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What's Between Optimal Multiplexing and Strongly Regular Graphs?

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Abstract

Measuring a sequence of quantities is central to many problems. Namely, it is very useful for imaging applications in a variety of modalities, e.g. *X-ray imaging, spectroscopy, infra-red (IR), multi-spectral imaging etc. Originally utilized for X-ray telescopy, multiplexing measurements is recognized by a growing number of methods as beneficial. For example, when multiplexing radiation sources, rather than measuring each source at a time, the benefits include increased signal-to-noise ratio and accommodation of scene dynamic range. However, existing multiplexing schemes are inhibited by fundamental limits set by noise characteristics and by sensor saturation. The prior schemes, including Hadamard-based codes may actually be counterproductive due to these effects. We aim to derive multiplexing codes that are* optimal *under these fundamental effects. Our approach is to find a lower bound on the mean square error (MSE) of the de-multiplexed data as well as the necessary conditions to attain this bound for every desired number of radiation sources. We then show a class of multiplexing codes that follow these conditions and can be used for optimal multiplexing. Our work is also applicable for verifying the optimality of any multiplexing code suggested in the future.*

1. Optical Multiplexing

A fundamental task in imaging is to minimize the measurement errors, expressed as image noise [2, 4, 9, 12, 14, 16, 17]. This is true for practically every imaging modality *e.g.* X-Ray [9, 16], spectroscopy [9], visible light [14, 17] and Infra-Red (IR) [2]. It is important to realize that measurement fluctuation are an inherent part of the imaging process, partly regardless of sensor quality. These fluctuations result from the quantum mechanical nature of photon flux itself.

A straightforward way of compensating for the measurement fluctuations is to measure the same scene repeatedly and average the acquired measurements. Alternatively, the integration time can be lengthen. Both these methods have the drawback of not being able to cope with highly dynamic scenes. A better way of handling measurement noise is to multiplex the measured sources [13, 14, 17]. This means that a combination of several energy sources are measured simultaneously in each measurement, then the results are computationally de-multiplexed to yield an estimate for the intensity of each individual source.

The question is, given all possibilities of simultaneous operation of sources, what is the optimal way to multiplex them. Ref. [14] suggested that Hadamard-based codes should be used. However, its analysis did not account for a very important problem: acquisition noise depends on the acquired irradiance itself. This might cause Hadamard multiplexing to become counter productive, as was later experienced by [17]. Ref. [18] has recently dealt with the problem of multiplexing under signal-dependent noise but it limited itself to a very small set of solutions. Namely, the multiplexing codes obtained by [18] are based on cyclic matrices only and are applicable to very specific numbers of sources. Ref. [13] devised a numerical optimization problem that yields multiplexing matrices, accounting for both saturation and photon noise. However, it is not guaranteed that those multiplexing matrices are indeed optimal.

Our approach to overcome the fundamental imaging limitations *i.e*. photon noise and saturation, agrees with the approach taken by Ref. [13]. We propose that optimal multiplexing codes must be constructed such that they are adaptable to every desired amount of radiance in each measurement. Such codes will allow the adjustment of the radiance used in each measurement to avoid saturation and minimize the degradation from *photon noise*.

In this work we make a crucial step towards achieving optimal multiplexing matrices. We derive necessary conditions on the optimality of a multiplexing code in the general case of C out of N radiating sources used in each measurement. Once formed, these conditions are used to construct optimal multiplexing codes. These conditions can also be used to verify the optimality of any offered multiplexing codes. For example, the codes obtained by the numerical optimization procedures offered by Ref. [13].

2. Theoretical Background

2.1. Multiplexing

Consider a setup of N radiation sources. Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)^t$ be a set of object radiance values as measured from a certain location. Each element in this set corresponds to radiating by any *individual* source in this setup. Here, t denotes transposition.

In general, several sources can be turned on in each measurement (multiplexing). Define an $N \times N$ multiplexing matrix $\mathbf{W}$. It is often referred to as a "multiplexing code". Each element of its mth row represents the radiation power of a source in the mth measurement. The power is measured relative to its maximum value. Hence, 0 states that the source is completely off while 1 indicates a fully activated source. The sequential measurements acquired at a detector are denoted by the vector $\mathbf{a} = (a_1, a_2, \ldots, a_N)^t$. It is given by

$$\mathbf{a} = \mathbf{W}\mathbf{i} + \upsilon \quad , \tag{1}$$

where υ is the measurement noise. Any bias to this noise is assumed to be compensated for. Let the noise υ be uncorrelated in different measurements with variance of σ_a^2.

Once measurements have been acquired under multiplexed sources, they are de-multiplexed computationally. This derives estimates for the radiance values of the individual radiation sources $\hat{\mathbf{i}}$. The best linear estimator in the sense of mean square error (MSE) for the radiance of the individual-sources is

$$\hat{\mathbf{i}} = \mathbf{W}^{-1}\mathbf{a} \quad . \tag{2}$$

The MSE of this estimator [9, 14] is

$$\text{MSE}_{\hat{\mathbf{i}}} = \frac{\sigma_a^2}{N}\text{trace}\left[\left(\mathbf{W}^t\mathbf{W}\right)^{-1}\right] \quad . \tag{3}$$

This is the expected noise variance of the recovered sources. In this paper we seek a lower bound on Eq. (3) and derive conditions on $\mathbf{W}$ to attain this bound, hence minimizing $\text{MSE}_{\hat{\mathbf{i}}}$.

