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ON THE CLASSICAL – AND NOT SO CLASSICAL – SHANNON SAMPLING THEOREM

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ABSTRACT. We proceed with our recently-introduced geometric approach to sampling of manifolds and investigate the relationship that exists between the classical, i.e. Shannon type, and geometric sampling concepts and formalism. Some aspects of coding and the Gaussian channel problem are considered. A geometric version of Shannon's Second Theorem is introduced. Applications to Pulse Code Modulation and Vector Quantization of Images are provided. An extension of our sampling scheme to a certain class of infinite dimensional manifolds is also considered. The relationship between real functions of bounded curvature and classical band-limited signals is investigated.

1. General background

1.1. **Introduction.** A sampling theorem for differentiable manifolds was recently presented and applied in the context image processing ([22], [23]):

Theorem 1.1. *Let $\Sigma^n \subset \mathbb{R}^{n+1}, n \geq 2$ be a connected, not necessarily compact, smooth hypersurface, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\Big(\frac{1}{k(p)}\Big)$, where $k(p) = \max\{|k_1|, ..., |k_n|\}$, and where $k_1, ..., k_n$ are the principal curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Moreover, the following corollary is also applicable to this problem:

Corollary 1.2. *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling scheme of Σ^n of finite density everywhere. In particular, if Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

The constructive proof of this theorem is based on the existence of the so-called fat triangulations (see [21]). The density of the vertices of the triangulation (i.e. of the sampling) is given by the inverse of the maximal principal curvature. (A concise outline of the proof of Theorem 1.1 is presented in Appendix 1.) As was shown, the resultant sampling scheme is in accord with the classical Shannon theorem, at least for the large class of

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(bandlimited) signals that are also $\mathcal{C}^2$ curves. In our proposed geometric approach, the radius of curvature substitutes the condition of the Nyquist rate.

Here we further investigate the extent and implications of this analogy, and of the geometric approach in general. We begin by making a few observations regarding the extent of our results, by finding the largest space of signals wherein our results may be applied effectively. Next, we establish the proper analogies considering concepts originating from the classical sampling and coding theory of Gaussian channels. In doing so, we attempt to construct a "dictionary" of geometric sampling and concepts originating from Shannon's fundamental approach [25].

In the the last section we extend our investigation to (a class of) infinite dimensional manifolds. Such manifolds, and the need for a sampling theory for this class of geometrical objects naturally arise, for instance, in the context of continuous variations and deformations, of signals, e.g video, perceived as infinite series of trigonometric functions.

Finally, in Appendix 2, we focus on classical (1-dimensional) bandlimited signals and determine relationships, if any, between such signals and curves of curvature bounded from above.

1.2. **General geometric signals.** We begin our investigation by noting that, by the Paley-Wiener Theorem (see, e.g. [20]), any bandlimited signal is of class $\mathcal{C}^\infty$. We have already shown in [22] that our geometric sampling method applies not only to bandlimited signals, but also to more general blackboard signals, i.e. L^2 functions whose graphs are smooth $\mathcal{C}^2$ curves, not necessarily planar.[1] In fact, the geometric sampling approach can be extended to a far larger class of manifolds. Indeed, every piecewise linear (PL) manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) *smoothing* (see e.g. [27]), and every topological manifold of dimension $n \leq 3$ admits a PL structure (cf., e.g., [27]). In particular, for curves and surfaces, one can first consider a smoothing of class $\geq \mathcal{C}^2$ (so that curvature can be defined properly), which can then be sampled with sampling rate given by the maximal curvature radius. Since the given manifold and its smoothing are arbitrarily close [18], one obtains the desired sampling result. (This very scheme is developed and applied for gray-scale images in [22].)

While numerical schemes for practical computation of smoothing exist, they are not necessarily computationally satisfactory. For practical applications, one can circumvent this problem by applying numerical schemes based on the finite element method ([24]). However, for the sake of mathematical correctness and in order to be able to tackle more general applications, one would like to consider more general curvature measures (see, e.g. [28]) and avoid smoothing altogether (see [24] for the full details of this approach and

[1]As already mentioned above, we shall further investigate the connections between bandlimited signals and functions of bounded second derivative in Appendix 2.

