

On the Compound MIMO Broadcast Channel

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Abstract— We consider the Gaussian multi-antenna compound broadcast channel where one transmitter transmits several messages, each intended for a different user whose channel realization is arbitrarily chosen from a finite set. Our investigation focuses on the behavior of this channel at high SNRs and we obtain the multiplexing gain of the sum capacity for a number of cases, and point out some implications of the total achievable multiplexing gain region.¹

I. INTRODUCTION

With the advent of 3rd generation cellular systems, multi-antenna systems are becoming common place. Even though many theoretical questions on the downlink channel (alternatively, broadcast channel, BC) in general are still open, the capacity region of multi-antenna downlink channel as well as some other questions have been resolved [1]. However, many practical questions still remain open. For example, the capacity region of a fading BC (scalar as well as multi-antenna) with no or partial channel state information (CSI) at the transmitter [2]. Another open problem is that of the capacity region of a multi-antenna BC with private and common messages. Recently, these questions have attracted attention [3], [4], [5], [6], [7]. In this paper we address and give theoretical bounds for a related and yet unsolved problem of a compound multi-antenna BC.

We consider a memoryless compound multi-antenna BC and focus on the case where the transmitter has M transmit antennas and each of the receivers has only one receive antenna. More precisely, we assume that the fading vector of user i takes one of J_i (finite) values. In addition, the transmitter has precise knowledge of all J_i fading vectors but not of the index of the actual realization of the fading vector. Therefore, a time sample of the channel can be defined as follows:

$$y_i^j = \mathbf{h}_i^{j\dagger} \mathbf{x} + n_i^j \quad i = 1, \dots, K \quad j = 1, \dots, J_i \quad (1)$$

where

- \mathbf{x} is a complex input vector. We assume that the input is power limited such that $E\mathbf{x}^\dagger \mathbf{x} \leq P$.
- y_i^j is the signal received by the j 'th realization/instance of user i .
- $n_i^j \sim \mathcal{CN}(0, 1)$ is an additive white circularly symmetric Gaussian noise which is present at the j 'th realization/instance of user i .

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- K is the number of users and hence, the number of different messages to be simultaneously transmitted.
- \mathbf{h}_i^j is the complex fading vector of the j 'th realization/instance of the i 'th user.

Each of the receivers has exact knowledge of the actual realization of the channel. We wish to transmit in such a manner that no matter what is the actual realization of the users' channels (\mathbf{h}_i^1 or \mathbf{h}_i^2 or ... or $\mathbf{h}_i^{J_i}$), our transmitter will be able to successfully send its messages.

The case where the channel is scalar, i.e. $M = 1$, is well understood. As in the case of $M = 1$ the channel is degraded, the capacity is determined by the worst realization (with the smallest fading realization $|\mathbf{h}_i^j|$) of each user group. Thus, the capacity region is that of a scalar Gaussian BC with one realization per message as follows:

$$R_{\pi_i} \leq \log \left(1 + \frac{\gamma_{\pi_i} P_{\pi_i}}{1 + \gamma_{\pi_i} \sum_{l=1}^{i-1} P_{\pi_l}} \right), \quad i = 1, \dots, K \quad (2)$$

where P_i 's are the power allocations per user such that $\sum_{i=1}^K P_i = P$. The $\gamma_i = \min_{j=1, \dots, J_i} |\mathbf{h}_i^j|^2$ are the fading power of the worst realization of user i and π_i is a permutation matrix which orders the users according to γ_i from the largest to the smallest one.

However, we consider the case where $M > 1$ which is not degraded and the capacity region of this problem is yet unknown. Therefore, we concentrate on the high SNR regime and obtain new results regarding the multiplexing gain of the sum-capacity of the above compound BC for a number of cases. We use the sum-capacity as a measure of the capabilities of the channel and define the multiplexing gain as the maximum value of

$$\lim_{P \rightarrow \infty} \frac{R_1 + R_2 + \dots + R_K}{\log(P)}$$

where the maximum is taken over all transmission strategies.