2.2. Eigenvalues and Singular Values

In this section we briefly review elementary definitions and results from linear algebra that will later be used for our analysis.

Definition 2.1. Let $\Lambda = \{\lambda_f\}_{f=1}^N$ be the set of the eigenvalues (EVs) of a matrix $\mathbf{W}$. The *multiplicity* of $\lambda_f \in \Lambda$ is the number of repetitions of the value of λ_f in Λ.

Lemma 2.1. *If* $\mathbf{R}$ *and* $\mathbf{S}$ *are matrices such that* $\mathbf{RS}$ *is a square matrix, then* [11]

$$\text{trace}(\mathbf{RS}) = \text{trace}(\mathbf{SR}) \quad . \tag{4}$$

Lemma 2.2. *Let* $\mathbf{W}$ *be a non-singular* $N \times N$ *matrix. Its EVs are* $\lambda_1 \leqslant \ldots \leqslant \lambda_N$. *Then (See for example Ref.* [10]*)*
i)

$$\text{trace}(\mathbf{W}) = \sum_{f=1}^{N} \lambda_f \tag{5}$$

ii) The EVs of $\mathbf{W}^{-1}$ *are* $\lambda_N^{-1} \leqslant \ldots \leqslant \lambda_1^{-1}$.

Definition 2.2. The singular values (SVs) of $\mathbf{W}$ are the square root of the EVs of $\mathbf{W}^t\mathbf{W}$.

Ref. [5] quotes the following theorem.

Theorem 2.3. *(Weyl-Horn) Let* $\xi_1 \leqslant \ldots \leqslant \xi_N$ *be the SVs of* $\mathbf{W}$, *then*

$$\prod_{m=f}^{N} \xi_m \geqslant \prod_{m=f}^{N} |\lambda_m| \qquad \forall f \in \{2, \ldots, N\} \tag{6}$$

and

$$\prod_{m=1}^{N} \xi_m = \prod_{m=1}^{N} |\lambda_m| \quad . \tag{7}$$

Note that if $\mathbf{W}$ is symmetric, then

$$\xi_m = |\lambda_m| \qquad \forall m \quad . \tag{8}$$

2.3. Strongly Regular Graphs

We now refer to some elementary definitions from graph theory. We will use them when seeking optimal solutions to the multiplexing problem. We quote some basic definitions from Ref. [7].

Consider a graph $G = (V, E)$, where V is a set of N *vertices*. Here E is the set of *edges* connecting a pair of vertices.

Definition 2.3. Two vertices p, q are said to be *adjacent* or *neighbors* if they are connected by an edge.

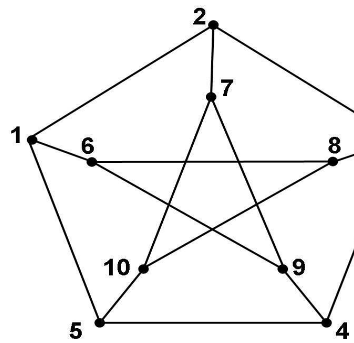

Figure 1. An example of a strongly regular graph (Peterson) [3]. This graph has the parameters $(N = 10; k = 3; \alpha = 0; \beta = 1)$.

Definition 2.4. The $N \times N$ *adjacency* matrix $\mathbf{\Omega}$ of the graph G is composed of elements

$$\omega_{p,q} = \begin{cases} 1 & \text{if } p \text{ and } q \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Definition 2.5. The *complement* of a graph G is a graph $\bar{G}$ where its adjacency matrix of $\bar{\mathbf{\Omega}}$, is composed of elements

$$\bar{\omega}_{p,q} = \begin{cases} 1 & \text{if } \omega_{p,q} = 0 \text{ and } p \neq q \\ 0 & \text{otherwise} \quad . \end{cases} \tag{10}$$

Definition 2.6. If all the vertices of G have the same number of neighbors k, then G is k*-regular*. In this case

$$\sum_{q=1}^{N} \omega_{p,q} = k \qquad \forall p \quad . \tag{11}$$

Definition 2.7. A *strongly regular graph* (SRG) [3] with parameters $(N; k; \alpha; \beta)$ is a k-regular graph that has the following properties:

- Any two adjacent vertices have exactly α common neighbors (neighbors of both vertices).
- Any two non-adjacent vertices have exactly β common neighbors.

For example, consider the graph in Fig. 1. The adjacent vertices 5 and 10 have no common neighbors and this relation also applies to all the other adjacent pairs in the graph. Hence, here $\alpha = 0$. Moreover, vertices 5 and 3 have a single common neighbor, 9, and so are all other analogous pairs. Hence, here $\beta = 1$.

