CCIT Report #665 November 2007

Expected Distortion for Gaussian Source with a Broadcast Transmission Strategy over a Fading Channel

Chao Tian, *Member, IEEE*, Avi Steiner, *Student Member, IEEE*,
Shlomo Shamai (Shitz), *Fellow, IEEE* and Suhas N. Diggavi, *Member, IEEE*

Abstract

We consider the problem of transmitting a Gaussian source on a slowly fading Gaussian channel, subject to the mean squared error distortion measure. The channel state information is known only at the receiver but not at the transmitter. The source is assumed to be encoded in a successive refinement manner, and then transmitted over the channel using the broadcast strategy. In order to minimize the expected distortion at the receiver, optimal power allocation is essential. We propose an efficient algorithm to compute the optimal solution in linear time $O(M)$, when the total number of possible discrete fading states is M. Moreover, we provide a derivation of the optimal power allocation when the fading state is a continuum, using the classical variational method. The proposed algorithm as well as the continuous solution is based on an alternative representation of the capacity region of the Gaussian broadcast channel.

I. Introduction

Fading channel occurs naturally as a model in wireless communications. For slow fading, the receiver can usually recover the channel state information (CSI) accurately, however the transmitter only knows the probability distribution of CSI, but not the realization. Such uncertainty can cause significant system performance degradation, and the broadcast strategy was used in [1], [2] as an approach to combat this detrimental effect. In this strategy, some information can only be decoded when the fading is less severe, which is superimposed on the information that can be decoded under more severe fading. Thus the receiver can decode the information adaptively according to the realization of the channel state. The similarity to the degraded broadcast channel [3] (see also [4]) is clear in this context, particularly for channels with a finite number of fading states. Generalizing this view, when the fading gain can take continuous values, the receiver can be taken as a continuum of users in a broadcast channel.

The broadcast strategy naturally matches the successive refinement (SR) source coding framework [5]–[7], as the information decodable under the most severe fading is protected the most, and should be used to convey the base layer information in the SR source coding. As more information can be decoded when the channel is subject to less fading, more SR encoded layers can be decoded, and the reconstruction quality improves. In this work, we consider this scheme for a quadratic Gaussian source on a single input single output (SISO) channel. In order to minimize the expected distortion at the receiver, it is essential to find the optimal power allocation in the broadcast strategy, and this is indeed our focus. It is worth noting here that though in [2] the objective function to be maximized is the expected rate, the cross layer design approach of combining SR source coding with broadcast strategy was in fact suggested (though not treated) in that work.

Initial effort on this problem was made by Sesia *et al.* in [8], where the broadcast strategy coupled with SR source coding was compared with several other schemes. The optimization problem was formulated by discretizing the continuous fading states, and an algorithm was devised when the source coding layers are assumed to have the same rate. This algorithm, however, does not directly yield the optimal power allocation when the fading states are discrete and pre-specified, nor does it give a closed-form solution for the continuous case. Etemadi and Jafarkhani also considered this problem in [9], and provided an iterative algorithm by separating the optimization problem into two sub-problems. In two more recent works [10], [11], Ng *et al.* provided a recursive algorithm to compute the optimal power allocation for the case with M possible fading states, with worst case complexity of $O(2^M)$; moreover, by directly taking the limit of the optimal solution for the discrete case, a solution was given for the continuous case optimal power allocation, under the assumption that the optimal power allocation is concentrated in a single interval. Similar problems were considered in [12]–[15] in the high SNR regime from the perspective of distortion exponent.

Our contribution in the present work is two-fold: firstly, we propose a new algorithm that can compute in linear time, i.e., of $O(M)$ complexity, the optimal power allocation for the case with M fading states; secondly, we provide a derivation of the continuous case optimal power allocation solution by the classical variational method [16]. Our derivation for the continuous case solution is more general than that in [11] as it removes the restriction that the optimal power allocation is concentrated in a single interval. Both the algorithm and the derivation rely on an alternative representation of the Gaussian broadcast channel capacity, which appeared in [17]. The dual problem of minimizing power consumption subject to a given expected distortion constraint is also discussed. Several specific examples are given as illustrations.

The broadcast strategy coupled with SR source coding can be considered as a source-channel separation approach with optimized cross-layer resource allocation. More explicit joint source-channel coding approaches, for example those in [13], [18]–[20], may provide better performance in the scenario being considered, however this aspect is beyond the scope of the present work. We nevertheless note that the algorithm proposed in the current work is extremely efficient, which makes it possible to use the broadcast strategy coupled with SR coding as a benchmark for future investigation into this joint source-channel coding problem. This is indeed useful since an outer bound which is non-trivial yet simple to compute is so far lacking despite extensive research (see [20] and the comments therein).

The rest of the paper is organized as follows. In Section II we give the system model and some preliminaries, and in Section III the new algorithm is provided and its optimality is proved; the dual problem of minimizing the power consumption subject to an expected distortion constraint is also considered. In Section IV we give the derivation for the continuous case solution, and Section V some special cases of fading distributions are considered. Section VI concludes the paper.

II. System model and Preliminaries

We assume a memoryless source $\{X_i\}_{i=1}^{\infty}$ is generated independently and identically according to a zero-mean unit variance real-valued Gaussian distribution. The channel is given by the model

$$Y_c = hX_c + N_c, \tag{1}$$

where X_c is the complex-valued channel input and Y_c is the channel output, h is the (random) multiplicative channel fading coefficient with $|h|^2 = s$, and $N_c \sim \mathcal{CN}(0,1)$ is the zero-mean complex circularly-symmetric independently and identically distributed (i.i.d.) Gaussian additive noise.

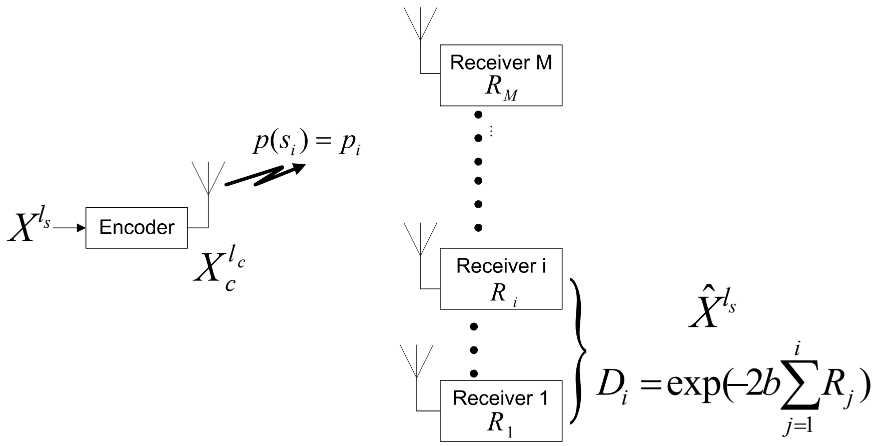

Fig. 1. The broadcast approach for minimizing the expected distortion.

We consider a slowly fading channel model, where each channel codeword consists of a length-l_c channel symbol block, and the realization of the multiplicative fading coefficient is independent across blocks. Source symbols of block length-l_s are encoded into a single channel codeword, and there is a source channel mismatch factor defined as $b = 2l_c/l_s$; this mismatch factor can also be interpreted as the bandwidth expansion/compression factor. This system is illustrated in Fig. 1. Each channel block is assumed to be sufficiently long to approach channel capacity, as well as the rate-distortion limit, however still much shorter than the dynamics of the slowly fading process. It is clear that without loss of generality each channel use in the complex domain is equivalent to two channel uses in the real domain subject to the same power constraint as

$$Y_r = \sqrt{s}X_r + N_r, \tag{2}$$

where X_r is the real-value channel input, and $s \in \mathbb{R}$ is the (random) channel power gain, and $N_r \sim \mathcal{N}(0, 1)$ is the zero-mean unit-variance real-valued Gaussian additive noise in the channel. From here on, we shall adopt this equivalent channel, and it is clear that the definition of source-channel mismatch factor b already takes this into consideration.

For the case with a finite number of fading states, the M possible power gains in an increasing order $s_1 < s_2 < ... < s_M$ are distributed according to a probability mass function p_i such that $\sum_{i=1}^{M} p_i = 1$. The transmitter has an average power constraint P, and if power P_i is allocated to the i-th layer in the broadcast strategy, the i-th layer channel rate R_i is given by

$$R_i = \frac{1}{2}\log(1 + \frac{s_i P_i}{1 + s_i \sum_{j=i+1}^{M} P_j}) = \frac{1}{2}\log(1 + \frac{P_i}{1/s_i + \sum_{j=i+1}^{M} P_j}), \tag{3}$$

where we use natural log. From the second expression, the equivalence to broadcast on a set of channels with different noise variances is clear. Let $n_i \triangleq 1/s_i$, which implies $n_1 > n_2 > ... > n_M$ are the equivalent noise power on the channels. The layers corresponding to smaller values of s_i's (and larger values of n_i's) will be referred to as the lower layers, which is consistent with the intuition that they are used to transmit the better protected lower layers of the SR source coding.

Since the Gaussian source is successively refinable [6], the receiver with power gain s_i can thus reconstruct the source within distortion

$$D_i = \exp(-2b \sum_{j=1}^{i} R_j). \tag{4}$$

Combining (3) and (4), the problem we wish to solve is essentially the following minimization over the power allocation $(P_1, P_2, ..., P_M)$,

$$\min \sum_{i=1}^{M} p_i \left(\prod_{j=1}^{i} (1 + \frac{P_j}{1/s_j + \sum_{k=j+1}^{M} P_k}) \right)^{-b} \tag{5}$$

$$\text{subject to:} \quad P_i \geq 0, \quad i = 1, 2, ..., M,$$
$$\textstyle\sum_{i=1}^{M} P_i \leq P.$$

When the fading state is continuous, the density of the power gain distribution is then given by $f(s)$, which is assumed to be continuous, and differentiable almost everywhere. In this case, the task is to find a power allocation density function $P(s)$, or its cumulative function, which minimizes the expected distortion.