An alternative view of this channel is that of a broadcast channel with common messages. The different realizations of the channel can be considered as different users to which a common message is being transmitted. This is actually quite a realistic model as third generation cellular systems transmit TV broadcasts over the downlink channel [8]. This application also motivates the consideration of the high SNR regime, where the impact of multi-antenna downlink systems is more pronounced.

The problem where each user has only one possible fading realization (i.e. $J_i = 1 \forall i$) is well understood. The capacity region for this case was recently established in [9] and it is well known that the multiplexing gain of this channel, under a full rank assumption, is equal to the minimum of the number of users and the number of transmit antennas ($\min(K, M)$) [10], which also equals the multiplexing gain of a single link of M transmit and K receive antennas. However, the case where some of the messages are common to a number of users is still unsolved. Some initial research into the capacity region of the more general case with common messages can be found in [11], [12], [13].

This problem is also related, though certainly not equivalent, to the problem of the fading multi-antenna BC with limited feedback to the transmitter (e.g. [3], [4]). In [3], Jindal considers the case where the receivers feedback an estimation of the channel fading vectors to the transmitter, using a digital channel. The feedback relies on a quantized version of the estimated fading vectors at the transmitter. In a recent contribution [4], Caire et-al considered an analog feedback and compare the performance of the digital and analog feedback schemes at high SNRs.

To relate the fading case to our compound case, consider a piecewise constant fading BC where the feedback is limited such that the transmitter only knows that the fading takes one of a finite number of possibilities. We can use the multiplexing gain of the channel in (1) to obtain an upper bound on the ultimate capabilities of this case. Lapidath et al [7] considered the fading BC with $K = 2$ and $M = 2$ where the fading has an infinite number of realizations and changes between one time sample to the next, under the ergodic assumption. They showed that the multiplexing gain over a single user channel with a single receive antenna is upper bounded by $\frac{4}{3}$ and they conjectured that the actual multiplexing gain is 1^2 .

We investigated a related case in our compound setting where $K = M = J_1 = J_2 = 2$. That is, the fading vector takes only one of two possible values at each receiver. Note that the case investigated in [7] and our case not only differ in the number of fading vector realizations but also in the fact that in [7] the fading vectors change from one time sample to the next while in our setting it remains constant throughout the transmission. As in the fading case in [7], we show that for the case of $K = M = J_1 = J_2 = 2$ we obtain an upper bound on the multiplexing gain of $\frac{4}{3}$. Unlike the fading case, here we also suggest a transmission scheme that obtains this multiplexing gain. We conjecture that as J_1 and J_2 increase, the multiplexing gain decreases and approaches 1.

The transmission schemes we suggest here are linear and are certainly not optimal but obtain a multiplexing gain larger than 1. Using some of the key ideas that we present later, it is possible to use dirty paper coding (DPC) like methods to obtain higher rates but not higher multiplexing gains. In [5],

²In [7], the authors considered a fading BC with fading vectors with real elements and not complex elements. Hence, they showed an upper bound on the multiplexing gain of $\frac{2}{3}$ which gives a factor of $\frac{4}{3}$ over the single link with a single receive antenna.

Bennatan and Burshtein suggest a coding scheme, which they called linear assignment fading paper (LAFP) coding, for the fading MIMO BC. This coding scheme is applicable also when the channel fading has a limited number of realizations and when the fading remains constant throughout the transmission. As in DPC, they rely on the result by Gelfand and Pinsker [14] and assign a linear combination of the channel input and the interference to the auxiliary random variable. Unlike the Costa result [15], they use a linear assignment to simultaneously optimize the rates of all users. Another related paper is [6] where the authors considered a carbon copying scenario where we wish to transmit the same message simultaneously to two users, each user suffering from an independent interference.

Throughout the rest of this paper we shall concentrate on the case where there are only two messages ($K = 2$). We shall use M to denote the number of transmit antennas. Therefore, we shall only state J_1 and J_2 to characterize the channel. This paper is organized as follows: in the following section we briefly state our main results and present some conclusions. In Sections III and IV we obtain lower and upper bounds on the sum capacity and multiplexing gain. In Section V we give an illustrative example for the suggested upper bound on the sum-rate and the transmission scheme. The last section summarizes our results.

II. MAIN RESULTS AND CONCLUSIONS

In the following sections we obtain lower and upper bounds on the multiplexing gains. However, we can set apart two cases for which these bounds are tight and we present them in the following theorems.