Section 2 below, for a brief discussion of this topic in a slightly different context).

It is worthwhile to note yet another aspect of our geometric sampling method: Shannon's Sampling theorem relates to bandlimited signals, that are, necessarily, unbounded in time (space). Obviously, unbounded signals are not possible in Image Processing, nor in any other practical implementations. In contrast, geometric sampling presents no unboundedness restrictions, quite the opposite: it is far easier to apply for bounded manifolds. The importance of this fact is far from being purely theoretical. Indeed, by eliminating the need to produce periodic signals (surfaces) it drastically reduces *aliasing* effects. There is, of course, a fundamental constraint of uncertainty (see also [9]) in the background of Shannon's Sampling Theorem. This will be addressed in the context of our geometrical approach elsewhere [24].

1.3. **Pulse Code Modulation for Images.** We note that our sampling result offers, as a direct application, a new PCM (*pulse code modulation*) method for images, considered as such and not as 1-dimensional signals. This has as an intrinsic advantage in that the sampling points are associated with relevant geometric features (via curvature) and are not chosen randomly via the Nyquist rate. Moreover, such a sampling is adaptive and, indeed, compressive (see discussion above), lending itself to obvious technological benefits.

1.4. **Vector Quantization for Images.** A supplementary effortless byproduct of the constructive proof of Theorem 1.1 resides in a precise method of *vector quantization* (or *block coding*). Indeed, one of the steps in the proof of Theorem 1.1 consists in the construction of a Voronoi (Dirichlet) cell complex $\{\bar{\gamma}^n_k\}$ (whose vertices will provide the sampling points). The centers a_k of the cells (satisfying a certain geometric density condition) represent, as usual, the *decision vectors*. The advantage of this approach, besides its immediacy, resides in the possibility of estimating the error in length and angle distortion when passing from the cell complex $\{\bar{\gamma}^n_k\}$ to the Euclidean cell complex $\{\bar{c}^n_k\}$ having the same set of vertices as $\{\bar{\gamma}^n_k\}$ (see [19]).

2. Sampling and Codes

2.1. **Packings, Coverings and Lattice codes.** Recall that in classical signal processing, $W = \eta/2$, where W is the frequency of the signal and η represents the *Nyquist rate*. This admits an immediate (and rather trivial) generalization to periodic signals, or, in geometric terms, for signals over a *lattice*: $\Lambda = \{\lambda_i\}$. In this case, one can even interpret the sides of the lattice as the various coordinates in a multi-dimensional (warped) space or time (see, e.g. [26], [15]).[2] Note that such signals can be viewed as distributions on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. According to this interpretation, the (n-dimensional!) period is the *fundamental cell* λ of the lattice. Two scalars are naturally associated with this cell: its diameter $\mathrm{diam}(\lambda)$ (or, alternatively, the length of the longest edge) and its volume $\mathrm{Vol}(\lambda)$. Either of them can be used as a measure of the n-dimensional period. However, they are both interrelated and associated to one geometric feature, the so-called "*fatness*":

Definition 2.1. Let $\gamma = \gamma^k$ be an k-dimensional cell. The *fatness* (or *aspect-ratio*) of γ is defined as:

$$\varphi(\gamma) = \min_{\lambda} \frac{\mathrm{Vol}(\lambda)}{\mathrm{diam}^l(\lambda)}\,,$$

where the minimum is taken over all the l-dimensional faces of γ, $0 \leq k$. (If $\dim\lambda = 0$, then $\mathrm{Vol}(\lambda) = 1$, by convention.)

This definition of fatness is equivalent (see [19]) to the following one:

Definition 2.2. A k-dimensional cell $\gamma \subset \mathbb{R}^n$, $2 \leq k \leq n$, is *φ-fat* if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$; where r denotes the radius of the inscribed sphere of γ and R denotes the radius of the circumscribed sphere of γ. A cell-complex $\Gamma = \{\gamma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $\varphi \geq 0$ such that all its cells are *φ-fat*.