Theorem 2.4. *The parameters* $(N; k; \alpha; \beta)$ *of a strongly regular graph satisfy* [3] *the constraint*

$$k(k - \alpha - 1) = (N - k - 1)\beta \tag{12}$$

In the following, we make use of a theorem due to Seidel [15]:

Theorem 2.5. *Let G be a strongly regular graph with parameters $(N; k; \alpha; \beta)$. Its adjacency matrix $\mathbf{\Omega}$ has generally three distinct EVs,*

$$\lambda_1^{\mathbf{\Omega}} = \frac{(\alpha - \beta) + \sqrt{\Delta}}{2} \tag{13}$$

$$\lambda_2^{\mathbf{\Omega}} = \frac{(\alpha - \beta) - \sqrt{\Delta}}{2} \tag{14}$$

$$\lambda_3^{\mathbf{\Omega}} = k \quad , \tag{15}$$

were,

$$\Delta \equiv (\alpha - \beta)^2 + 4(k - \beta) \quad . \tag{16}$$

The multiplicity of $\lambda_3^{\mathbf{\Omega}}$ is 1.

From Eq. (9), $\mathbf{\Omega}$ is symmetric. Thus, Eq. (8) applies to the SVs of $\mathbf{\Omega}$.

$$\xi_f^{\mathbf{\Omega}} = |\lambda_f^{\mathbf{\Omega}}| \qquad \forall f \quad . \tag{17}$$

Since the EVs of $\mathbf{\Omega}$ indicate its SVs, Theorem 2.5 can be applied to the SVs of $\mathbf{\Omega}$. In particular, the multiplicity of EVs in Theorem 2.5 generally applies to the SVs of $\mathbf{\Omega}$.

3. Optimal Power-Regulated Multiplexing

3.1. Problem Formulation

We seek multiplexing codes under fixed power. Such a property is desired to avoid the saturation of the detector. Here, saturation means that the number of electrons generated in the detector exceeds the capacity of the circuitry that translates it into gray levels. The property of fixed power is useful for other reasons, such as curbing photon noise, as we shall detail later.

A trivial way to multiplex sources such that the total power is fixed is to equally reduce the power of each of the sources. However, as the number of sources increases, Refs. [13, 14] proved that such a step should be avoided.[1] A better solution is to decrease the number of radiation sources C used in each measurement. If all measured sources radiate similarly, then C expresses units of sources and is equivalent to a restriction on the power. This decrease, however, is not easy to accomplish. The reason is that current codes in the literature [9, 18] do not support multiplexing of arbitrary number of C out of N sources. Rather, optimal codes were derived for special cases. For instance, in Hadamard based codes, $C = (N + 1)/2$. This raises the need to extend the set of multiplexing codes, that comply with a constraint on the number of simultaneously activated sources C.[2]

[1] This is true under the assumption that the power limitation is insensitive to the specific identities of the radiation sources. Hence, the saturation concerns only the total measured power.

[2] We note that Ref. [13] suggests a numerical search for such codes. However, here we show a closed form solution and an analysis of the limitations of this solution.

Power is fixed by setting

$$\sum_{s=1}^{N} w_{m,s} = C \quad \forall m \in \{1,2,\ldots,N\} \quad . \tag{18}$$

Recall that each source can be activated with some portion of its maximum power *i.e.*

$$0 \leqslant w_{m,s} \leqslant 1 \qquad \forall m,s \in \{1,2,\ldots,N\} \quad . \tag{19}$$

We use Eq. (3) to formulate a minimization task for the reconstruction $\mathrm{MSE}_{\hat{\mathbf{i}}}$. We slightly modify $\mathrm{MSE}_{\hat{\mathbf{i}}}$ for the cost function. To simplify the analysis, define

$$\widetilde{\mathrm{MSE}} \triangleq \frac{\mathrm{MSE}_{\hat{\mathbf{i}}}}{\sigma_a^2} = \frac{1}{N}\mathrm{trace}\left[\left(\mathbf{W}^t\mathbf{W}\right)^{-1}\right]\} \quad . \tag{20}$$

The minimization of $\widetilde{\mathrm{MSE}}$ is the same as minimization of MSE itself, if σ_a^2 is constant. The influence of σ_a^2 will be later discussed in Sec. 5. The constraints for our problem are taken from Eqs. (18,19). Thus, the optimization problem is

$$\min_{\mathbf{W}} \widetilde{\mathrm{MSE}} \triangleq \frac{1}{N}\mathrm{trace}\left[\left(\mathbf{W}^t\mathbf{W}\right)^{-1}\right] \tag{21}$$

$$\text{s.t.} \quad \sum_{s=1}^{N} w_{m,s} - C \quad = 0 \quad \forall m \in \{1,\ldots,N\} \tag{22}$$

$$-w_{m,s} \quad \leqslant 0 \ \ \forall m,s \in \{1,\ldots,N\} \tag{23}$$

$$w_{m,s} - 1 \quad \leqslant 0 \ \ \forall m,s \in \{1,\ldots,N\} \quad . \tag{24}$$

We shall now derive sufficeint conditions for a matrix $\mathbf{W}$ to solve Eqs. (21,22,23,24).