Theorem 1: Consider a complex compound BC with $K = 2$ users, $J_1 = 1$ and $J_2 = M$. Furthermore, assume that any set of M vectors taken from the set of $\mathbf{h}_1^1, \mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^M$ has rank M . The overall multiplexing gain is given by:

$$1 + \frac{M-1}{M}$$

Proof: This is a direct result of the upper and lower bounds proved in Theorems 5 and 7 in the following sections. \square

Theorem 2: Consider a complex compound BC with $K = 2$ users and $J_1 = J_2 = M$. Furthermore, assume that any set of M vectors taken from the set of $\mathbf{h}_1^1, \mathbf{h}_1^2, \dots, \mathbf{h}_1^M, \mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^M$ has rank M . The overall multiplexing gain is given by:

$$\frac{2M}{M+1}$$

Proof: This is a direct result of the upper and lower bounds proved in Theorems 6 and 8 in the following sections. \square

Note that the requirement of linear independence of any M vectors \mathbf{h}_2^j and \mathbf{h}_1^j in the above theorems is not too restricting. If the fading vectors are chosen uniformly (in direction), almost surely, any M fading vectors of size M will turn out to be linearly independent.

Though we could not establish a tighter upper bound for the cases where $J_1, J_2 > M$, we believe that the lower bounds given in Theorems 7 and 8, to follow, are actually tight. We summarize this in the following conjecture that generalizes the above two theorems.

Conjecture 1: Consider a complex compound BC with $K = 2$ users, $J_2 = J \geq M$ and assume that any set of M vectors taken from the set of $\mathbf{h}_1^1, \mathbf{h}_1^2, \dots, \mathbf{h}_1^{J_1}, \mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^{J_2}$ has rank M . The multiplexing gain is $1 + \frac{M-1}{J}$ if $J_1 = 1$ and $\frac{2J}{2J-M+1}$ if $J_1 = J_2 = J$.

Beyond the sum-rate multiplexing gain we can also define an entire region of multiplexing gains. The notion of a multiplexing gain region was introduced in [16] and is defined as the set of all achievable limit points

$$\lim_{P \rightarrow \infty} \left(\frac{R_1(P)}{\log(P)}, \frac{R_2(P)}{\log(P)} \right).$$

This region must be convex as we may always use time-sharing to obtain all convex combinations of multiplexing gain pairs. The following theorem summarizes our results for the multiplexing gain region of the MIMO compound BC.

Theorem 3: Consider a complex compound BC with $K = 2$ users, M transmit antennas and J_1 and J_2 realizations for the first and second user, respectively. Furthermore, assume that any set of M vectors taken from the set of $\mathbf{h}_1^1, \mathbf{h}_1^2, \dots, \mathbf{h}_1^{J_1}, \mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^{J_2}$ has rank M . Then,

- 1) For $J_1 = 1$ and $J_2 = M$ the multiplexing gain region is given by

$$\begin{aligned} \lim_{P \rightarrow \infty} \frac{R_2}{\log(P)} &\leq 1 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)} \\ \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)} &\leq 1 \end{aligned}$$

- 2) For $J_1 = M$ and $J_2 = M$ the multiplexing gain region is given by

$$\begin{aligned} \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)} &\leq 1 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_2}{\log(P)} \\ \lim_{P \rightarrow \infty} \frac{R_2}{\log(P)} &\leq 1 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)} \end{aligned}$$

Proof: Using Theorem 4, in Section III, we can show that these regions must be outer bounds. In order to show that these regions are achievable, we need to show that all corner points of the regions are achievable. In Figure 1 we plotted the these regions and their corner points. We can obtain points A and D by transmitting only to one of the users each time. We need to show that point B is achievable in the case of $J_1 = J_2 = M$ and that point C is achievable in the case of $J_1 = 1$ and $J_2 = M$. In the proof of Theorems 8 and 7, in Section IV, we describe transmission schemes that obtain exactly these points. \square

It is clear from Figure 1 that points B and C correspond to the maximum sum-rate multiplexing gain points. Throughout the rest of this text we shall limit our attention to these points ($\lim_{P \rightarrow \infty} \frac{R_1 + R_2}{\log(P)}$).