Recall that the in- and circum-radius are important in lattice problems: given a lattice Λ with (dual) Voronoi cell Π (of volume 1), one has to minimize the inradius to solve the packing problem, and to minimize the circumradius for solving the covering problem (see [6]). Note that Λ and Π are simultaneously fat. It follows that fat cell-complexes and, in particular, fat triangulations, represent a mini-max optimization for both the packing and the covering problem. Moreover, since fat triangulations are essential for the sampling theorem for manifolds, it appears that there exists an intrinsic relation between the sampling problem for manifolds and the covering and packing problems.

2.2. **Average Power, Rate of Code and Channel Capacity.** Note that in the context of lattices with fundamental cell λ it is natural to extend the classical definitions of *average power* in the signal:

$$P = \frac{1}{T}\int_0^T f^2(t)dt\,,$$

and the *rate* of the code:

$$R = \frac{1}{T}\log_2 N\,,$$

[2]Alternatively, one can interpret the dimensions as representing wave length, or even as mixed fundamental quantities, e.g. space-time or even space-time-wave-length, as they arise in Medical Imaging (CT).

where N represents the number of code points, in the following manner:

$$P = \frac{1}{\mathrm{Vol}(\Lambda)} \int_{\lambda} f^2(t)dt = \frac{1}{N_1 \mathrm{Vol}(\lambda)} \int_{\lambda} f^2(t)dt \,,$$

and

$$R = \frac{1}{\mathrm{Vol}(\Lambda)} \log_2 N = \frac{1}{N_1 \mathrm{Vol}(\lambda)} \log_2 N,$$

respectively, N_1 being the number of cells.

Similarly, one can adapt the classical definition of the *channel capacity*:

$$C = \lim_{T\to\infty} R = \lim_{T\to\infty} \frac{\log_2 N}{T}\,,$$

as:

$$C = \lim_{T\to\infty} \frac{\log_2 N}{\mathrm{Vol}(\Lambda)} = \lim_{T\to\infty} \frac{1}{N_1 \mathrm{Vol}(\lambda)} \log_2 N.$$

Since the numbers N and N_1 are related by $N_1 = \alpha(N)$, where α is the *growth function* of the manifold, the expression of C becomes:

$$C = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\lambda)} \frac{\log_2 N}{\alpha(N)}.$$

It follows immediately that $C = \infty$ for non-compact Euclidean and Hyperbolic manifolds, and $C = 0$ for their Elliptic counterparts. Unfortunately, no such immediate estimates can be readily produced for manifolds of variable curvature.

Note that by putting $1/T = \mathrm{Vol}(M)$, the definitions above apply for any sampling scheme of any manifold of finite volume, not just for lattices. In this case N and N_1 represent the number of vertices, respective simplices, of the triangulation.

The interpretation of frequency considered above does not extend, however, to general geometric signals. For a proper generalization we have to look into the geometric analogue of W. By [23], Theorem 5.2 on curves, i.e. 1-dimensional (geometric) signals, W equals the *curvature rate* $k/2$, were k represents the maximal absolute curvature of the curve. This, and the sampling Theorem 4.11 of [23] naturally leads us to the following definition of W for general geometric signals:

Definition 2.3. Let $M = M^n$ be an n-dimensional manifold $n \geq 2$. $W = W_M = 1/k_M$, where $k_M = \max k_i$ and $k_i, i = 1, \ldots, n$ are the principal curvatures of M.

According to classical considerations, the energy E of the signal $f : \mathbb{R} \to \mathbb{R}$ is considered to be equal to its L^2 norm:

$$E = E(f) = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2W} \sum_{k=-\infty}^{+\infty} f^2\Big(\frac{k}{2W}\Big)\,.$$

One would like, of course, to find proper generalizations of the notion of energy for more general (geometric) signals. In view of the discussion above, it is clear that a first step is to replace $2W$ by its proper generalization. However, when considering more general function spaces of specific relevance (s.a. *bounded variation* (BV), *bounded oscillation* (BO), *bounded mean oscillation* (BMO)), one should consider energies fitting the specific norm of the space under consideration. This discussion is, of course, also valid with regard to the best way to define average power P, and rate R, of a geometric signal.