3.2. Conditions for a Global Optimum

A numerical procedure has been tailored to the optimization problem (21) in Ref. [13]. It is preferable however, to reach a closed-form solution, if it exists. This is done by deriving sufficient conditions for the optimality of a given $\mathbf{W}$. Such conditions allow us to identify an optimal solution to the problem, if encountered. Indeed, in this paper we show that these conditions are satisfied by matrices $\mathbf{W}$ originally developed in graph theory. We can also apply these conditions to verify that a matrix reached by numerical optimization is indeed the global optimum.

Our approach to deriving the optimality conditions is as follows: first, we find a tight lower bound on $\widetilde{\mathrm{MSE}}$. Then, we formulate a necessary and sufficient condition to attain this bound. Finally, we minimize the bound itself, with respect to the elements of $\mathbf{W}$.

3.2.1 The Cost as a Function of Singular Values

First, we express Eq. (21) in terms of the SVs of $\mathbf{W}$. We use Lemma 2.2. Hence,

$$\widetilde{\mathrm{MSE}} \triangleq \frac{1}{N}\mathrm{trace}\left[\left(\mathbf{W}^t\mathbf{W}\right)^{-1}\right] = \frac{1}{N}\sum_{f=1}^{N}\frac{1}{\mu_f} \quad , \tag{25}$$

where $\mu_1 \leqslant \ldots \leqslant \mu_N$ are the EVs of $\mathbf{W}^t\mathbf{W}$. Recall the definition in Theorem 2.3, that $\{\xi_f\}_{f=1}^N$ are the SVs of $\mathbf{W}$. Thus, in light of definition 2.2.

$$\mu_f \equiv \xi_f^2 \quad \forall f \in \{1,\ldots,N\} \quad . \tag{26}$$

We show an implication of constraints (22,23,24) on the SVs of $\mathbf{W}$. This will allow us to better streamline (22,23,24) into Eq. (25), forming a lower bound on the MSE. To understand the connection between these constraints and the SVs of $\mathbf{W}$, we cite the following theorem [8]:

Theorem 3.1. *Let* $\mathbf{W}$ *be constrained by Eqs.* (22,23,24). *Then,* C *is an EV of* $\mathbf{W}$. *Furthermore, let* λ_m *be an EV of* $\mathbf{W}$. *Then,* $|\lambda_m| \leqslant C$. The proof is given in App. A.

Theorem 3.1 deals with the EVs of $\mathbf{W}$. This has an implication on the corresponding SVs, which are used in (25). This implication stems from Theorem 2.3. Let us set $f = N$ in Eq. (6). Then,

$$\xi_N \geqslant |\lambda_N| \quad . \tag{27}$$

Now, following Theorem 3.1, $\lambda_N = C$. Hence, using Eqs. (26,27)

$$\mu_N \triangleq \xi_N^2 \geqslant C^2 \quad . \tag{28}$$

Eq. (28) constrains the largest SV of $\mathbf{W}$. Therefore, we separate it from the rest of the SVs in Eq. (25). The advantage of this move will be clarified shortly. Thus we rewrite Eq. (25) as

$$\widetilde{\mathrm{MSE}} = \frac{1}{N\mu_N} + \frac{1}{N}\sum_{f=1}^{N-1}\frac{1}{\mu_f} \quad . \tag{29}$$

for a reason that will be clarified next.

3.2.2 Optimality of the Singular Values

In this paper we seek the tight lower bound on $\mathrm{MSE}_{\hat{\mathbf{i}}}$. To achieve this we have first expressed $\widetilde{\mathrm{MSE}}$ by Eq. (29), in terms of the SVs of $\mathbf{W}$. We now seek to minimize Eq. (29) as a function of these SVs. Apparently, $\widetilde{\mathrm{MSE}}$ in Eq. (29) is reduced if we simply increase any single SV, μ_f, while the rest are constant. Can we do this arbitrarily? The answer is *no*. The reason is that Eqs. (22,23,24) bound the domain of $\mathbf{W}$, hence, bounding the domain of its SVs. Therefore, any

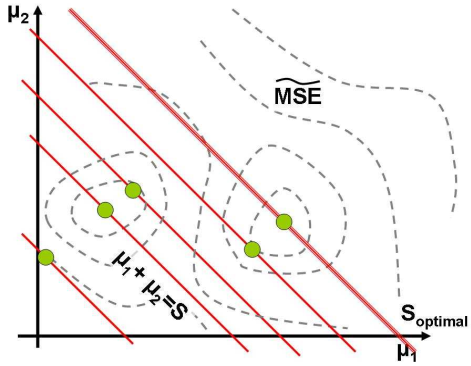

Figure 2. $\widetilde{\mathrm{MSE}}$ as a function of $\{\mu_f\}_{f=1}^{N}$. The green dots mark the vectors in μ domain that minimize $\widetilde{\mathrm{MSE}}$, when S is fixed. The highlighted line marks the optimal value of S. The green dot along this line marks the global minimum $\widetilde{\mathrm{MSE}}$. The minimum is derived in closed form and thus, not affected by local minima of $\widetilde{\mathrm{MSE}}$.