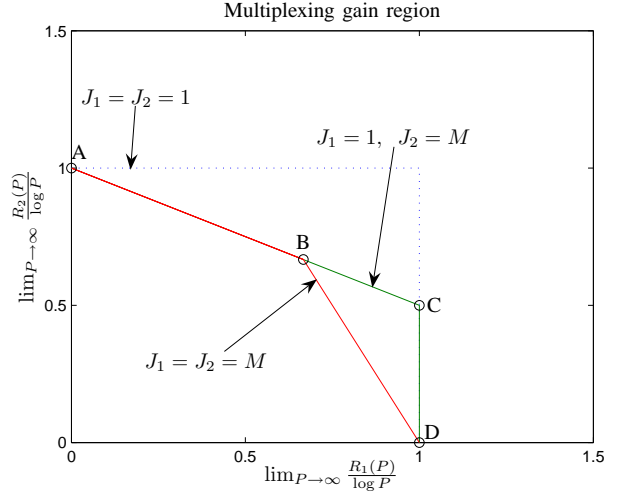


Fig. 1. The multiplexing gain regions for $J_1 = J_2 = 1$ and $J_1 = 1, J_2 = M$ and $J_1 = J_2 = M$

We can draw several conclusions from these results. The first conclusion concerns the choice of the transmission scheme in the case where there are common messages. In [13] we considered a MIMO BC where we transmit two messages. Each message is sent to a different set of users where in each set all users are expected to decipher their respective message. In [13] we found the capacity region for the case where all users in one set are degraded with respect to all users in the other set. We have shown that for this case, a superposition of Gaussian codes and successive decoding achieves the capacity region.

In the case where there is no set of users which is degraded w.r.t the other, we might consider using a generalized form of dirty paper coding as described in [5]. Bennatan and Burshtein referred to this generalized DPC as linear assignment fading paper (LAFP) coding. Though they use it in an ergodic fading environment, it may also be used in the compound (or common messages) case. However, it is not difficult to verify that in a direct application of LAFP coding, one set of users always suffers from the signals intended to the other set of users. Therefore, the multiplexing gain obtained by this method is bounded by 1. We suggest that in order to obtain higher multiplexing gains one should use LAFP coding on more than one time slot simultaneously. This was shown to be effective even for simple linear methods in Theorem 8.

Another conclusion concerns the fading channel with equiprobable realizations. In the compound case, the channel realizations remain constant throughout the entire transmission. In a fading channel the realizations change from one time instant to the next. For both cases we assume that the realizations of the channel are taken from a finite set such that users 1 and 2 have J_1 and J_2 possible realizations.

In the case of $J_1 = 1$ and $J_2 \geq M$ we can obtain a multiplexing gain of $1 + \frac{M-1}{J_2}$ also in the fading case. This is done by choosing $M - 1$ out of J_2 realizations of the

second user and simultaneously transmitting to user 1 and all $M - 1$ realizations of the second user using zero forcing. As the number of transmit antennas is M , we can obtain a multiplexing gain of 1 between the input and the outputs of each of the realizations. User 1 sees a constant channel and therefore, the multiplexing gain associated with message #1 is 1. User 2 sees a fading channel. This user may opt to disregard the channel output whenever the realization of the channel is not one of the $M - 1$ chosen realizations. Therefore, this user sees an equivalent of a single user fading channel which suffers from channel noise and which is in outage $\frac{J_2 - M + 1}{J_2}$ of the time. Therefore, the multiplexing gain associated with this user is $\frac{M-1}{J_2}$. Overall, we get a multiplexing gain of $1 + \frac{M-1}{J_2}$.

For the case of $J_1 = J_2 = J \geq M$ it is not difficult to see that if we assume that the channel is piecewise constant over $2J - M + 1$ time samples, one can use the transmission scheme described in the proof of Theorem 8 in Section IV to obtain a multiplexing gain of $\frac{2J}{2J - M + 1}$. This might suggest that the upper bound on the multiplexing gain that was obtained in [7] is actually tight if there are only two realizations for each user, also in the ergodic fading environment.