We can now look into the first definition of code efficiency: the *nominal coding gain* of a code c_1 over another (c_2) is:

$$\mathrm{ncg}(C1, C2) = 10 \log_{10} \left(\frac{\mu_1}{E_1} \Big/ \frac{\mu_2}{E_2} \right),$$

where μ is the square of the minimal squared-distance between coding points. For geometric codes of bounded curvature (hence compact ones), the expression for μ is particularly simple: $\mu = 1/\min k$ (k denoting again principal curvature).

2.3. **The Channel Coding Problem.** It is only natural to attack the problem of the *Gaussian white noise channel* in the context of "geometric signals", i.e. manifolds. Recall that in the classical context, a *received signal* is represented by a vector $X = F + Y$, where $F = (f_1, \ldots, f_N)$ is the *transmitted signal*, and $Y = (y_1, \ldots, y_n)$ represents the noise, whose components y_i are independent Gaussian random variables, of *mean* 0 and *average power* σ^2. The main, classical result for the Gaussian channel is the following:

Theorem 2.4 (Shannon's Second Theorem, [25])**.** *For any rate R not exceeding the* capacity *C_0:*

$$C_0 = \frac{1}{T} \log_2 \left(1 + \frac{P}{\sigma^2} \right),$$

there exists a sufficiently large T, such that there exists a code of rate R and average power $\leq P$, and such that the probability of a decoding error is arbitrarily small. Conversely, it is not possible to obtain arbitrarily small errors for rates $R > C_0$.

In the case of geometric signals, F is given by the sampling (code) points on the manifolds and, since the mean equals 0, the noisy transmitted signal $F + Y$ lies in the *tube* $\mathrm{Tub}_\sigma(M)$. Recall that $\mathrm{Tub}_\sigma(S) = \bigcup_{p \in S} I_{p,\sigma_p}$, where I_{p,σ_p} is the open symmetric interval through p, in the direction of unit normal $\bar{N}_p$ of M at p, of length $2\sigma_p$, where σ_p is chosen to be small enough such that $I_{p,\sigma_p} \cap I_{q,\sigma_q} = \emptyset$, for any $p, q \in S$.

While not being evident, the existence of tubular neighborhoods is assured both locally, for any regular, orientable manifold, and globally for regular, compact, orientable manifold (see [10]). In addition, the regularity of the manifolds $\partial\mathrm{Tub}^-_\sigma(M) = M - \bigcup_{p \in M} \varepsilon \bar{N}_p$, $\partial\mathrm{Tub}^+_\sigma(M) = M + \bigcup_{p \in M} \varepsilon \bar{N}_p$, where $\partial\mathrm{Tub}^-_\sigma(M) \cup \partial\mathrm{Tub}^+_\sigma(M) = \partial\mathrm{Tub}_\sigma(M)$, is at least as high as that of M: If M is convex, then $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ are piecewise $\mathcal{C}^{1,1}$ manifolds (i.e. they admit parameterizations with continuous and bounded derivatives), for all $\sigma > 0$. Also, if M is a smooth enough manifold with a boundary, that is, at least piecewise $\mathcal{C}^2$, then $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ are piecewise $\mathcal{C}^2$ manifolds, for all small enough σ (see [8]).

In the geometric setting, σ can be taken, of course, to be the maximal Euclidean deviation. However, a better deviation measure is, at least for compact manifolds, the Haussdorf Distance (between M and $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$):

Definition 2.5. Let (X, d) be a metric space and let $A, B \subseteq (X, d)$. The *Hausdorff distance* between A and B is defined as:

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}\,.$$

For non-compact manifolds one has to consider the more general *Gromov-Hausdorff distance* (see, e.g., [1]).