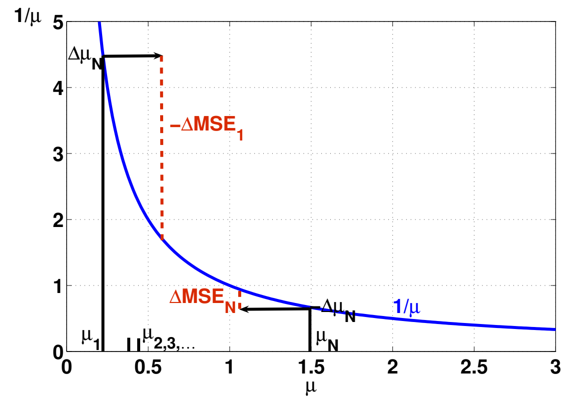

Figure 3. An illustration for the solution of Eq. (31). The black vertical lines indicate an instantaneous state of the SVs of $\mathbf{W}$. If the largest SVs is reduces by ϵ and the smallest SV is increased by the same amount, the total sum of the cost function (31) is reduced

μ_f cannot be arbitrarily increased. The SVs are mutually coupled. To express the coupling, we first use a normalization, by setting

$$S \triangleq \sum_{f=1}^{N} \mu_f \tag{30}$$

and setting it to be a constant. Later we alleviate the need for this normalization as demonstrated in Fig. 2.

Now, we show that under the constraint $S \equiv \mathrm{Const}$,

$$\min_{\mu_N} \left\{ \frac{1}{N\mu_N} + \frac{1}{N}\sum_{f=1}^{N-1}\frac{1}{\mu_f} \right\} = \frac{1}{NC^2} + \frac{1}{N}\sum_{f=1}^{N-1}\frac{1}{\mu_f} \quad . \tag{31}$$

This observation is now clarified (see Fig. 3). The minimization in Eq. (31) is only over the value of μ_N. If μ_N is decreased by some small quantity $\Delta\mu_N$ then, to conserve S in Eq. (30), the value of at least one other μ_f should increase. An increase of μ_f by $\Delta\mu_N$ reduces $\widetilde{\mathrm{MSE}}$ since,

$$\Delta\widetilde{\mathrm{MSE}} = \frac{\partial\widetilde{\mathrm{MSE}}}{\partial\mu_f}\Delta\mu_N = -\frac{1}{N\mu_f^2}\Delta\mu_N \quad . \tag{32}$$

We seek the strongest reduction of $\widetilde{\mathrm{MSE}}$. Clearly, in Eq. (32), the reduction $\Delta\widetilde{\mathrm{MSE}}$ is strongest if $\Delta\mu_N$ affects the lowest μ_f, which is μ_1. Overall, decreasing μ_N and increasing μ_1 by $\Delta\mu_N$ yields a reduction of

$$\Delta\widetilde{\mathrm{MSE}} = -\frac{\Delta\mu_N}{N}\left(\frac{1}{\mu_1} - \frac{1}{\mu_N}\right) < 0 \quad . \tag{33}$$

Delivering a net benefit of $\widetilde{\mathrm{MSE}}$.

Since a benefit stems from a reduction of μ_N, then μ_N should be as low as possible. From Eq. (28), the lowest possible value of μ_N is C^2. To recap, in an optimal multiplexing code $\mathbf{W}$, the largest SV satisfies

$$\xi_N = C \tag{34}$$

i.e.,

$$\mu_N = C^2 \quad . \tag{35}$$

This leads to Eq. (31)

After determining μ_N, we turn to the other SVs. Trivially,

$$\frac{1}{N}\sum_{f=1}^{N-1}\frac{1}{\mu_f} \equiv \frac{N-1}{N}\left(\frac{1}{N-1}\sum_{f=1}^{N-1}\frac{1}{\mu_f}\right) \quad . \tag{36}$$

The parenthesis on the right hand side of (36) expresses the reciprocal harmonic mean of $\{\mu_f\}_{f=1}^{N-1}$. The *inequality of means* [1] states that:

$$\frac{1}{N-1}\sum_{f=1}^{N-1}\frac{1}{\mu_f} \geqslant \frac{N-1}{\sum_{f=1}^{N-1}\mu_f} \quad . \tag{37}$$

Using Eqs. (30,35) in Eq. (37) yields,

$$\frac{1}{N-1}\sum_{f=1}^{N-1}\frac{1}{\mu_f} \geqslant \frac{N-1}{S-C^2} \quad . \tag{38}$$

Combining Eqs. (31,36,38) into (29) yields

$$\widetilde{\mathrm{MSE}} \geqslant B \tag{39}$$

where B is the lower bound of $\widetilde{\mathrm{MSE}}$, given by

$$B = \frac{1}{NC^2} + \frac{(N-1)^2}{N(S-C^2)} \quad . \tag{40}$$

The lower bound B is constant as long as S in fixed. We wish that $\widetilde{\text{MSE}}$ will actually attain this fixed lower bound. Equality in Eq. (39) is obtained if equality holds in (37), which occurs when $\{\mu_f\}_{f=1}^{N-1}$ are all equal. We recap with the following corollary.