III. UPPER BOUNDS ON THE SUM-CAPACITY MULTIPLEXING GAIN

We first state a Theorem which will be useful in deriving some upper bounds on the multiplexing gain later on.

Theorem 4: Consider a two user complex compound multi-antenna BC with $J_1 = 1$, $J_2 = M$ and where \mathbf{h}_2^j , $j = 1, \dots, J_2$ are linearly independent. Then

$$\lim_{P \rightarrow \infty} \frac{R_2}{\log(P)} \leq 1 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)}.$$

Proof: The proof relies on giving the received signals at each of the instances of user 2 to user 1. Thus, we create a degraded broadcast channel that can be analyzed using the results stated in [17]. We also use the fact that due to linear dependence between all $M + 1$ received signals in the new receiver of user 1, we can remove one of the received signals (of user 1) without impacting the multiplexing gains. The details of the proof are omitted here and the reader is referred to the journal version of this paper [18]. \square

We can now use the above theorem to easily obtain some upper bounds on the sum-capacity multiplexing gain.

Theorem 5: Consider a complex compound BC with $K = 2$ users, $J_1 \geq 1$ and $J_2 \geq M$. Furthermore, assume that $\mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^{J_2}$ are linearly independent. The multiplexing gain is upper bounded by:

$$\lim_{P \rightarrow \infty} \frac{(R_1 + R_2)}{\log(P)} \leq 1 + \frac{M - 1}{M}$$

Proof: It is sufficient to show that the theorem holds for $J_1 = 1$ and $J_2 = M$. By Theorem 4 we have $\lim_{P \rightarrow \infty} \frac{R_2}{\log(P)} \leq 1 - \frac{1}{M} \cdot \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)}$. Therefore, by adding $\lim_{P \rightarrow \infty} \frac{R_1}{\log(P)}$ to both sides of the equation we

obtain

$$\lim_{P \rightarrow \infty} \frac{(R_1 + R_2)}{\log(P)} \leq 1 + \frac{M - 1}{M} \cdot \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)}.$$

However, $\lim_{P \rightarrow \infty} \frac{R_1}{\log(P)} \leq 1$ as this is the multiplexing gain of the rate achieved in point to point Gaussian channel with a single receive antenna. \square

Indeed, the above bound holds for $J_1 > 1$ and $J_2 > M$. However, it is not tight for these cases. A better bound is given in the following theorem for $J_1 \geq M$ and $J_2 \geq M$.

Theorem 6: Consider a complex compound BC with $K = 2$ users and $J_1 \geq M$ and $J_2 \geq M$. Furthermore, assume that $\mathbf{h}_k^1, \mathbf{h}_k^2, \dots, \mathbf{h}_k^{J_k}$ are linearly independent for $k = 1, 2$. The multiplexing gain of the sum capacity of this channel is bounded by

$$\lim_{P \rightarrow \infty} \frac{(R_1 + R_2)}{\log(P)} \leq \frac{2M}{M + 1}$$

Proof: It is sufficient to show that the theorem holds for $J_1 = J_2 = M$. Clearly, the capacity region of this compound BC is contained within that of the BC where the first user's fading is perfectly known and is taken to be \mathbf{h}_1^1 (i.e. $J_1 = 1$). This is the two user compound BC with $J_1 = 1$ and $J_2 = M$ which was considered in Theorem 4. Therefore, we may write

$$\lim_{P \rightarrow \infty} \frac{R_2}{\log(P)} \leq 1 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_1}{\log(P)}.$$

However, we can also consider the flipped channel where $J_1 = M$ and $J_2 = 1$. Applying Theorem 4 for this case we get

$$\lim_{P \rightarrow \infty} \frac{R_1}{\log(P)} \leq 1 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_2}{\log(P)}.$$

Thus, by summing the two results we have

$$\lim_{P \rightarrow \infty} \frac{R_1 + R_2}{\log(P)} \leq 2 - \frac{1}{M} \lim_{P \rightarrow \infty} \frac{R_1 + R_2}{\log(P)}.$$

and the proof of the above proposition follows immediately. \square

Again, the upper bound in the above theorem also holds for the case of $J_1 = J_2 > M$. However, we believe that it is not tight for that case.

