorem.

In a completely analogous manner one can approach the problem of image compression (see, e.g. [SZ], p. 280): here the samples represent the coarse pixel set and the surface the fine pixel set.

6. Discussion

6.1. **Sampling.** More important, one honestly has to ask himself the following question: "What is a signal?"

If the answer to the question above is given in the classical context, i.e. if a signal is viewed as an element f of $L^2(\mathbb{R})$, such that $supp\,(\hat{f}) \subseteq [-\pi, \pi]$, where $\hat{f}$ denotes the Fourier transform of f, then our result does not hold. Indeed, we have the following counterexample:

Counterexample 6.1. *There exist band limited signals (as above) f such that:*

(i) $f \in L^2(\mathbb{R})$, $f'' \in L^\infty(\mathbb{R})$;

but

(ii) f'' is not bounded.

Therefore, our approach refers to a more "intuitive" or "blackboard" interpretation of signals.

On the other hand, it is more broad, in the sense that it applies to any PL curve in the plane, not only for graphs of function.

Remark 6.2. *A result similar to ours (yet technically weaker) was recently proved by G. Meenakshisundaram (*[Mee]*).*

6.2. **Simplex Fatness and Future Study.** Since the fatness of the triangulation of $int\,M^n$ depends, by Formula 3.6, only on the dimension n of the given manifold, and since by Lemma 3.19, the fatness of the mash of the tringulations of ∂M^n and $int\, M^n$ is a function solely on the fatness of the given triangulation (and hence upon the dimension n), it follows that a lower bound for the fatness of any triangulations is achieved.

Remark 6.3. *The existence of a lower bound for the fatness of the simplices ensures the existence of lower bounds for the dilatation and distortion of quasi-conformal and quasi-isometric representations of the manifolds (see* [ASZ]*,* [S2]*). Such representations are relevant in Medical Imaging (see* [ASZZ]*).*

However, since the bound given by Formula 3.6 is achieved via the specific construction of [Pe], the following question arises naturally:

Question 1. *Is the lower bound of Formula 3.6 the lowest possible?*

The answer to the question above seems to be negative, since Peltonen's construction depends upon the specific isometric embedding employed.

More important, the diameters of the simplices obtained in our construction (i.e. the *mesh* of the triangulation) are a function of the curvature radii, hence an *extrinsic* constraint, hence again strongly dependent upon the embedding in higher dimensional Euclidean space. This fact immediately generates the following question:

Question 2. *Does the Nash embedding technique impose any restrictions upon the curvature radii?*

Remark 6.4. *For further problems related to the quality of the obtained triangulation and its relevance to he theory of quasiregular mappings, see* [S3].

We conclude with the following remarks and suggestions for further study:

We have obtained in Corollary 4.4 week intrinsic condition for the existence of fat triangulation with mesh bounded from below. As already noted, one would like to find such non-extrinsic (i.e. curvature restricting) conditions (perhaps coupled with fitting topological constraints) in higher dimension, as well. Indeed, in dimensions greater or equal three, even the problem of deciding which curvature (sectional, Ricci, scalar) is most relevant is a highly non-trivial problem, that we defer for further study.

Another direction of study stems from the need, both in the classical signal-processing context and in that of manifold sampling, for mashing and sampling methods of geometrical objects that are not even PL, and hence no smoothing techniques can be applied for them. In this general setting, *metric curvatures*, represent, in our view, the most promising tool. Indeed, research in this direction is currently undertaken.

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