Corollary 3.2. *An ideal multiplexing matrix* $\mathbf{W}$ *is such that all its SVs (but the largest one)* $\{\xi_f\}_{f=1}^{N-1}$, *are equal to each other. Its largest SV equals* C.

Note that we refer to matrices that attain the lower bound B as *ideal*. It is not guaranteed however, that such matrices exist for all values of N and C. Matrices that minimize $\widetilde{\text{MSE}}$ without reaching B are simply referred to as *optimal*.

3.2.3 The Optimal Variable S

We have shown in Sec. 3.2.2 that for a specific S, the MSE is minimized by matrices that comply with the terms in Corollary 3.2. We now relieve the constraint of a fixed S. Hence, S may vary (see Fig. 2). Furthermore, we derive the best value for S. This yields a condition on the elements of $\mathbf{W}$. Following Lemma 2.2,

$$\text{trace}\left(\mathbf{W}^t\mathbf{W}\right) = \sum_{k=1}^{N} \mu_k = S \quad . \tag{41}$$

Following Lemma 2.1,

$$\text{trace}\left(\mathbf{W}^t\mathbf{W}\right) = \text{trace}\left(\mathbf{W}\mathbf{W}^t\right) \quad . \tag{42}$$

The diagonal elements of $\mathbf{W}\mathbf{W}^t$ are

$$\left(\mathbf{W}\mathbf{W}^t\right)_{m,m} = \sum_{s=1}^{N} w_{m,s}^2 \quad . \tag{43}$$

Due to Eq. (19),

$$w_{m,s}^2 \leqslant w_{m,s} \quad . \tag{44}$$

From Eqs. (18,43,44),

$$\left(\mathbf{W}\mathbf{W}^t\right)_{m,m} \leqslant \sum_{s=1}^{N} w_{m,s} = C \quad . \tag{45}$$

From Eqs. (41,42,45)

$$S = \text{trace}\left(\mathbf{W}\mathbf{W}^t\right) = \sum_{m=1}^{N} (\mathbf{W}\mathbf{W}^t)_{m,m} \leqslant NC \quad . \tag{46}$$

Note that equality holds in (46) *if* and *only if* equality also holds in Eq. (44). Trivially, this happens if all elements of $\mathbf{W}$ are either 1's or 0's. In this case Eq. (46) yields

$$S_{\text{optimal}} = NC \quad . \tag{47}$$

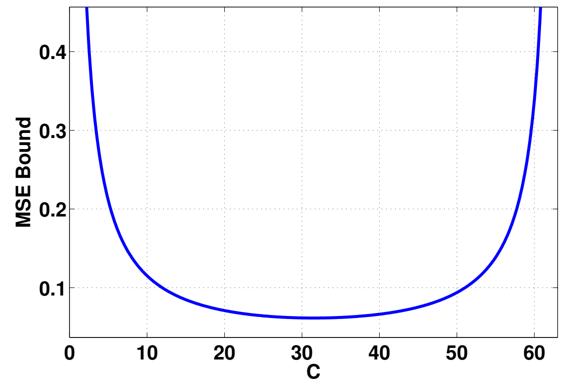

Figure 4. Bound of $\widetilde{\text{MSE}}$, for $N = 63$. Here C varies from 1 to 63.

Eq. (46) alleviates the need for a fixed S.

From Eqs. (40,46,47)

$$B \geqslant B_{\text{min}} \tag{48}$$

where

$$B_{\text{min}} = \left[\frac{1}{NC^2} + \frac{(N-1)^2}{N(NC-C^2)}\right] \quad . \tag{49}$$

Equality holds if $w_{m,s} \in \{0,1\} \quad \forall m,s$. In other words B reaches its lowest potential value B_{min}, if S is given by Eq. (47).

Corollary 3.3. *The lower bound on* $\widetilde{\text{MSE}}$ *is achieved using binary multiplexing matrices* $\mathbf{W}$.

We now combine the results of Corollaries 3.2 and 3.3. From Eqs. (3,39,40,48,49),

$$\text{MSE}_{\hat{\mathbf{i}}} \geqslant \sigma_a^2 B_{\text{min}} \quad . \tag{50}$$

The low bound $\sigma_a^2 B_{\text{min}}$ is obtained by a matrix $\mathbf{W}$, if this matrix complies with both Corollaries 3.2 and 3.3.

The result in Eqs. (49,50) is very useful. Being a tight bound, B_{min} determines the behavior of $\text{MSE}_{\hat{\mathbf{i}}}$ as a function of both N and C. Furthermore, if $\mathbf{W}$ satisfies the optimality conditions, then $\sigma_a^2 B_{\text{min}}$ is exactly the expected value of $\text{MSE}_{\hat{\mathbf{i}}}$. In Fig. 4 we illustrate the behavior of $\widetilde{\text{MSE}}$ for a specific value of N and a rage of values of C.

As a closure for this section, we find the value of C that minimizes the bound B_{min}. We denote this value by C_{opt}. It is stressed out that this value does not necessarily correspond to a multiplexing matrix that meets the bound B_{min}. Such a matrix might not at all exist *e.g.* if C_{opt} is not an integer. However, it might give us a clue about the best possible multiplexing matrix, given no other considerations in selecting C *e.g.* photon noise.