IV. LOWER BOUNDS ON THE SUM-CAPACITY MULTIPLEXING GAIN

We now obtain lower bounds on the multiplexing gain by establishing transmission schemes that actually obtain these gains. These transmission schemes obtain the upper bounds of the previous section for some of the cases.

We begin with the simpler case where one of the users has only one instant ($J_1 = 1$) and the second has several. The following theorem gives us a lower bound on the multiplexing gain for such a case:

Theorem 7: Consider a complex compound BC with $K = 2$ users, $J_1 = 1$ and $J_2 \geq M$. Furthermore, assume that any set of M vectors taken from the set of $\mathbf{h}_1^1, \mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^{J_2}$ has rank

M . The following is a lower bound on the multiplexing gain of the sum capacity:

$$\lim_{P \rightarrow \infty} \frac{(R_1 + R_2)}{\log(P)} \geq 1 + \frac{M-1}{J_2}.$$

Proof: We shall describe a linear transmission scheme which relies on zero-forcing and which achieves the above multiplexing gain.

Choose arbitrarily a set of indexes j_1, j_2, \dots, j_{M-1} such that $1 \leq j_k \leq J_2 \quad \forall k = 1, \dots, M-1$ and such that $j_m \neq j_n \quad \forall m \neq n$. We define $\mathbf{H} = (\mathbf{h}_1^1 \mathbf{h}_2^{j_1} \mathbf{h}_2^{j_2} \dots \mathbf{h}_2^{j_{M-1}})$. As required by the theorem, \mathbf{H} is full ranked and therefore we can use zero-forcing to transmit M different messages, each with a multiplexing gain of 1. We choose to transmit the same message to the last $M-1$ users and hence the multiplexing gain for this scheme is 2. However, note that $J_2 - M + 1$ of the realizations of user 2 do not receive data. To overcome this problem we use time domain multiple access, switching in a cyclic way and selecting $(M-1)$ users out of J_2 possible ones. Thus, each of the realizations of user 2 does not receive data $\frac{J_2 - M + 1}{J_2}$ of the time and we obtain a multiplexing gain of $2 - \frac{J_2 - M + 1}{J_2} = 1 + (M-1)/J_2$. \square

Note that the zero-forcing technique used in the above proof can be replaced by the ranked known interference technique described in [10] and which relies on dirty paper coding. For example, we may transmit to user 1 at a direction which is orthogonal to the fading vectors of the $M-1$ chosen instances of user 2, using DPC to cancel out the interference from the signals intended to user 2. The signals to the $M-1$ chosen instances of user 2 are encoded using simple Gaussian coding. As the signal to user 1 is orthogonal to the fading vectors of the chosen instances of user 2, at their receivers there will be no interference from user's 1 signal. Thus, we obtain an overall multiplexing gain of $1 + (M-1)/J_2$ without requiring that the signals transmitted to user 2 will be orthogonal to the fading vector of user 1. Alternatively, for the case where $J_2 = M = 2$, we can use the "carbon copy" transmission method described in [6]. Here the message of user 2 is transmitted in an orthogonal direction to user 1, while message 1 impacts the second user in two different ways dictated by the realization of $(\mathbf{h}_2^1, \mathbf{h}_2^2)$, thus giving rise to the carbon copy approach. As shown in [6] in terms of multiplexing gain, the carbon copy approach yields in this case a multiplexing gain of $\frac{1}{2}$, when incorporated in our setting gives the same multiplexing gain of $\frac{3}{2}$. Both methods allow higher rates than those obtained with zero forcing. However, all methods obtain the same multiplexing gain. Furthermore, all methods rely on a time sharing argument.

Next, we consider the more complex case where for both users the number of instances is larger than or equal to the number of transmit antennas. More specifically, we shall assume that $J_1 = J_2 \geq M$. Note that in the case where $J_1 < M$ and $J_2 < M$, the multiplexing gain is 2, which remains undegraded. This multiplexing gain is obtained by transmitting one signal (corresponding to message #1) in a direction which

is orthogonal to all J_2 fading vector realizations of user 2 and the second signal (corresponding to message #2) is transmitted in a direction which is orthogonal to all J_1 fading vector realizations of user 1.