III. The new algorithm and its optimality

In this section, we consider the optimization problem when the channel has a finite number of fading states. The difficulty is that the optimization problem as defined in (5) is not convenient for computation due to its complicated form. In fact it is not immediately clear that the function being optimized is a convex function of $(P_1, P_2, ..., P_M)$, nevertheless it is indeed clear that the rate-region of the degraded broadcast channel is always convex. Thus our approach is to convert the problem given in (5) into a convex problem, by utilizing another characterization of $\boldsymbol{R} = (R_1, R_2, ..., R_M)$ for the Gaussian broadcast channel capacity. Such an alternative characterization was given in [17], and we start by translating it into our notation.

A. Rederivation of the equivalent representation

The rate vector $\boldsymbol{R}$ on the boundary of the Gaussian broadcast capacity region corresponds to a power allocation $P_1, P_2, ..., P_M$. Solving the value of P_i in terms of the rate vector $\boldsymbol{R}$ gives an equivalent system of equations, and through further simplification, we have[1]

$$\sum_{i \geq m} P_i = \sum_{i \geq m} (n_i - n_{i+1}) \exp\left(2 \sum_{j=m}^{i} R_j\right) - n_m, \quad m = 1, 2, 3, ..., M, \tag{6}$$

where we define $n_{M+1} \triangleq 0$. The RHS of the above equation is monotonically decreasing in m, hence for any non-negative rate vector $\boldsymbol{R}$, provided that (6) is satisfied for $m = 1$, i.e.,

$$P = \sum_{i \geq 1} (n_i - n_{i+1}) \exp\left(2 \sum_{j=1}^{i} R_j\right) - n_1,$$

[1]The equations given in [17] appears to have a minor typographical error that the inner summation was given as $\sum_{j=1}^{i} R_j$.

there must be a power allocation such that (6) is satisfied for all m, and it is on the boundary of the capacity region. Thus we can alternatively characterize the capacity region as

$$\mathcal{C}=\left\{(R_1,R_2,...,R_M): R_m\geq 0,\ m=1,2,...,M,\ \sum_{i=1}^{M}(n_i-n_{i+1})\exp\left(2\sum_{j=1}^{i}R_j\right)-n_1\leq P\right\}.$$

Moreover, the function on the left hand side of this inequality is convex in $(R_1,R_2,...,R_M)$, which is clear by observing its sum-exponential form.

We can now reformulate the optimization problem as a convex programming problem:

$$\min\sum_{i=1}^{M}p_i\exp(-2b\sum_{j=1}^{i}R_j)$$

$$\text{subject to:}\qquad R_i\geq 0,\quad i=1,2,...,M,$$
$$\textstyle\sum_{i=1}^{M}(n_i-n_{i+1})\exp\left(2\sum_{j=1}^{i}R_j\right)-n_1\leq P.$$

If the optimal rate allocation $\boldsymbol{R}$ can be found, then the corresponding optimal power allocation can be recovered from it.

B. The Lagrangian formulation and the algorithm

Now consider the Lagrangian form

$$L=\sum_{i=1}^{M}p_i\exp(-2b\sum_{j=1}^{i}R_j)-\sum_{i=1}^{M}\nu_iR_i+\lambda\left(\sum_{i=1}^{M}(n_i-n_{i+1})\exp\left(2\sum_{j=1}^{i}R_j\right)-n_1-P\right).$$

The Karush-Kuhn-Tucker (KKT) condition requires that $\frac{\partial L}{\partial R_m}=0$ for the optimal solution [21], which subsequently yields

$$\frac{\partial L}{\partial R_m}=-\sum_{i=m}^{M}2bp_i\exp(-2b\sum_{j=1}^{i}R_j)+2\lambda\sum_{i=m}^{M}(n_i-n_{i+1})\exp\left(2\sum_{j=1}^{i}R_j\right)-\nu_m=0,\quad m=1,2,...,M.\tag{7}$$

Taking the difference between $\frac{\partial L}{\partial R_m}=0$ and $\frac{\partial L}{\partial R_{m+1}}=0$ gives

$$-2bp_m\exp(-2b\sum_{j=1}^{m}R_j)+2\lambda(n_m-n_{m+1})\exp\left(2\sum_{j=1}^{m}R_j\right)=\nu_m-\nu_{m+1},\quad m=1,2,...,M,\tag{8}$$

where for convenience we define $\nu_{M+1}\triangleq 0$. $\lambda\geq 0$ and $\nu_m\geq 0$, $m=1,2,...,M$ are the Lagrangian multipliers. Furthermore, the complementary slackness requires $\nu_mR_m=0$, $m=1,2,...,M$; the power constraint should be satisfied with equility, or $\lambda=0$. Note that the set of equations in (8) also implies the set of equations (7), and thus they are equivalent.

Since the optimization problem is a convex programming problem, the KKT condition is both necessary and sufficient for an optimal solution [21]. Clearly if the quantity

$$\kappa_m \triangleq \frac{bp_m}{n_m - n_{m+1}}$$

is monotonically increasing, we can set $\nu_i = 0$ for $i = 1, 2, ..., M$ and find an explicit solution, provided that the power constraint is not violated; however this is not true in general. Nevertheless, the κ factor plays an extremely important role, which in fact reveals certain inherent structure of the layers.

We observe the following simple fact to motivate the algorithm: whenever two consecutive layers yield two κ factors that are not increasing, some combination of the layers must occur such that one layer is assigned zero rate. This is because otherwise the resulting rate vector will include negative rates, thus invalid. To be more precise, two consecutive layers are said to be combined when the higher layer is not allocated any positive rate, such that the two layers can be treated as a single aggregated layer with the sum-probability mass, and the power gain of the lower layer.

The following algorithm can be used to find the optimal rate allocation. In the sequel, when a layer is assigned zero rate, it will be called *ineffective*; otherwise it will be called *effective*. For simplicity, define $\kappa_{M+1} = \infty$. A layer is labeled *active* if it is a result of combination of layers in the immediate previous loop. An intuitive explanation is given in the next subsection.

---

1) Combination of layers to reach a monotonic κ sequence.
   a) Assign $\Delta n_m = n_m - n_{m+1}$ and calculate κ_m for $m = 1, 2, ..., M$. Label all the layers effective and active. Let $r = 1$.
   b) Denote the lower effective neighbor of layer i as i^-, and its upper effective neighbor layer as i^+. Start from $i_k = i_1$, for all the a_r active layers $i_1, i_2, ..., i_{a_r}$:
      i) If $i_k > 1$ and $\kappa_{i_k^-} \geq \kappa_{i_k}$: label layer i_k ineffective and combine it with its **current** lower effective neighbor layer j. Update $p_j = p_j + p_{i_k}$, $\Delta n_j = \Delta n_j + \Delta n_{i_k}$, as well as κ_j values accordingly. Label j as active.
      ii) If $\kappa_{i_k} \geq \kappa_{i_k^+}$: label layer i_k^+ ineffective and combine it with its **current** lower effective neighbor layer j. Update $p_j, \Delta n_j$ and κ_j values accordingly. Label j as active.
      iii) If $k < a_r$, increment k by 1 and return to step (1(b)i).
   c) If after the above loop, any layer remains active: increment r by 1 and return to step (1b).
2) Denote the number of effective layers by K. For all the effective layers i_k, $k = 1, 2, ..., K$, let $\exp(2R_{i_k}) = \kappa_{i_k}^{1/(b+1)} / \kappa_{i_{k-1}}^{1/(b+1)}$; in other words for all the effective layers we have $\exp(2\sum_{j=1}^{i_k} R_j) = \kappa_{i_k}^{1/(b+1)}$. Assign the ineffective layers rate zero.
3) Check power consumption.
   a) Let i_{k_o} be the lowest effective layer, and define $P_n = P + n_{i_{k_o}}$.
   b) Let

$$\lambda^{1/(b+1)} = \frac{\sum_{k=k_o}^{K} (n_{i_k} - n_{i_{k+1}}) \exp\left(2\sum_{j=1}^{i_k} R_j\right)}{P_n}.$$

c) If

$$\frac{bp_{i_{k_o}}}{\lambda(n_{i_{k_o}} - n_{i_{k_o+1}})} \geq 1, \tag{9}$$

then reduce $R_{i_{k_o}}$ by $\frac{1}{2(b+1)} \log \lambda$, and the algorithm terminates; otherwise, label i_{k_o} ineffective ($R_{i_{k_o}} = 0$), increment k_o by 1, update $P_n = P + n_{i_{k_o}}$, and return to step (3b).

C. *The correctness of the algorithm and its complexity*

Intuitively, the layers are classified into two kinds: those with κ value lower than or equal to their lower neighbors (*the first kind*), and those with κ value higher than their lower neighbors (*the second kind*); see Fig. 2. The algorithm combines the first kind layers to their lower neighbors in each step, and then continues this operation until no layers of the first kind exist in the resulting sequence. The resulting rate allocation is valid, if the κ sequence is indeed monotonically increasing, the rates are non-negative and the power is used up.

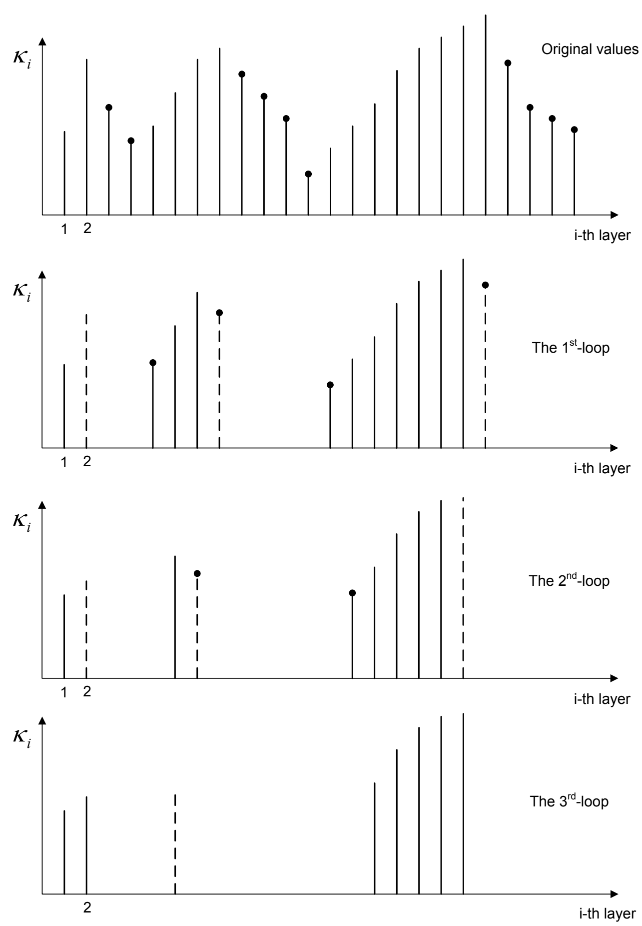

Fig. 2. An example of the algorithm: the lines with dots on the top are the layers of the first kind, and dashed lines are the active layers after the previous loop; the active layers are not labeled before the first step. In the example, Step-1 of the algorithm terminates after three loops. At each loop, some layers are combined with its lower neighbor layers (and removed).