We now see a proper time to remind that if C is free to vary, the optimal multiplexing matrices for a subset of values of N are already known. These matrices are based on the Hadamard matrices and are known as the **S**-matrices [9]. It can easily be shown that those matrices comply with our conditions for attaining the lower multiplexing bound. An interesting characteristic of these matrices is that if they exist, they have exactly $\frac{N+1}{2}$ elements valued 1 in each row and the rest are 0. In other words they have $C_{\rm Had} = \frac{N+1}{2}$.

We now turn to find $C_{\rm opt}$. This is done by differentiating $B_{\rm min}$ with respect to C and equalizing it to 0.

$$\frac{\partial B_{\rm min}}{\partial C} = -\frac{2}{NC^3} - \frac{(N-1)^2(N-2C)}{N(NC-C^2)^2} = 0 \quad . \tag{51}$$

Eq. (51) yields $C_{\rm opt}$.

$$C_{\rm opt} = \frac{-3+N^2-2N \pm \sqrt{(N^2-2N+9)(N-1)^2}}{4N-8} \tag{52}$$

A careful examination of $C_{\rm opt}$ in Eq. (52) reveals that it is actually not very far from $\frac{N+1}{2}$. This value, as mentioned earlier is proven to be the best C for multiplexing in cases where **S**-matrices exist. In fact, as N approaches infinity we have

$$\lim_{N\to\infty} C_{\rm opt} = \frac{N}{2} \quad . \tag{53}$$

So asymptomatically

$$\lim_{N\to\infty} C_{\rm opt} = \lim_{N\to\infty} C_{\rm Had} \quad . \tag{54}$$

This last observation sums up nicely our analytical derivations. It relates our lower bound to multiplexing matrices that are long ago known to be optimal. We see that not only the **S**-matrices attain our MSE bound, they also attain this bound very close to its lowest point. From the optimality of the **S**-matrices we can conclude that the lowest point of our bound is indeed very close to the global optimum of the MSE.

4. Some Ideal Solutions

4.1. Strongly Regular Graphs as A Solution

Sec. 3 derived a lower bound on $\text{MSE}_{\hat{\mathbf{i}}}$, which can be obtained by multiplexing. We also presented the conditions that a multiplexing code should satisfy in order to attain this bound. However, it is not obvious that such *ideal* codes exist. Here, we show that indeed there is a class of matrices that satisfy these optimality conditions.

The adjacency matrix $\mathbf{\Omega}$ described in Theorem 2.5 can be used as the desired multiplex matrix $\mathbf{W}$, sought in Sec. 3, as we now detail.

- The matrix $\mathbf{\Omega}$ is is an adjacency matrix, therefore it is binary. Hence, it satisfies Corollary 3.3.

Figure 5. An example for a strongly regular graph with parameters $(45;12;3;3)$ developed by Ref. [6]. Here 1s appear as white squares while 0s appear as black squares. Since in this graph $\alpha=\beta=3$, it satisfies the conditions of Theorem 4.1.

- The highest SV of the desired $\mathbf{W}$ is C, with multiplicity which is generally 1 (See Corollary 3.2). Similarly, the highest SV of $\mathbf{\Omega}$ is k, with the same multiplicity.

- Setting the parameters $\alpha = \beta$ in Eqs. (13,14) yields $|\lambda_1^{\mathbf{\Omega}}| = |\lambda_2^{\mathbf{\Omega}}| = \frac{\sqrt{\Delta}}{2}$. Using Eq. (17), all SVs of $\mathbf{\Omega}$ are equal to $\frac{\sqrt{\Delta}}{2}$ but the largest SV, which equals k. Namely,

$$\xi_1^{\mathbf{\Omega}} = \sqrt{C-\beta} \tag{55a}$$
$$\xi_2^{\mathbf{\Omega}} = C \quad . \tag{55b}$$

Therefore, $\mathbf{\Omega}$ satisfies the conditions of Corollary 3.2.

To summarize,

Theorem 4.1. *Let $\mathbf{\Omega}$ be the adjacency matrix of a $(N;C;\alpha;\alpha)$ strongly regular graph. Then, $\mathbf{\Omega}$ is an ideal matrix $\mathbf{W}$ solving Eqs.* (21, 22, 23, 24) .

Recall that an SRG is subject to a constraint given in (12) on the values of its parameters. If $\alpha=\beta$ as in Theorem 4.1, then Eq. (12) takes the following form

$$\alpha = \frac{k(k-1)}{(N-1)} \quad . \tag{56}$$

As an example, consider the $(45;12;3;3)$ SRG [6]. This is a class of graphs, determined up to a isomorphism. One representative of this class is shown as a binary image in Fig. 5. Note that the parameters of this graph satisfy Eq. (56).