Since, by the remarks above, for σ small enough, both the distance between M and $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ and the deviations of their curvature measures are arbitrarily small, we can state a first "soft" geometric version of Shannon's Theorem for the Gaussian channel. While a perfect analogy is not available, we can nevertheless formulate the following theorem:

Theorem 2.6 (Shannon's Second Theorem, qualitative version)**.** *Let M^n be a smooth geometric signal (manifold) and let σ be small enough, such that $\mathrm{Tub}_\sigma(M)$ is a submanifold of $\mathbb{R}^{n+1}$. Then, given any noisy signal $M + Y$, such that the average noise power σ_Y is at most σ, there exists a sampling of $M + Y$ with an arbitrarily small probability of decoding error.*

Remark 2.7. For geometric approach to codes, the analogue of the capacity is $C_0 = C_0(k, \sigma, r)$, where r represents the differentiability class of M.

Remark 2.8. For compact manifolds, the existence of tube $\partial\mathrm{Tub}^+_\sigma(M)$ is, as we have already remarked, assured globally. Hence it follows that the sampling scheme is also global and necessitates $O(N)$ points, $N = N_M$. However, for non-compact manifolds (in particular non-band limited geometric signals), the existence of $\partial\mathrm{Tub}^+_\sigma(M)$ is guaranteed only locally. Therefore "gluing "of the patches is needed, operation which requires the insertion of additional vertices (i.e. sampling points), their number being a function of the dimension of M. Hence, in this case, $N_{M+Y} = O(N^n_M)$.

It is important to note that, again, this result is not restricted to smooth manifolds, but rather extends to much more general signals: Indeed, for any compact set $M \in \mathbb{R}^n$, the $(n-1)$-dimensional sets $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$, are *Lipschitz manifolds*[3] for almost any ε (see [11]). Moreover, the generalized curvatures measures of $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ are arbitrarily close

[3] i.e. topological manifolds equipped with a maximal atlas for which the changes of coordinates are Lipschitz functions.

to the curvature of M, for small enough σ ([5], [11]). It follows that the generalization above befits not only the case of the Gaussian noise, but to more general types of noise, as well (see, [25], [12], [14]).

The full details of a *quantitative* version, including the general case, are laborious and warrant a separate discussion (see [24]).

3. Geometric Sampling of Infinite Dimensional Signals

Since in the classical context band-limited signals are viewed as elements f of $L^2(\mathbb{R})$, such that $supp\,(\hat{f}) \subseteq [-\pi, \pi]$, where $\hat{f}$ denotes the Fourier transform of f, one is led naturally, to the following question: can one extend the sampling theorem proven in [22] to infinite dimensional manifolds? Using an example developed in [16], one may conclude that not only this question is far from naive, but rather the answer is positive in that our geometric sampling method translates directly into the context of infinite dimensional manifolds, at least for a class of functions that naturally arise in the the context of signal and image processing. However, since the full proofs required in the example below are rather technical, we refer the reader to the original paper [16], and limit ourselves here solely to a brief presentation

Consider the following spaces:

$$C_1^\infty = \left\{e \in \mathcal{C}^\infty(\mathbb{R}) \,\middle|\, e(x+1) = e(x)\right\},$$

$$C_+^\infty = \left\{e \in \mathcal{C}^\infty(\mathbb{R}) \,\middle|\, e(x+1) = e(x), \int_0^1 e^2 = 1\right\},$$

$$M \subset C_+^\infty, M = \left\{\lambda_0 = 0 \,|\, \lambda_0 \text{ first eigenvalue of} Q\right\},$$

where Q denotes the Hill operator: $Q = -D^2 + q\,, q(x+1) = q(x)$, and where $D = \partial/\partial x, 0 \leq x < 1$.

Since 1 an 0 are regular values for their respective functions, M is a smooth, co-dimension one hyper-surface in C_1^∞, that is endowed with the topology inherited from that of C^∞ endowed (as usual) with the *sup* norm. Further, exactly as in the finite-dimensional case, for any 2-dimensional section determined by unit tangent vectors to M at q, one can define (and compute) the maximal principal curvature (of the section).

Moreover, since the co-dimension of M in C_1^∞ is 1, it follows that, together with a tangent plane, a normal to M at q is also defined. It follows that one can use the same method as in the finite dimensional case to find a sampling of M.