A few more comments are in order: 1) In step 1, we seek to form a monotonic sequence of κ_i by combining consecutive layers, such that step 2 can provide meaningful rates. To do this, we combine (remove) all the layers that are monotonically non-increasing. Only the neighbors of those layers whose κ values were updated in the immediate previous loop need to be considered, because this is the only case that a change of classification may occur. 2) In step 3, we need to assure that the total power is used up by adjusting the value of λ. However, this has to be done such that the lowest layer still has positive rate, which is the condition in (9). If this is not possible, the lowest effective layer is eliminated; this condition is checked repeatedly for the reduced layers until it is satisfied. 3) In the loop of step (1b), we emphasize that the layer is combined with its current effective lower layer, because the layer i_k^- (or i_k) may become ineffective in the previous steps.

The complexity of the algorithm is $O(M)$. Step 2 is clearly of $O(M)$ complexity. In Step 3, λ is updated less than M times. A close inspection of the summation in the numerator reveals that each time it can be done with $O(1)$ complexity, and thus Step 3 is of $O(M)$ complexity. The complexity of Step 1 is more subtle. The value of κ_i can be computed in $O(M)$ complexity. Denote the number of loops in Step 1 as r_0; denote the number of layers with κ value lower than or equal to its lower neighbor (*the first kind*) in the r-th loop as b_r.The complexity in the r-th loop is bounded by linear term of b_r, however $b_r \leq 2a_r$, because only the active layers and their lower effective neighbor layers can be of the first kind. Moreover, notice that $\sum_{r=2}^{r_0} a_r \leq M$, since it is upper bounded by the total number of layers made ineffective, further implied by the fact that a layer is active only when a layer is made ineffective and combined into it. Clearly we have $a_1 = M$, and thus $\sum_1^{r_0} a_r \leq 2M$. The overall complexity is thus $O(M)$, and then the conversion into power allocation is of $O(M)$ complexity.

We note that in order to achieve the $O(M)$ complexity of the given algorithm, a fairly involved data structure is needed. More precisely, a doubly-linked list to update effective layers, coupled together with a singly-linked list to update the active layers (which can be combined into one linked-list) appears most appropriate. However, even a naive implementation without such data structure is of $O(M^2)$ complexity, since Step 1 terminates within at most M iterations, and each iteration has maximum complexity $O(M)$.

D. The optimality of the algorithm

Since we are solving a convex optimization problem, and it obviously satisfies Slater's condition, the KKT conditions are sufficient for optimality. Thus the proof for optimality reduces to finding $\nu_i \geq 0$, that satisfy (8) and the complementary slackness condition $\nu_i R_i = 0$, $i = 1, 2, ..., M$, with the solution found by the algorithm; note that the power constraint is already satisfied with equality.

***Theorem** 1:* The algorithm given above finds the optimal rate allocation.

Proof: Since for the effective layers $R_{i_k} > 0$ by definition, we may set $\nu_{i_k} = 0$ to satisfy the complementary slackness condition. There are several cases that we need to consider next:

1) The ineffective layers above the lowest effective layer.
2) The (originally effective) layers which are rendered ineffective by the power constraint, i.e., the layers that become ineffective in step 3.
3) Other ineffective layers below the lowest effective layers, i.e., the layers that become ineffective in step 1 and are below the lowest effective layer.

For the first case, suppose some of these layers are between two effective layers I and J, $I \leq J$; if there are ineffective layers above the highest effective layer, we take $J = M + 1$. From step 3 of the algorithm we can essentially assign the value of $\rho \triangleq \exp(\sum_{j=1}^{I} 2R_j)$ such that

$$-2b[p_I + p_{I+1} + ... + p_{J-2} + p_{J-1}]\rho^{-b} + 2\lambda(n_I - n_J)\rho = 0. \tag{10}$$

Since layer I and J are effective, we set $\nu_I = \nu_J = 0$. By expanding the condition in (8), we have

$$-2bp_k\rho^{-b} + 2\lambda(n_k - n_{k+1})\rho - \nu_k + \nu_{k+1} = 0, \quad k = I, I+1, ..., J-1.$$

Though the above equations (under the solution found by the algorithm) uniquely specify ν_i, $I < i < J$, it is not clear yet whether those values are indeed non-negative. We need the following lemma to proceed, the proof of which is given after the proof of the theorem.

***Lemma** 1:* For the combined layers between layer-I and layer-J, given any j^* such that $I \leq j^* \leq J-2$, we have

$$\kappa_{j^*}^{-} \triangleq \frac{b\sum_{i=I}^{j^*} p_i}{\sum_{i=I}^{j^*} \Delta n_i} \geq \frac{b\sum_{i=j^*+1}^{J-1} p_i}{\sum_{i=j^*+1}^{J-1} \Delta n_i} \triangleq \kappa_{j^*}^{+}. \tag{11}$$

Now we are ready to prove the non-negativeness of ν_i's for the first kind of ineffective layers. The value of ν_i, where $I < i^* < J$, is also specified by

$$-2b\sum_{i=i^*}^{J-1} p_i\rho^{-b} + 2\lambda\sum_{i=i^*}^{J-1} \Delta n_i\rho - \nu_{i^*} = 0, \tag{12}$$

$$-2b\sum_{i=I}^{i^*-1} p_i\rho^{-b} + 2\lambda\sum_{i=I}^{i^*-1} \Delta n_i\rho + \nu_{i^*} = 0. \tag{13}$$

To see ν_{i^*} thus specified is indeed non-negative, suppose $\nu_{i^*} < 0$ was true, then

$$-2b\sum_{i=i^*}^{J-1} p_i\rho^{-b} + 2\lambda\sum_{i=i^*}^{J-1} \Delta n_i\rho = \nu_{i^*} < 0 \tag{14}$$

Lemma 1 asserts that for i^* we have

$$\kappa_{i^*-1}^{-} = \frac{b\sum_{i=I}^{i^*-1} p_i}{\sum_{i=I}^{i^*-1} \Delta n_i} \geq \kappa_{i^*-1}^{+} = \frac{b\sum_{i=i^*}^{J-1} p_i}{\sum_{i=i^*}^{J-1} \Delta n_i}. \tag{15}$$

which implies that

$$-2b\sum_{i=I}^{i^*-1} p_i\rho^{-b} + 2\lambda\sum_{i=I}^{i^*-1} \Delta n_i\rho \leq -2b\sum_{i=i^*}^{J-1} p_i\rho^{-b} + 2\lambda\sum_{i=i^*}^{J-1} \Delta n_i\rho < 0 \tag{16}$$

However this subsequently implies

$$-2b\sum_{i=I}^{J-1} p_i\rho^{-b} + 2\lambda\sum_{i=I}^{J-1} \Delta n_i\rho < 0, \tag{17}$$

which would contradict (10); alternatively we see that (16) would contradict (13) if the supposition was true. This proves that for the first kind of ineffective layers, the given ν_i's are non-negative.

We next consider the second kind of ineffective layers. Suppose the originally effective layers i_k, i_{k+1},...,i_{k+h} become ineffective due to the power constraint. Since they are all effective originally, we have by the monotonicity of the κ factor

$$\frac{bp^*_{i_k}}{n_{i_k}-n_{i_{k+1}}} \le \frac{bp^*_{i_{k+1}}}{n_{i_{k+1}}-n_{i_{k+2}}} \le ... \le \frac{bp^*_{i_{k+h}}}{n_{i_{k+h}}-n_{i_{k+h+1}}}, \tag{18}$$

where we have used p^* to denote the accumulated probability mass after the combining of layers in Step 1 but before Step 3. By step 3 of the algorithm we have

$$\lambda > \frac{bp^*_{i_{k+h}}}{n_{i_{k+h}}-n_{i_{k+h+1}}}. \tag{19}$$

Thus we only need to show the following equations specify a set of non-negative ν_i's for $i = i_k, i_{k+1}, ..., i_{k+h}$:

$$\begin{aligned}
-2bp^*_{i_{k+h}} \quad & +2\lambda(n_{i_{k+h}}-n_{i_{k+h+1}}) \quad & -\nu_{i_{k+h}} = 0,\\
-2bp^*_{i_{k+h-1}} \quad & +2\lambda(n_{i_{k+h-1}}-n_{i_{k+h}}) \quad & -\nu_{i_{k+h-1}} + \nu_{i_{k+h}} = 0,\\
& ... & ...\\
-2bp^*_{i_{k+1}} \quad & +2\lambda(n_{i_{k+1}}-n_{i_{k+2}}) \quad & -\nu_{i_{k+1}} + \nu_{i_{k+2}} = 0,\\
-2bp^*_{i_k} \quad & +2\lambda(n_{i_k}-n_{i_{k+1}}) \quad & -\nu_{i_k} + \nu_{i_{k+1}} = 0.
\end{aligned} \tag{20}$$

From the first equation, we get

$$\nu_{i_{k+h}} = -2bp^*_{i_{k+h}} + 2\lambda(n_{i_{k+h}}-n_{i_{k+h+1}}),$$

and it is non-negative because of (19). From the second equation, we have

$$\nu_{i_{k+h-1}} = -2bp^*_{i_{k+h-1}} + 2\lambda(n_{i_{k+h-1}}-n_{i_{k+h}}) + \nu_{i_{k+h}}, \tag{21}$$

which is also non-negative, because $-2bp^*_{i_{k+h-1}} + 2\lambda(n_{i_{k+h-1}}-n_{i_{k+h}}) \ge 0$ due to (18) and (19), and the last term is non-negative from the proceeding argument. Continue this line of argument, then it is clear

$$\nu_{i_k} \ge \nu_{i_{k+1}} \ge ... \ge \nu_{i_{k+h}} \ge 0. \tag{22}$$

By this we ν_i's for the second kind of ineffective layers are indeed non-negative.