4.2. Solutions from Complement Graphs

In Sec. 4.1 we have devised a set of solutions for the sought optimal $\mathbf{W}$. In the following section we shall see that this set directly yields yet another set of solutions. Refs. [3, 6] formalize the following theorem:

Theorem 4.2. *A graph G is strongly regular with parameters $(N;k;\alpha;\beta)$ if and only if its complement $\bar{G}$ is also strongly regular, with parameters $(N;N-k-1;N-2k+\beta-2;N-2k+\alpha)$.*

In our context, Theorem 4.2 means that given an SRG, $\bar{G}$ with parameters $(N;\bar{C};\bar{\alpha};\bar{\alpha}+2)$, there exists a complement SRG, G. This graph G has the parameters $(N;C;\alpha;\alpha)$, where $C=N-\bar{C}-1$ and $\alpha=N-2\bar{C}+\bar{\alpha}$. Hence, using Theorem 4.1, G is a solution for Eqs. (21, 22, 23, 24).

5. Photon noise

In this section, we present a multiplexing application that benefits from our analytical derivation. Namely, it applies the MSE lower bound derived in Sec. 3 to illumination multiplexing.

Consider a setup of N light sources, spread in various directions relative to a scene. Such a setup is often used for the acquisition of images of a scene under variable illumination [13, 14].

It has already been recognized Refs. [14] that multiplexing the light sources in each image acquisition, is highly beneficial to the SNR of the desired, single source images.

Although this is generally true, the presence of photon noise in the acquired images inhibits the multiplexing gain. Intuitively, when the noise is signal dependent, increasing the radiance put in each measurement, also increases the noise. This effect downgrades the multiplexing gain, sometimes even causing it to be counterproductive.

As in [13] we use an affine noise model

$$\sigma_a^2=\kappa_{\text{gray}}^2+C\eta \quad . \tag{57}$$

Ref. [13] minimizes the MSE of the decoded images, given that they were acquired under the affine noise in eq. (57). This minimization is based on a numerical procedure with its cost function given by eq.(3). Since this cost function is not unimodal, it is not guaranteed that its global minimum will be achieved. We use the analytic bound to examine the quality of the multiplexing codes obtained by the procedure depicted in [13].

6. Discussion

We have formulated an optimization problem that is the key in finding optimal multiplexing codes that are tailored to fundamental inherent constraints in radiometric measurements. This optimization problem is multimodal. Our work finds the necessary and sufficient conditions for a given multiplexing code to be an optimal solution for the problem. Those conditions are general in reference to both the total number of radiation sources N and the radiance allowed for every single measurement.

In addition we have presented a set of matrices that obey these constraints and can be used as optimal multiplexing codes. These matrices are taken from the discipline of graph theory. Specifically they are the adjacency matrices of strongly regular graphs.

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A. Proof of Theorem 3.1

To make the paper self-contained, we prove Theorem 3.1. This derivation is a variation of a proof that appears in [8]. Define

$$\mathbf{W}^{\text{offset}} \triangleq \mathbf{W}-C\mathbf{I} \quad . \tag{58}$$

Let $w_{m,s}^{\text{offset}}$ be the elements of $\mathbf{W}^{\text{offset}}$. Following Eq. (18),

$$\sum_{s=1}^{N} w_{m,s}^{\text{offset}}=0 \qquad \forall m\in\{1,2,\ldots,N\} \quad . \tag{59}$$

Hence, one of the EVs of $\mathbf{W}^{\text{offset}}$ is 0. In other words, $\det(\mathbf{W}^{\text{offset}})=0$. By the definition in (58), this means that $\det(\mathbf{W}-C\mathbf{I})=0$, *i.e.*, C is an EV of $\mathbf{W}$. This proves the first part of Theorem 3.1.

Suppose that $\mathbf{u}^f=(u_1^f,\ldots,u_N^f)^t$ is an eigenvector corresponding to λ_f; without loss of generality we may normalize $\mathbf{u}^f$ such that

$$\max_{m\in\{1,\ldots,N\}} |u_m^f|=1 \quad . \tag{60}$$

Hence, $u_m^f=1$ for a certain $m\in\{1,\ldots,N\}$. From Eqs. (19,60)

$$|w_{m,s}u_m^f|\leqslant 1 \quad \forall m,s\in\{1,\ldots,N\} \quad . \tag{61}$$

Eqs. (18,61) directly lead to a constraint on the absolute value of the sth component of $\mathbf{W}\mathbf{u}^f$.

$$|(\mathbf{W}\mathbf{u}^f)_m|=|\sum_{s=1}^{N} w_{m,s}u_s^f|\leqslant\sum_{s=1}^{N}|w_{m,s}u_s^f|\leqslant C \quad . \tag{62}$$

In addition, $\mathbf{u}^f$ is an eigenvector. Hence,

$$|(\mathbf{W}\mathbf{u}^f)_m|=|\lambda_f\mathbf{u}^f|_m=|\lambda_f u_m^f| \quad . \tag{63}$$

Recall that $u_m^f=1$. Hence (63) becomes

$$|(\mathbf{W}\mathbf{u}^f)_m|=|\lambda_f| \quad . \tag{64}$$

Using Eq. (64) in (62) yields

$$|\lambda_f|\leqslant C \quad . \tag{65}$$

This proves the second half of Theorem 3.1.

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