The following theorem shows that a multiplexing gain higher than 1 is achievable also when $J_1 = J_2 \geq M$ and shows that the upper bound on the multiplexing gain is tight when $J_1 = J_2 = M$.

Theorem 8: Consider a complex compound BC with $K = 2$ users and $J_1 = J_2 = J \geq M$. Furthermore, assume that any set of M vectors taken from the set of $\mathbf{h}_1^1, \mathbf{h}_1^2, \dots, \mathbf{h}_1^J, \mathbf{h}_2^1, \mathbf{h}_2^2, \dots, \mathbf{h}_2^J$ has rank M . Then, the multiplexing gain is lower bounded by

$$\lim_{P \rightarrow \infty} \frac{(R_1 + R_2)}{\log(P)} \geq \frac{2J}{2J - M + 1}$$

Proof: Again, we describe a linear transmission scheme which obtains the above multiplexing gain. For the sake of brevity, we give a heuristic explanation of this scheme for the case of $J_1 = J_2 = M = 2$. For a detailed proof of the general case, the reader is referred to [18].

In order to describe the transmission scheme we consider our channel over more than one time sample and look at it as an extended MIMO system with a higher number of transmit and receive antennas. In the case of $J_1 = J_2 = M = 2$ we consider three time slots simultaneously. The extended transmitter has $2 \times 3 = 6$ transmit antennas (two for each time slot). On the side of the receivers we assume that for each realization we calculate two different linear combinations of the three received symbols.

Let $\bar{\mathbf{x}}$ denote the transmitted signal over three time slots (i.e. $\bar{\mathbf{x}}$ is a matrix of size 2×3) and let $\mathbf{f}_i^j(1)$ and $\mathbf{f}_i^j(2)$ be column vectors of size 3×1 . Furthermore, let $\bar{\mathbf{n}}_i^j$ denote the three consecutive noise samples at the receiver of the j 'th realization of the i 'th user (i.e. $\bar{\mathbf{n}}_i^j$ is a vector of size 1×3). We use $s_i^j(1) = (\mathbf{h}_i^{j \dagger} \bar{\mathbf{x}} + \bar{\mathbf{n}}_i^j) \mathbf{f}_i^j(1)$ and $s_i^j(2) = (\mathbf{h}_i^{j \dagger} \bar{\mathbf{x}} + \bar{\mathbf{n}}_i^j) \mathbf{f}_i^j(2)$ to denote two linear combinations of the received signal at instance j of user i . Thus, we have created a virtual extended MIMO system with 6 transmit antennas and 2 users with 2 instances each, where at each instance we receive 2 symbols. Therefore, each user in the extended MIMO system is defined by 4 fading vectors of size 6×1 standing for all linear combinations between the input and each of the outputs related to that user (two vectors for each of the two instances).

In order to transmit such that user 1 is not affected, we need to transmit in a direction which is orthogonal to all vectors which are related to that user. As the dimension of the input vector is 6 and there are only four vectors related to user 1 we can find two such vectors, \mathbf{v}_1 and \mathbf{v}_2 . Similarly, we can find two vectors, \mathbf{u}_1 and \mathbf{u}_2 , which are orthogonal to all four vectors associated with user 2. Thus, we can transmit to user 1 by sending signals over \mathbf{u}_1 and \mathbf{u}_2 and to user 2 over \mathbf{v}_1 and \mathbf{v}_2 . Note that user 1 does not receive any interference from the signal directed to user 2 and similarly, user 2 is oblivious of the transmission to user 1. Therefore, we obtain a multiplexing gain of 4 (as for each user we have constructed