For the third kind of ineffective layers, we might have to split any given layer in the $i_k, i_{k+1}, ..., i_{k+h}$ layers, in order to recover the original layers as well as the corresponding ν values. However, from any one of equations in (20), it is seen that

$$-2bp^*_{i_{k+a}} + 2\lambda(n_{i_{k+a}}-n_{i_{k+a+1}}) = \nu_{i_{k+a}} - \nu_{i_{k+a+1}} \ge 0. \tag{23}$$

Now following the same line of argument as the first kind ineffective layers, we can indeed find the desired ν values for all the split layers in step 1 of the algorithm. This completes the proof for the optimality of the algorithm. ■

It now remains to prove Lemma 1, before which we first give the following useful facts.

***Lemma** 2:* When all the quantities are positive

- If

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq ... \geq \frac{a_n}{b_n} \geq \frac{e_1}{f_1} \geq \frac{e_2}{f_2} \geq ... \geq \frac{e_n}{f_n}, \tag{24}$$

then we have

$$\frac{\sum a_i}{\sum b_i} \geq \frac{\sum e_i}{\sum f_i}. \tag{25}$$

- If

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq ... \geq \frac{a_n}{b_n} \geq \frac{c_1 + c_2}{d_1 + d_2} \geq \frac{e_1}{f_1} \geq \frac{e_2}{f_2} \geq ... \geq \frac{e_m}{f_m}, \quad \text{and} \quad \frac{c_1}{d_1} \geq \frac{c_2}{d_2}, \tag{26}$$

then we have

$$\frac{\sum a_i + c_1}{\sum b_i + d_1} \geq \frac{\sum e_i + c_2}{\sum f_i + d_2}. \tag{27}$$

These facts are straightforward by elementary calculations and thus the detailed proof is omitted.

Proof of Lemma 1

We use an induction approach and show this fact is true after any loop in Step 1 of the algorithm. The statement is clearly true after the first loop, by the given monotonicity of the κ sequence for any combined layer before the operation, and the first fact in Lemma 2. Then consider r-th step for step (1b). For a given index j^*, let $\kappa^-_{j^*,r-1}$ and $\kappa^+_{j^*,r-1}$ denote the appropriate quantities after step $r-1$; i.e., the quantities defined in Lemma 1 in the interval I to J for which layer j^* is in. Denote the updated layer that the original layer j^* is in as i^*. By the procedure, suppose the aggregated layers after the $(r-1)$-th loop $i_1 < i_2 < ... < i_n < i^* < j_1 < j_2 < ... < j_m$ are to be combined into a new aggregated layer in the r-th loop, which satisfy

$$\kappa_{i_1} \geq \kappa_{i_2} \geq ... \geq \kappa_{i_n} \geq \kappa_{i^*} \geq \kappa_{j_1} \geq \kappa_{j_2} \geq ... \geq \kappa_{j_m}, \tag{28}$$

as well as

$$\kappa^-_{j^*,r-1} \geq \kappa^+_{j^*,r-1}. \tag{29}$$

Then by the second fact in Lemma 2, we have $\kappa^-_{j^*,r} \geq \kappa^+_{j^*,r}$. This induction is apparently true for any steps in Step 1 of the algorithm, thus the lemma is proved. ■

E. The dual problem

We can also consider the dual problem of minimizing the power consumption for a given expected distortion value. This can be done with essentially no change to the problem as

$$\min \sum_{i=1}^{M} (n_i - n_{i+1}) \exp\left(2 \sum_{j=1}^{i} R_j\right) - n_1 \triangleq P^*,$$

subject to: $R_i \geq 0, \quad i = 1, 2, ..., M,$

$\sum_{i=1}^{M} p_i \exp(-2b \sum_{j=1}^{i} R_j) \leq D.$

The Lagrangian form is almost without change as

$$L = \lambda \sum_{i=1}^{M} p_i \exp(-2b \sum_{j=1}^{i} R_j) + \left(\sum_{i=1}^{M} (n_i - n_{i+1}) \exp\left(2 \sum_{j=1}^{i} R_j\right) - n_1 - P \right) - \sum_{i=1}^{M} \nu_i R_i.$$

The following condition similar to (8) can be derived, where again $\nu_{M+1} \triangleq 0$

$$-2\lambda b p_m \exp(-2b \sum_{j=1}^{m} R_j) + 2(n_m - n_{m+1}) \exp\left(2 \sum_{j=1}^{m} R_j\right) - \nu_m + \nu_{m+1} = 0, \quad m = 1, 2, ..., M. \tag{30}$$

From this form we see that the first and second step of the algorithm can be used without any change, and in the third step of the algorithm, only very minor changes are needed for this dual problem. For simplicity we combine the layers into the aggregated ones and take the aggregated probability mass as p'_i.

3a) Let i_{k_o} be the lowest effective layer, and define $D' = D - p'_{i_{k_o}}$.

3b) Let

$$\lambda^{1/(b+1)} = \frac{D}{\sum_{k=k_o}^{k_K} p_i \exp\left(-2b \sum_{j=1}^{i_k} R_j\right)}. \tag{31}$$

3c) If

$$\frac{\lambda b p_{i_{k_o}}}{(n_{i_{k_o}} - n_{i_{k_o}+1})} \geq 1,$$

then reduce $R_{i_{k_o}}$ by $\frac{1}{2(b+1)} \log \lambda$; otherwise, label i_{k_o} ineffective ($R_{i_{k_o}} = 0$), increment k_o by 1, update $D' = D' - p'_{i_{k_o}}$, and return to step 3b).

The proof of correctness and optimality remains virtually unchanged, and the complexity is also unchanged.

IV. Variational derivation of the continuous case solution

We next turn our attention to the case of continuum of layers, which is in fact the case considered in [2]. To facilitate understanding, we first give a less technical derivation under the assumption that the optimal power allocation concentrates on a single interval of the power gain range, and show that this is indeed true for some probability density function $f(s)$. This simple derivation provides important intuitions for the general case, based on which a more general derivation is then given. For simplicity, we first assume $f(s)$ has support on the entire non-negative real line $[0, \infty)$; later it is shown that this assumption can be relaxed.

A. A simple derivation for the single interval solution

In this sub-section we give a simple derivation under the assumption that there is a unique interval $[s_1, s_2]$ which the power allocation concentrates on. The optimization problem can be reformulated as follows. Define

$$I(i) = \exp(\sum_{j=1}^{i} 2R_j). \tag{32}$$

We take the number of layers to infinity, and the objective function the constraint become integrals. The function $I(s)$ we need to find is

$$I(s) = \exp(2 \int_0^s R(u)du) \tag{33}$$

where we convert back to the power gain s instead of noise power n, and $R(s)$ is the rate density associated with a fading gain s. It is clear we can replace the inequality constraint by equality constraint without loss of optimality

$$\int_0^\infty s^{-2}\exp(2\int_0^s R(u)du)ds = \int_0^\infty \frac{I(s)}{s^2}ds = P. \tag{34}$$

The term to be minimized is

$$\overline{D}(I) = \int_0^\infty f(s)\exp\left(-2b\int_0^s R(u)du\right)ds = \int_0^\infty \frac{f(s)}{I(s)^b}ds. \tag{35}$$

Note the additional condition that $I(s)$ has to be monotonically non-decreasing, and the the boundary conditions $I(0) = 1$.

Ignoring the positivity constraint $I'(s) \geq 0$ for now, take

$$J(s, I, I') = \frac{f(s)}{I(s)^b}, \quad G(s, I, I') = \frac{I(s)}{s^2},$$

the optimization problem can thus be written in the usual variational notation as

$$\text{minimize} \qquad \int_0^\infty J(s, I, I')ds, \tag{36}$$

$$\text{subject to} \qquad \int_0^\infty G(s, I, I')ds = P. \tag{37}$$

Next we assume there is a unique interval $[s_1, s_2]$ for which power allocation is non-zero. Under this assumption, the objective function reduces to

$$\overline{D}(I) = \int_0^\infty J(s, I, I')ds = \int_{s_1}^{s_2} \frac{f(s)}{I(s)^b}ds + F(s_1) + \frac{1 - F(s_2)}{I(s_2)^b}, \tag{38}$$

where $F(s)$ is the cumulative distribution function of the fading gain random variable, i.e. $F(s) = \int_0^s f(u)du$, and the constraint becomes

$$P(I) = \int_0^\infty G(s, I, I')ds = \int_{s_1}^{s_2} I(s)\frac{1}{s^2}ds + \frac{I(s_2)}{s_2} - \frac{1}{s_1} = P. \tag{39}$$

We can write the Lagrangian form $L(I) = \overline{D}(I) + \lambda(P(I) - P)$. To find the extremal solution, we consider an increment $q(s)$ on $I(s)$, and the increment of the Lagrangian functional is given by $\Delta(q) = L(I+q) - L(I)$. Take an arbitrary increment $q(s)$ with $q(s) = 0$ for $s \notin [s_1, s_2]$ and $q(s_1) = 0$, and the principal linear part of the increment given in the following equation should be zero (see (27) p. 25, and pp. 42-50 in [16])

$$\begin{aligned} \delta L(I) &= \int_{h_1}^{h_2} \left(J_I + \lambda G_I - \frac{d}{ds}[J_{I'} + \lambda G_{I'}]\right) q(s)ds \\ &\quad + (J_{I'} + \lambda G_{I'})|_{s=s_2} q(s_2) + \left[\frac{-b(1-F(s_2))}{I(s_2)^{b+1}} + \lambda \frac{1}{s_2}\right] q(s_2) = 0 \end{aligned} \tag{40}$$

where the last term in the summation is due to the terms outside of the integration in (38) and (39). Since $q(s)$ can be arbitrary, we have

$$J_I + \lambda G_I - \frac{d}{ds}[J_{I'} + \lambda G_{I'}] = 0, \tag{41}$$

with

$$J_I = \frac{-bf(s)}{I^{b+1}(s)}, \quad G_I = \frac{1}{s^2}, \quad J_{I'} = G_{I'} = 0, \tag{42}$$

which further simplifies to

$$I(s) = \left(\frac{bf(s)s^2}{\lambda}\right)^{1/(b+1)}. \tag{43}$$

At this point, it is clear that for $I'(s) \geq 0$ to be true, which is necessary for $I(s)$ to be a valid solution, $f(s)s^2$ should have non-negative derivative in any interval such that (43) holds; in fact for any interval that positive rate is allocated to, $f(s)s^2$ should have strictly positive derivative such that $I(s)$ is strictly increasing. If there is only one interval over the support of $f(s)$ where $f(s)s^2$ has strictly positive derivative, then the single interval solution assumption is indeed true. Now since $q(s_2)$ can be arbitrary, at this variable end (pp. 25-29 in [16]) a necessary condition for an extremum is

$$\frac{-b(1-F(s_2))}{I(s_2)^{b+1}} + \lambda\frac{1}{s_2} = 0, \tag{44}$$

which gives

$$\lambda = \frac{bs_2(1-F(s_2))}{I(s_2)^{b+1}}. \tag{45}$$

Because $I(s_1) = 1$, $\lambda = bf(s_1)s_1^2$, the expression of $I(s)$ gives one boundary condition

$$1 - F(s_2) = f(s_2)s_2. \tag{46}$$

The lower bound s_1 is determined by the power constraint, from which we have

$$\int_{s_1}^{\infty} \frac{I(s)}{s^2} ds = \int_{s_1}^{s_2} \left(\frac{f(s)}{f(s_1)s_1^2} \right)^{1/(b+1)} s^{-2b/(b+1)} ds + \frac{1}{s_2} \left(\frac{f(s_2)s_2^2}{f(s_1)s_1^2} \right)^{1/(b+1)} = P + \frac{1}{s_1}, \tag{47}$$

where in the second equation we split the integral into two parts partitioned by $s = s_2$. We have thus found the unique extremal solution

$$I(s) = \left(\frac{f(s)s^2}{f(s_1)s_1^2} \right)^{1/(b+1)} \tag{48}$$

in $[s_1, s_2]$ with the boundary conditions specified by (46) and (47).

To find the corresponding power allocation, define $T(s) = \int_s^{\infty} P(r)dr$. We derive from (6) that

$$T(s) = \int_{s}^{\infty} \frac{I(r)}{I(s)} dr - \frac{1}{s} = \left(\frac{f(s_2)s_2^2}{f(s)s^2} \right)^{1/(b+1)} \frac{1}{s_2} + \int_s^{s_2} \left(\frac{f(r)r^2}{f(s)s^2} \right)^{1/(b+1)} \frac{1}{r^2} dr - \frac{1}{s}. \tag{49}$$

Through some basic algebra, it can be shown that this is in fact the same solution as that in [11]. Thus the limit of the optimal solution of the discrete case in [11] indeed converges to the extremal solution derived through the classical variational method. Moreover, the variational method derivation directly asserts that $f(s)s^2$ has a non-negative derivative for any positive power allocation interval; this condition was however lacking in the derivation in [11].

B. *Derivation of the general case solution*

Next we provide a more technical and complete derivation for the general case when the power allocation does not necessarily concentrate in a single interval of the range of power gain. From the physical meaning of $I(s)$, we assume that $I(s)$ is a piecewise smooth continuous and differentiable function, and it is within this function space we seek the optimal solution. With the positivity condition of $I'(s) \geq 0$, the Lagrangian functional subject to optimization can be written as

$$L(I) = \int_{0}^{\infty} \left(\frac{f(s)}{I(s)^b} + \lambda \frac{I(s)}{s^2} - I'(s)v(s) \right) ds \tag{50}$$

where $v(s)$ is an arbitrary non-negative function; see page 249 in [22]. The general extremum and subsidiary conditions are then similar as in the last sub-section, and we state here for completeness

$$-b \frac{f(s)}{I(s)^{b+1}} + \frac{\lambda}{s^2} + v'(s) = 0 \tag{51}$$

and the complementary slackness conditions are

$$\lambda \left(\int_{0}^{\infty} \frac{I(s)}{s^2} ds - P \right) = 0, \tag{52}$$

$$I'(s)v(s) = 0 \tag{53}$$

We thus have the general solution for $I(s)$ in some interval where $I'(s) \neq 0$,

$$I(s) = \left(\frac{bf(s)s^2}{\lambda}\right)^{1/b+1}, \tag{54}$$

which has the same form as given in the previous sub-section. We first observe from the complementary slackness in (53) that for any $I'(s) \neq 0$, it must be true that $v(s) = 0$. Thus for any interval $[a, b]$ for which $I'(s) \neq 0$, it is seen that

$$-\frac{bf(s)}{I^{b+1}(s)} + \lambda\frac{1}{s^2} = 0. \tag{55}$$

This implies that $\lambda \neq 0$, because otherwise, $I(s) = \infty$ in this interval, which clearly violates the power constraint. Then the first complementary slackness condition requires

$$\int_0^\infty I(s)\frac{1}{s^2}ds - P = 0. \tag{56}$$

Because for any interval $[a, b]$ for which $I'(s) > 0$, we have

$$\frac{bf(s)}{I^{b+1}(s)} = \lambda\frac{1}{s^2}, \tag{57}$$

it is clear that $f(s)s^2$ has to have positive derivatives in any interval which are allocated with positive rate (and power); if on the other hand this condition is not satisfied, then $I'(s) = 0$ for this interval, and accordingly $v(s)$ can be strictly positive.

Given the above discussion, it is clear that disjoint intervals with positive rate (power) allocation are separated by intervals where $I'(s) = 0$. Moreover, it will be shown in the sequel that only a single continuous allocation interval may occur within a single interval where the derivative of $f(s)s^2$ is strictly positive; we shall assume that the number of such intervals is finite. For simplicity, assume that the first such interval has the form $[0, a]$, i.e., the lower boundary is zero; this condition can be relaxed as shown in the next sub-section. Denote there are a total of K positive power allocation intervals, and let the i-th positive power allocation interval be specified by $[s_{i,l}, s_{i,u}]$, and label the positive $f(s)s^2$ derivative interval it belongs to as the i-th such interval, specified by $[l_i, u_i]$; i.e., $[s_{i,l}, s_{i,u}] \subseteq [l_i, u_i]$.

For a piecewise smooth continuous extremal solution, the Weierstrass-Erdmann (corner) conditions must be satisfied (page 63 in [16]). These are given for every corner point $s_c \in \{s_{i,l}, s_{i,u}\}_{i=1}^K$. The two conditions are

$$\begin{array}{c} L_{I'}|_{s=s_c^-} = L_{I'}|_{s=s_c^+} \\ (L - I'L_{I'})\,|_{s=s_c^-} = (L - I'L_{I'})\,|_{s=s_c^+} \end{array}. \tag{58}$$

By substituting $L(I)$ from (50) into the corner conditions (58), we get

$$v(s_c^-) = v(s_c^+) \tag{59}$$

and

$$\left(\frac{f(s)}{I(s)^b} + \lambda\frac{I(s)}{s^2} + I'(s)v(s) - I'(s)v(s)\right)\Big|_{s=s_c^-} = \left(\frac{f(s)}{I(s)^b} + \lambda\frac{I(s)}{s^2} + I'(s)v(s) - I'(s)v(s)\right)\Big|_{s=s_c^+} \tag{60}$$

which further simplifies into

$$\frac{f(s_c)}{I(s_c^-)^b} + \lambda \frac{I(s_c^-)}{s_c^2} = \frac{f(s_c)}{I(s_c^+)^b} + \lambda \frac{I(s_c^+)}{s_c^2} \tag{61}$$

where (59) and (61) impose certain continuity conditions on $v(s)$ and on $I(s)$, respectively. Thus $v(s) = 0$ for $s \in [s_{i,l}, s_{i,u}]$ for all $i = 1, 2, ..., K$, and is non-negative otherwise; from (59), we have $v(s_{i,u}^+) = 0$ and $v(s_{i,l}^-) = 0$ for all $i = 1, 2, ..., K$.

The continuity of $v(s)$ at the corner points does not rely on the presumption that in the extremal solution, the power allocation in a single interval $[l_n, u_n]$ with positive $s^2 f(s)$ derivative does not have within it more than one disjoint positive power allocation sub-interval. We now prove that the presumption is indeed true. Suppose otherwise, which implies the range $(c, d) \subseteq [l_n, u_n]$ is assigned with $I'(s) = 0$, and there exist $c' \in [l_n, c)$ and $d' \in (d, u_n]$ such that $I'(s) > 0$ in $[c', c]$ and $[d, d']$. We have that

$$v(c) = v(d) = 0, \tag{62}$$

$$v'(s) = \frac{bf(s)}{I^{b+1}(c)} - \lambda \frac{1}{s^2} \text{ for } s \in (c, d) \tag{63}$$

$$\frac{bf(c)}{I^{b+1}(c)} - \lambda \frac{1}{c^2} = 0, \tag{64}$$

where (62) is due to the preceding discussion, (63) is due to the (51), and (64) is due to (57), the continuity of $I(s)$ and the continuous assumption on $f(s)$. Since $s^2 f(s)$ has strictly positive derivative in (c, d), we have that in this interval

$$v'(s) = \frac{bf(s)}{I^{b+1}(c)} - \lambda \frac{1}{s^2} > 0. \tag{65}$$

But this contradicts with (62). Thus the supposition can not be true, and there can be only one effective allocation interval in each such first kind interval $[l_n, u_n]$.