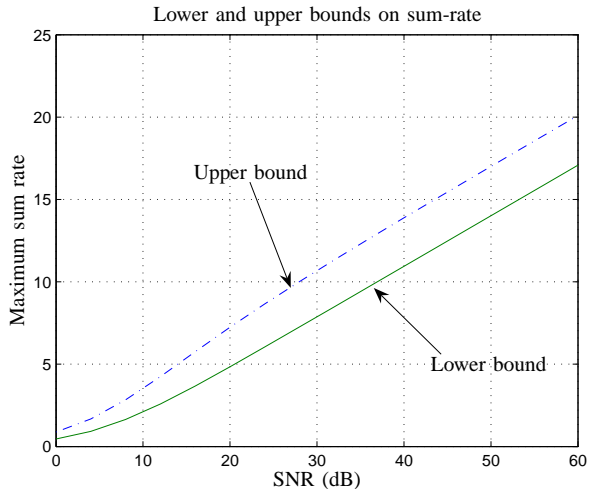


Fig. 2. Upper and lower bounds on the multiplexing gain of the channel in (3). Both bounds have a slope of $4/3$ at high SNRs

a 2×2 complex MIMO channel, with a multiplexing gain of 2). As we obtained this multiplexing gain over 3 time slots, we get an overall multiplexing gain of $4/3$.

The above heuristic explanation is likely to hold for any randomly generated linear combinations \mathbf{f}_i^j . In [18] we detail a specific choice of \mathbf{f}_i^j which is guaranteed to obtain a multiplexing gain of $\frac{2J}{2J-M+1}$ in the general case. \square

V. ILLUSTRATIVE EXAMPLE

In the following we present a specific example that illustrates the upper bound on the multiplexing gain as well as an achievable scheme. In this specific example we have $J_1 = J_2 = M = 2$ where the fading vectors are given by

$$\begin{aligned} \mathbf{h}_1^1 &= \frac{1}{\sqrt{9^2 + 1^2}} \begin{pmatrix} 9 \\ 1 \end{pmatrix}, & \mathbf{h}_1^2 &= \frac{1}{\sqrt{9^2 + (-2)^2}} \begin{pmatrix} 9 \\ -2 \end{pmatrix}, \\ \mathbf{h}_2^1 &= \frac{1}{\sqrt{1^2 + 9^2}} \begin{pmatrix} 1 \\ 9 \end{pmatrix}, & \mathbf{h}_2^2 &= \frac{1}{\sqrt{(-3)^2 + 9^2}} \begin{pmatrix} -3 \\ 9 \end{pmatrix}. \end{aligned} \quad (3)$$

We assumed that the noise power at each of the receivers is equal to 1. As all fading vectors are normalized, we shall use the total transmit power, $P = E[\mathbf{x}^\dagger \mathbf{x}]$, as a measure of the SNR.

In Figure 2 we plotted upper and lower bounds on the sum-rate versus the SNR. Note that here we obtain an explicit upper bound on the sum-rate for all SNRs and not only an upper bound on the multiplexing gain. This upper bound was obtained by using a result from [13] on the capacity region of a degraded compound MIMO BC instead of the informational formula in [17] (see the journal version of this paper [18] for more details). The lower bound was obtained using the scheme detailed in [18] and which was outlined in the proof of Theorem 8. As can be seen from Figure 2, the upper and lower bounds have the same slope of $4/3$ at high SNRs. There

is also a noticeable gap of 10dB between the bounds at high SNRs. In [18] we describe some methods of reducing this gap.

VI. CONCLUSIONS

In this text we investigated the multi-antenna compound broadcast channel and in particular, the case where there are only two messages. We present upper and lower bounds on the multiplexing gains of this channel as well as an entire multiplexing gain region. We showed that a multiplexing gain higher than 1 is achievable even if the number of channel realizations (J_1 and J_2) is greater than the number of transmit antennas (M) and showed that not all degrees of freedom are lost. We have also shown that the bounds we presented are tight when the number of channel realizations is equal to the number of transmit antennas, and conjecture the general behavior in terms of M , J_1 and J_2 .

We conclude by our results that the classic application of dirty paper coding or its generalized form, linear assignment fading paper (LAFP) coding ([5]), is not optimal. In order to obtain full multiplexing gain, it is necessary to use LAFP coding over several time slots simultaneously. The LAFP as well as the "carbon copy" techniques can be employed to further tighten the lower bounds presented in Section V. When confining attention to linear precoding and preprocessing at the receivers, operation over several time slots is fundamental in the compound regime, as is demonstrated in the suggested scheme in Section IV, proof of Theorem 8. In addition, we point out a linkage between the results obtained here for the compound channel and the multiplexing gain results obtained in [7] for the fading channel.

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