In order to determine all corner points $s_c \in \{s_{i,l}, s_{i,u}\}_{i=1}^K$, we rewrite the original functional (50), as a piecewise optimization problem, and derive the variation.

$$\begin{aligned} L(I) &= F(s_{1,l}) - \lambda \frac{1}{s_{1,l}} - \int_0^{s_{1,l}} I'(s)v(s)ds + \sum_{i=1}^K \left[\int_{s_{i,l}}^{s_{i,u}} \left(\frac{f(s)}{I(s)^b} + \lambda \frac{I(s)}{s^2} - I'(s)v(s) \right) ds \right] \\ &\quad + \sum_{i=1}^K \left[\frac{F(s_{i+1,l}) - F(s_{i,u})}{I(s_{i,u})^b} + \lambda I(s_{i,u}) \left(\frac{1}{s_{i,u}} - \frac{1}{s_{i+1,l}} \right) - \int_{s_{i,u}}^{s_{i+1,l}} I'(s)v(s)dh \right] \end{aligned} \tag{66}$$

where we define $s_{K+1,l} \triangleq \infty$, and thus $F(s_{K+1,l}) = 1$. The variation of $L(I)$ in (66) w.r.t. an arbitrary function

$q(s)$, which satisfies $q'(s) = 0$ for $s \notin [s_{i,l}, s_{i,u}]$, is given by

$$\begin{aligned}\delta L(I) &= \sum_{i=1}^{K}\left[\int\limits_{s_{i,l}}^{s_{i,u}}\left(D_I - \frac{d}{ds}D_{I'}\right)q(s)ds + D_{I'}|_{s=s_{i,u}}q(s_{i,u}) - D_{I'}|_{s=s_{i,l}}q(s_{i,l})\right] \\ &\quad + \sum_{i=1}^{K}\left[-b\frac{F(s_{i+1,l}) - F(s_{i,u})}{I(s_{i,u})^{b+1}} + \lambda\left(\frac{1}{s_{i,u}} - \frac{1}{s_{i+1,l}}\right)\right]q(s_{i,u}) \\ &= \sum_{i=1}^{K}\left[\int\limits_{s_{i,l}}^{s_{i,u}}\left(\frac{-bf(s)}{I^{b+1}(s)} + \lambda\frac{1}{s^2}\right)q(s)ds\right] + \sum_{i=1}^{K}\left[-b\frac{F(s_{i+1,l}) - F(s_{i,u})}{I(s_{i,u})^{b+1}} + \lambda\left(\frac{1}{s_{i,u}} - \frac{1}{s_{i+1,l}}\right)\right]q(s_{i,u})\end{aligned} \tag{67}$$

where $D(I) \triangleq \frac{f(s)}{I(s)^b} + \lambda\frac{I(s)}{s^2} - I'(s)v(s)$. Note that for $I(s)$ to remain a continuous function, it must be true that $q(s_{i,u}) = q(s_{i+1,l})$.

Hence the general solution extremal expression is the same as that afore-specified, with an additional condition, arising from the variable end-point problem similar as in the previous sub-section. This condition is given by

$$-b\frac{F(s_{i+1,l}) - F(s_{i,u})}{I(s_{i,u})^{b+1}} + \lambda\left(\frac{1}{s_{i,u}} - \frac{1}{s_{i+1,l}}\right) = 0, \tag{68}$$

By using the general solution for $I(s)$ within $[s_{i,l}, s_{i,u}]$ as given in (54), the condition (68) becomes

$$\frac{F(s_{i+1,l}) - F(s_{i,u})}{f(s_{i,u})s_{i,u}^2} - \left(\frac{1}{s_{i,u}} - \frac{1}{s_{i+1,l}}\right) = 0. \tag{69}$$

From the continuity conditions on the corner points, i.e. $I(s_{i,u}) = I(s_{i+1,l})$ the next condition is obtained

$$f(s_{i+1,l})s_{i+1,l}^2 = f(s_{i,u})s_{i,u}^2. \tag{70}$$

For $i = K$, the condition (69) simplifies into

$$1 - F(s_{i,u}) = f(s_{i,u})s_{i,u}, \tag{71}$$

which is (46) when $K = 1$.

To summarize, the extremal solution is given by

$$I(s) = \left(\frac{f(s)s^2}{f(s_{1,l})s_{1,l}^2}\right)^{1/(b+1)}, \tag{72}$$

in the intervals $[s_{i,l}, s_{i,u}]$, $i = 1, 2, ..., K$, and the boundary values are determined by

$$\frac{F(s_{i+1,l}) - F(s_{i,u})}{1/s_{i,u} - 1/s_{i+1,l}} = f(s_{i,u})s_{i,u}^2 = f(s_{i+1,l})s_{i+1,l}^2, \quad i = 1, 2, ..., K, \tag{73}$$

as well as the the power constraint

$$\int\limits_0^\infty \frac{I(s)}{s^2}ds = P \tag{74}$$

These are the necessary conditions to determine an extremum.

C. Comments on relaxing the support condition

Until this point, we have assumed that the pdf $f(s)$ has support on the entire non-negative real line $[0, \infty]$, such that the conditions (69) and (70) (or (46) and (47)) are sufficient to determine boundary values. This assumption can be relaxed to the case that $f(s)$ has a compact support, however certain complication will be introduced. To illustrate this, we next treat one special case where the support of $f(s)$ is on a single finite interval $[s_a, s_b]$ and $s^2 f(s)$ has a unique positive derivative interval $[l_1, u_1]$, such that $s_a \leq l_1$ and $s_b \geq u_1$; the same approach can be extended to more general support. Without loss of generality, we can assume $s_a = l_1$ and $s_b > u_1$. We shall follow the less general derivation as given in Section IV-A for simplicity. The main difference now is that $I(s_1) = 1$ is not necessarily true if $s_1 = s_a$, i.e., it is possible for $I(s)$ to be discontinuous at s_a.

In order to resolve this difficult, first assume $s_1 > s_a$, then indeed we have $I(s_1) = 1$, from which (46) and (47) follow. If the solution given by them indeed satisfies $l_1 < s_1 \leq s_2 \leq u_1$, then an extremal solution is specified by these boundaries. If on the other hand this is not true, then $s_1 = s_a$ and the condition $I(s_1) = 1$ is not necessarily true. Denote the value as $I(s_a) \triangleq I_0$. In this case, i.e. $s_1 = s_a$, the solution in $[s_1, s_2]$ can be written as

$$I(s) = I_0 \left(\frac{f(s)s^2}{f(s_a)s_a^2} \right)^{1/(b+1)}. \tag{75}$$

The boundary condition (46) can still be derived. The other boundary condition derived from the power constraint is now given by

$$I_0 \int_{s_1}^{s_2} \left(\frac{f(s)s^2}{f(s_a)s_a^2} \right)^{1/(b+1)} \frac{1}{s^2} ds + I_0 \left(\frac{f(s_2)s_2^2}{f(s_a)s_a^2} \right)^{1/(b+1)} \left(\frac{1}{s_2} - \frac{1}{s_b} \right) - \frac{1}{s_1} - P = 0. \tag{76}$$

Thus for this case that $s_1 = s_a$, the solution is given by (75) with the value of s_2 and boundary value I_0 determined by (46) and (76).

V. Numerical examples

In this section, several examples are solved using the solution developed in the previous sections. We start with one example motivated by multiple access with user interference, then discuss the Rayleigh fading case, finally a simple example is given for the general solution when the optimal power allocation does not concentrate in a single interval.

A. Interference multiple access channel

Consider here an additive white Gaussian noise (AWGN) channel with interfering users. The goal by a single user is to find the optimal power allocation of transmitting in the presence of unknown interference by the other users. The level of interference depends on the number of users using the multiple access channel at a given transmission block. Such channel models were considered, for example, in [23]–[25]. More precisely, consider the following model

$$y = x + \sum_{i=1}^{N} z_i \Delta_i + w \tag{77}$$

where y is the received signal, and x is the multi-layer transmitted real-valued signal the user can design subject to an average power constraint P. The additive interference z_i's are assumed to be Gaussian distributed, which are the signals sent by other users in this time slot. Finally, $w \sim \mathcal{N}(0, 1)$ is the additive noise. Every interference element z_i is associated with an average power level σ_i^2, and a random binary variable Δ_i to determine whether user i is transmitting at this time slot. Δ_i is assumed to be i.i.d. Bernoulli random variable with $Pr(\Delta_i = 1) = p_{on}$. The exact realization of the Δ_i's are not known to user $j \neq i$, however it is assumed that the value of p_{on} is known. Since the receiver has no knowledge of z_i and cannot attempt joint decoding, it has to treat z_i as AWGN when decoding x. Thus an equivalent channel model representation may be useful,

$$y = \frac{1}{\sqrt{1 + \sum\limits_{i=1}^{N} \sigma_i^2 \Delta_i}} x + w', \tag{78}$$

where $w' \sim \mathcal{N}(0, 1)$. The equivalent fading gain s_j is then

$$s_j = \left(1 + \sum_{i=1}^{N} \sigma_i^2 \Delta_i\right)^{-1}, \quad j = 1, .., 2^N. \tag{79}$$

It is clear that there are 2^N possible fading gain states, which correspond to all possible states of Δ_i. The optimal power allocation for multi-layer coding can be found by the the discrete layering algorithm presented in Section III-B.

We demonstrate by examples of the optimal power allocation for two cases of interference distributions. Figures 3.(a) - 3.(d) show the fading gain discrete density p_i, the optimal discrete power allocation P_i and $I_i \triangleq \exp(-2 \sum_{j=1}^{i} R_j)$, respectively, where there are $N = 4$ interfering users; the transmit power is $P = 0$ dB; the bandwidth expansion is $b = 2$; σ_i, and p_{on} are

$$\sigma = (0.95, 0.45, 0.55, 0.81), \quad p_{on} = 0.3 \tag{80}$$

The second example is given in Figures 4.(a) - 4.(d), where there are $N = 6$ interfering users, with σ_i, and p_{on} specified by

$$\sigma = (0.49, 0.91, 0.51, 0.87, 0.81, 0.46), \quad p_{on} = 0.5 \tag{81}$$

where the transmit power and bandwidth expansions are the same in both examples.

The results may just as well be generalized to the case that p_{on} is different for every user, depending on the scheduling method in the system.

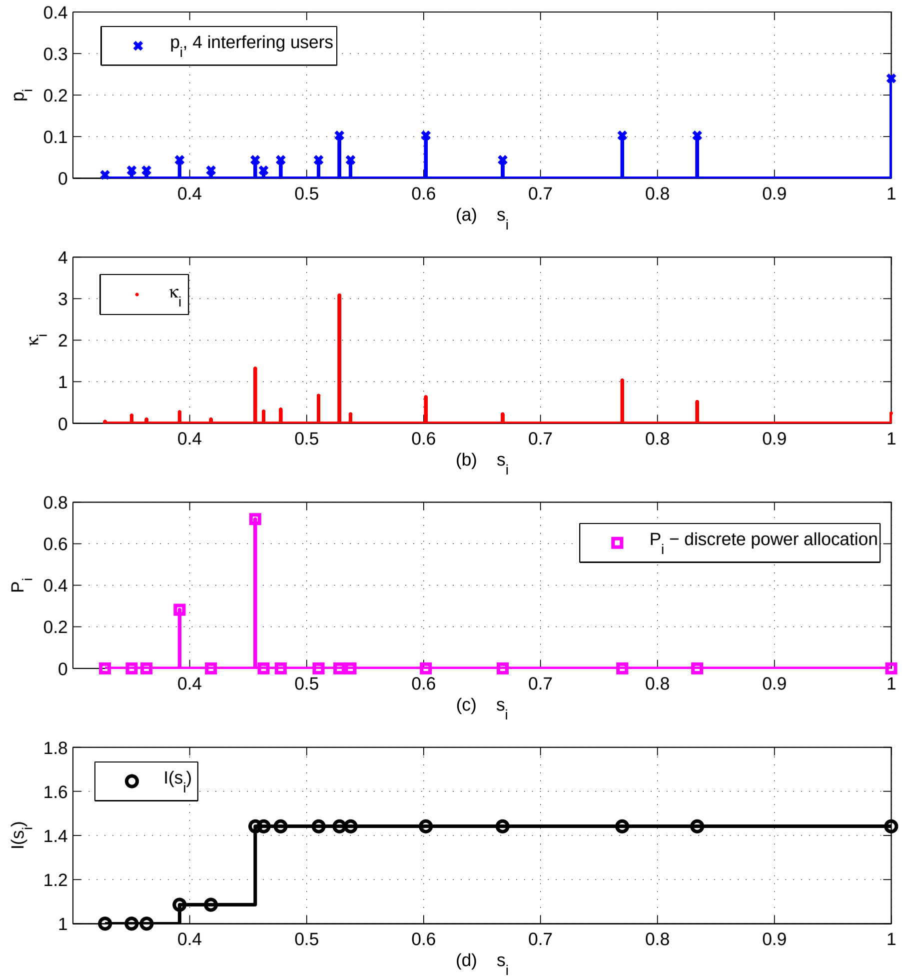

Fig. 3. An example of AWGN interference channel with $N = 4$ interfering users ($p_{on} = 0.3$, $b = 2$, $P = 0$dB). (a) pdf of the fading gain s_i, denoted by p_i. (b) The associated κ_i. (c) Optimal discrete power allocation P_i. (d) Optimal cumulative rate exponent $I(s_i)$.

B. The Rayleigh fading case

We consider here the SISO Rayleigh fading channel. Starting with the outage approach minimal distortion, which is a single level coding distortion bound. This is later compared to the minimal average distortion with continuous broadcasting.

1) Outage approach: In case there is only a single source code and channel code, the transmission scheme is known as the outage approach. When channel conditions allow, the data can be completely recovered, otherwise an outage event occurs. Let the transmission rate be R_1

$$R_1 = \frac{1}{2}\log(1 + s_1 P), \tag{82}$$

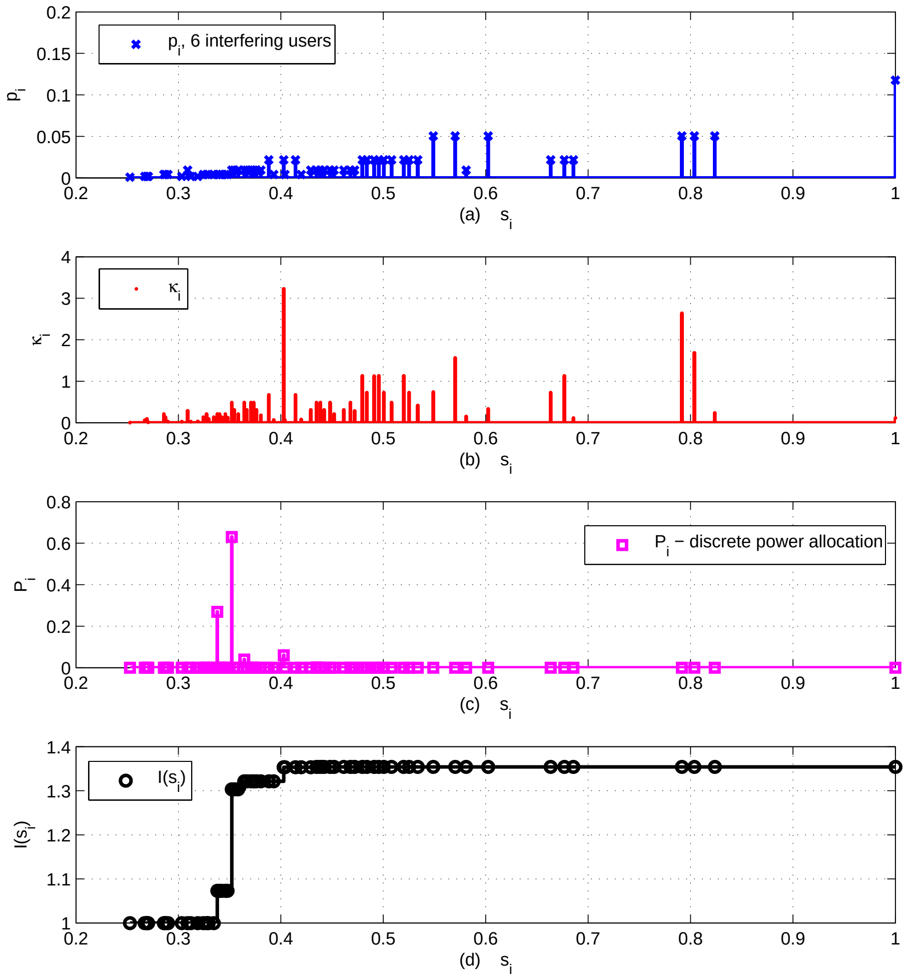

Fig. 4. An example of AWGN interference channel with $N = 6$ interfering users ($p_{on} = 0.3$, $b = 2$, $P = 0$dB). (a) pdf of the fading gain s_i, denoted by p_i. (b) The associated κ_i. (c) Optimal discrete power allocation P_i. (d) Optimal cumulative rate exponent $I(s_i)$.

where s_1 can be considered as the fading gain threshold. For any $s \geq s_1$, a rate R_1 may be achieved, otherwise a failure occurs such that $D_1 = 1$. This is in contrast with the continuous broadcasting, where there is an outage region with different decodable rates depending on the fading gain realization. The obtained distortion when decoding R_1 is $D_1 = (1 + s_1 P)^{-b}$. Under this coding strategy, one can also optimize the expected distortion

$$D_{1,opt} = \min_{s_1 \in [0,\infty)} F(s_1) + (1 + P s_1)^{-b} (1 - F(s_1)) . \tag{83}$$

For a flat Rayleigh fading channel, the fading power distribution is $F(s_1) = 1 - e^{-s_1}$, the average distortion then corresponds to

$$D_{1,opt} = \min_{s_1 \in [0,\infty)} D(s_1) = \min_{s_1 \in [0,\infty)} 1 - e^{-s_1} + (1 + P s_1)^{-b} e^{-s_1} . \tag{84}$$

The condition for extremum, i.e. $\frac{d}{ds_1}D(s_1)=0$, yields the following polynomial,

$$x^{b+1}-x+Pb=0 \tag{85}$$

where $x \triangleq 1+Ps_1$. An explicit solution for $s_{1,opt}$ cannot be derived analytically in general for every b. However, for particular cases it can be analytically solved. For example, for $b=1$, solving (85) gives,

$$s_{1,opt}(b=1)=\frac{1}{2P}\left(1+\sqrt{1+4P}\right)-1/P. \tag{86}$$

2) Continuous case broadcast: We consider the average achievable distortion for a SISO Rayleigh fading channel. Consider the following fading gain distribution,

$$F(s)=1-\exp\left(-\frac{s}{\overline{s}}\right), \tag{87}$$

where $\overline{s}$ is the expected fading gain power. For this distribution the optimal power allocation is single interval continuous, and zero outside the interval $[s_1,s_2]$. That can be immediately observed from $f(s)s^2$ by taking its first derivative

$$\frac{d}{ds}f(s)s^2=(\frac{2s}{\overline{s}}-\frac{s^2}{\overline{s}^2})\exp\left(-\frac{s}{\overline{s}}\right), \tag{88}$$

where $\frac{d}{ds}f(s)s^2\geq 0$ on a single interval $s\in[0,2\overline{s}]$. Then the upper bound $s_2\in[0,2\overline{s}]$ is determined by (46), which reduces to

$$\exp\left(-\frac{s_2}{\overline{s}}\right)=\frac{s_2}{\overline{s}}\exp\left(-\frac{s_2}{\overline{s}}\right) \tag{89}$$

yielding $s_2=\overline{s}$. Solving (47) gives the other boundary value s_1, denoted as $s_{1,opt}$; the condition (47) does not lead to an analytical expression, but can be solved numerically. Then the general expression for $I(s)$, for the Rayleigh fading channel is given by

$$I(s)=\begin{cases} 1 & s\leq s_{1,opt} \\ \left(\frac{s^2}{s_{1,opt}^2}\exp\left(-\frac{s-s_{1,opt}}{\overline{s}}\right)\right)^{1/(b+1)} & s_{1,opt}<s\leq\overline{s} \\ \left(\frac{\overline{s}^2}{s_{1,opt}^2}\exp\left(-\frac{\overline{s}-s_{1,opt}}{\overline{s}}\right)\right)^{1/(b+1)} & s>\overline{s}\end{cases}. \tag{90}$$

In Figure 5, the average distortion bounds for Rayleigh fading channels are demonstrated for three different values of bandwidth expansion values ($b=0.5,1,2$). For every bandwidth expansion, the minimal average distortion of the outage approach and broadcast approach are compared. It can be noticed that the smaller b is, the larger is the broadcast gain, which can be defined as the SNR gain of the broadcast approach over the outage approach for the same average distortion value. Thus the benefit of the broadcast approach compared to the outage directly depends on the system design parameter b.

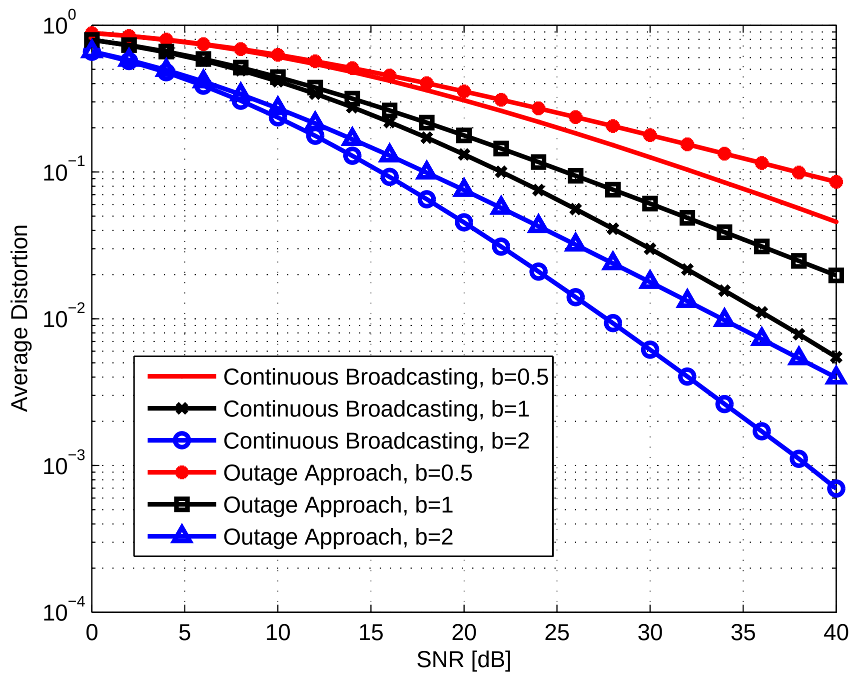

Fig. 5. Minimal average distortion, a comparison of outage approach and broadcast approach, for $b = 0.5, 1, 2$.

C. An example with more than one power allocation interval

Consider the following piece-wise smooth pdf $f(s)$

$$f(s) = \begin{cases} (70c_f(s-0.1)+10)/s^2 & s \in [0.1, 1.1) \\ (-70c_f(s-1.1)+80)/s^2 & s \in [1.1, 2.1) \\ (390c_f(s-2.1)+10)/s^2 & s \in [2.1, 3.1) \\ (-80c_f(s-3.1)+400)/s^2 & s \in [3.1, 4.1] \end{cases} \tag{91}$$

where c_f is a normalizing constant such that $\int_0^\infty f(s)ds = 1$, which can be computed to be $c_f \approx 274.0645$. Fig. 6 shows pdf $f(s)$, $f(s)s^2$ and the optimal continuous power allocation $P(s)$ with $P = 0$ dB. It is worth noting that at the upper bound for the first effective power allocation interval and the lower bound of the second interval, the values of $f(s)s^2$ is exactly the same, as required by (73).

VI. Conclusion

We considered the optimal power/rate allocation in the broadcast strategy, in order to minimize the expected distortion of a quadratic Gaussian source transmitted over a fading channel. A linear complexity algorithm is proposed, and its correctness and optimality are proved. Moreover, a derivation for the optimal allocation with a continuum of layers is given, using the classical variational method.

References

[1] S. Shamai, "A broadcast strategy for the Gaussian slowly fading channel," in *Proc. 1997 IEEE International Symposium on Information Theory*, Ulm, Germany, 29 June-4 July 1997, p. 150.

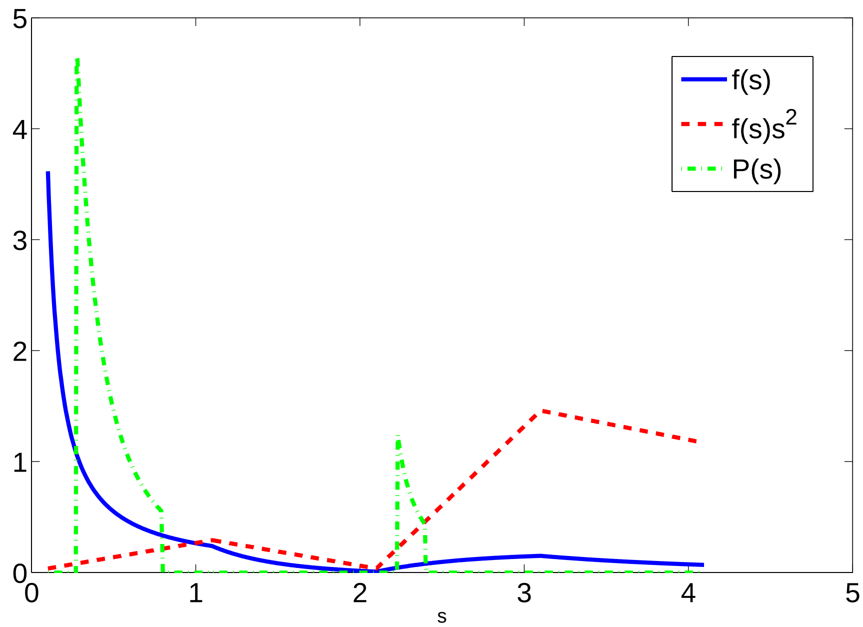

Fig. 6. Optimal power allocation $P(s)$ for $f(s)$ given in (91) with the total power $P = 0$dB.

[2] S. Shamai and A. Steiner, "A broadcast approach for single-user slowly fading MIMO channel," *IEEE Trans. Information Theory*, vol. 49, no. 10, pp. 2617–2635, Oct. 2003.

[3] T. Cover, "Broadcast channels," *IEEE Transactions on Information Theory*, vol. 18, no. 1, pp. 2–14, Jan. 1972.

[4] T. M. Cover and J. A. Thomas, *Elements of information theory*. New York: Wiley, 1991.

[5] V. N. Koshelev, "Hierarchical coding of discrete sources," *Probl. Pered. Inform.*, vol. 16, no. 3, pp. 31–49, 1980.

[6] W. H. R. Equitz and T. M. Cover, "Successive refinement of information," *IEEE Trans. Information Theory*, vol. 37, no. 2, pp. 269–275, Mar. 1991.

[7] B. Rimoldi, "Successive refinement of information: characterization of achievable rates," *IEEE Trans. Information Theory*, vol. 40, no. 1, pp. 253–259, Jan. 1994.

[8] S. Sesia, G. Caire, and G. Vivier, "Lossy transmission over slow-fading awgn channels: a comparison of progressive, superposition and hybrid approaches," in *Proc. IEEE International Symposium on Information Theory*, Adelaide, Australia, Sep. 2005.

[9] F. Etemadi and H. Jafarkhani, "Optimal layered transmission over quasi-static fading channels," in *Proc. IEEE Symposium Information Theory*, Jul 2006.

[10] C. Ng, D. Gündüz, A. Goldsmith, and E. Erkip, "Recursive power allocation in gaussian layered broadcast coding with successive refinement," in *Proc. IEEE International Conference on Communications (ICC)*, Glasgow, Scotland, Jun. 2007.

[11] ——, "Minimum expected distortion in gaussian layered broadcast coding with successive refinement," in *Proc. IEEE International Symposium on Information Theory*, Nice, France, Jun. 2007.

[12] G. Caire and K. Narayanan, "On the SNR exponent of hybrid digital-analog space time coding," in *Proc. Allerton Conf. Commun., Contr. and Comp.*, UIUC, IL, Sep. 2005.

[13] G. Caire and K. Narayanan, "On the SNR exponent of hybrid digital-analog space time coding," *IEEE Transactions on Information Theory,*, vol. 53, no. 8, pp. 2867–2878, Aug. 2007.

[14] D. Gündüz and E. Erkip, "Source and channel coding for quasi-static fading channels," in *Proc. Asilomar Conf. on Signals, Systems and Computers*, Pacific grove, CA, Oct.-Nov. 2005.

[15] K. Bhattad, K. R. Narayanan and G. Caire, "On the Distortion SNR Exponent of Some Layered Transmission Schemes," submitted to *IEEE Transactions on Information Theory*, available at http://arxiv.org/abs/cs/0703035.

[16] I. M. Gelfand and S. V. Fomin, *Calculus of variations*. Englewood Cliffs, NJ: Prentice-Hall, 1963.

[17] D. N. C. Tse, “Optimal power allocation over parallel gaussian broadcast channels,” in *U.C. Berkeley technical report UCB/ERL M99/7. http://www.eecs.berkeley.edu/Pubs/TechRpts/1999/3578.html*, 1999.

[18] U. Mittal and N. Phamdo, “Hybrid digital-analog (HDA) joint source-channel codes for broadcasting and robust communications,” *IEEE Transactions on Information Theory,*, vol. 48, no. 5, pp. 1082–1102, May. 2002.

[19] M. Skoglund, N. Phamdo, and F. Alajaji, “Hybrid digital-analog source-channel coding for bandwidth compression/expansion,” *IEEE Transactions on Information Theory,*, vol. 52, no. 8, pp. 3757–3763, Aug. 2006.

[20] Z. Reznic, M. Feder, and R. Zamir, “Distortion bounds for broadcasting with bandwidth expansion,” *IEEE Transactions on Information Theory,*, vol. 52, no. 8, pp. 3778–3788, Aug. 2006.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[22] D. G. Luenberger, *Optimization by vector space methods*. John Wiley and Sons: New York, 1969.

[23] M. Medard and A. Goldsmith, “Capacity of time-slotted ALOHA systems,” *IEEE Wireless Communications and Networking Conference (WCNC’99), New Orleans, LA, USA*, pp. 490–494, September 21-24 1999.

[24] M. Medard, S. Meyn, J. Huang, and A. Goldsmith, “Capacity of time-slotted ALOHA systems,” *IEEE Int. Symp. Inform. Theory (ISIT’00), Sorrento, Italy*, p. 407, June 25-30 2000.

[25] M. Medard, J. Huang, A. Goldsmith, S. Meyn, and T. Coleman, “Capacity of time-slotted ALOHA packetized multiple-access systems over the AWGN channel,” *IEEE Trans. Wireless Communications*, vol. 3, no. 2, pp. 486–499, March 2004.