It follows that a sampling scheme identical to that developed for the finite dimensional case can be applied for the manifold M, as well. Unfortunately, no uniform sampling is possible for the entire manifold: the maximal curvature, associated with functions approximating "saw-tooth" functions can be made as large as desired (see [16]).

One would like to extend these considerations, in a systematic manner, to general infinite dimensional manifolds (e.g. l_2 and Hilbert cube manifolds). However, even if the appropriate geometric differential notions are defined and computed, the fundamental problem of constructing fat triangulations for infinite dimensional manifolds still has to be solved. On the positive side is the fact that triangulations of such manifolds exist (see [3]). However, even finding a notion analogous to that of fatness in the ∞-dimensional case represents a challenge. (For a different approach to a differential geometry of some infinite dimensional spaces see for example [17].)

Appendix 1 – A Concise Version of Proof of Theorem 1.1

We begin by introducing the necessary definitions and notations:

Let M^n denote an n-dimensional complete Riemannian manifold, and let M^n be isometrically embedded into $\mathbb{R}^\nu$ ("ν"-s existence is guaranteed by Nash's Theorem – see, e.g. [19]).

Let $\mathbb{B}^\nu(x,r) = \{y \in \mathbb{R}^\nu \,|\, d_{eucl} < r\}$; $\partial\mathbb{B}^\nu(x,r) = \mathbb{S}^{\nu-1}(x,r)$. If $x \in M^n$, let $\sigma^n(x,r) = M^n \cap \mathbb{B}^\nu(x,r)$, $\beta^n(x,r) = exp_x\big(\mathbb{B}^n(0,r)\big)$, where: exp_x denotes the exponential map: $exp_x : T_x(M^n) \to M^n$ and where $\mathbb{B}^n(0,r) \subset T_x\big(M^n\big)$, $\mathbb{B}^n(0,r) = \{y \in \mathbb{R}^n \,|\, d_{eucl}(y,0) < r\}$.

The following definitions generalize in a straightforward manner classical ones used for surfaces in $\mathbb{R}^3$:

Definition 3.1. (1) $\mathbb{S}^{\nu-1}(x,r)$ is *tangent* to M^n at $x \in M^n$ iff there exists $\mathbb{S}^n(x,r) \subset \mathbb{S}^{\nu-1}(x,r)$, s.t. $T_x(\mathbb{S}^n(x,r)) \equiv T_x(M^n)$.
(2) Let $l \subset \mathbb{R}^\nu$ be a line, then l is *secant* to $X \subset M^n$ iff $|\, l \cap X| \geq 2$.

Definition 3.2. (1) $\mathbb{S}^{\nu-1}(x,\rho)$ is an osculatory sphere at $x \in M^n$ iff:
 (a) $\mathbb{S}^{\nu-1}(x,\rho)$ is tangent at x;
 and
 (b) $\mathbb{B}^n(x,\rho) \cap M^n = \emptyset$.
(2) Let $X \subset M^n$. The number $\omega = \omega_X = \sup\{\rho > 0 \,|\, \mathbb{S}^{\nu-1}(x,\rho)$ osculatory at any $x \in X\}$ is called the *maximal osculatory* (*tubular*) *radius* at X.

Remark 3.3. There exists an osculatory sphere at any point of M^n (see [2]).

Definition 3.4. Let $U \subset M^n, U \neq \emptyset$, be a relatively compact set, and let $T = \bigcup_{x\in\bar{U}} \sigma(x,\omega_U)$. The number $\kappa_U = \max\{r \,|\, \sigma^n(x,r)$isconnected for all $s \leq \omega_U$, $x \in \bar{T}\}$, is called the *maximal connectivity radius* at U, defined as follows:

Note that the maximal connectivity radius and the maximal osculatory radius are interconnected by the following inequality ([19], Lemma 3.1):

$$\omega_U \leq \frac{\sqrt{3}}{3}\kappa_U\,.$$

We are now able to present the main steps of the proof:

The Compact Case. This case was proved in Cairns' seminal paper [2]. His method is to produce a point set $A \subseteq M^n$, that is maximal with respect to the following density condition:

$$d(a_1, a_2) \geq \eta\,, \text{for all } a_1, a_2 \in A\,; \tag{3.1}$$

where

$$\eta < \omega_M\,. \tag{3.2}$$

One makes use of the fact that for a compact manifold M^n we have $|A| < \aleph_0$, to construct the finite cell complex "cut out of M" by the ν-dimensional Dirichlet complex, whose (closed) cells are given by:

$$\bar{c}_k = \bar{c}_k^{\nu} = \{x \in \mathbb{R}^{\nu} \,|\, d_{eucl}(a_k, x) \leq d_{eucl}(a_i, x),\ a_i \in A\,,\ a_i \neq a_k\}, \tag{3.3}$$

i.e. the (closed) cell complex $\{\bar{\gamma}_k^n\}$, where:

$$\{\bar{\gamma}_k^n\} = \bar{\gamma}_k = \bar{c}_k \cap M^n \tag{3.4}$$

(see [2] for details).

Open Riemannian Manifolds. In adapting Cairns' method to the non-compact case, one has to allow for a number of required modifications. We proceed to present below the construction devised by Peltonen, which consists of two parts:

Part 1

Step A

Construct an exhaustive set $\{E_i\}$ of M^n, generated by the pair (U_i, η_i), where:

(1) U_i is the relatively compact set $E_i \setminus \bar{E}_{i-1}$ and
(2) η_i is a number that controls the fatness of the simplices of the triangulation of E_i, constructed in Part 2, such that it will not differ to much on adjacent simplices, i.e.:
   (i) The sequence $(\eta_i)_{i \geq 1}$ descends to 0;
   (ii) $2\eta_i \geq \eta_{i-1}$.

The numbers η_i are chosen such that they satisfy the following bounds:

$$\eta_i \leq \frac{1}{4}\min_{i\geq 1}\{\omega_{\bar{U}_{i-1}}, \omega_{\bar{U}_i}, \omega_{\bar{U}_{i+1}}\}\,.$$

Step B

(1) Produce a maximal set A, $|A| \leq \aleph_0$, s.t. $A \cap U_i$ satisfies:
   (i) a density condition, namely:

$$d(a, b) \geq \eta_i/2\,, \text{for all } i \geq 1\,;$$

and
(ii) a "gluing" condition for U_i, U_{i+1}, i.e. their intersection is large enough.

Note that according to the density condition (i), the following holds:

For any i and for any $x \in \bar{U}_i$, there exists $a \in A$ such that $d(x,a) \leq \eta_i/2$.

(2) Prove that the Dirichlet complex $\{\bar{\gamma}_i\}$ defined by the sets A_i is a cell complex and every cell has a finite number of faces (so that it can be triangulated in a standard manner).

Part 2

Consider first the dual complex Γ, and prove that it is a Euclidian simplicial complex with a "good" (i.e. proper) density. Project then Γ on M^n (using the normal map). Finally, prove that the resulting complex $\widetilde{\Gamma}$ can be triangulated by fat simplices. Indeed, the fatness of any n-dimensional simplex $\gamma \in \widetilde{\Gamma}$, contained in the set U_i is given by the following bound:

$$\frac{r_\gamma}{R_\gamma} \geq \frac{1}{2^{5n+1}} \frac{(n+2)^{\frac{n+1}{2}}}{(n+1)^{n+1}}\,. \tag{3.5}$$

Manifolds With Boundary of Low-Differentiability. The idea of the proof of this case is to build first two fat triangulations: $\mathcal{T}_1$ of a product neighbourhood N of ∂M^n in M^n and $\mathcal{T}_2$ of $int\,M^n$ (its existence follows from Peltonen's result), and then to "mash" the two triangulations into a new triangulation $\mathcal{T}$, while retaining their fatness. While the mashing procedure of the two triangulations is basically the one developed in the original proof of Munkres' theorem, the triangulation of $\mathcal{T}_1$ has been modified, in order to ensure the fatness of the simplices of $\mathcal{T}_1$. The method we have employed for fattening triangulations is the one developed in [4]. For the technical details, see [21].

3.0.1. *Curvature Radii.* As mentioned above, a sampling scheme exists, the sampling points being provided by the vertices of the fat triangulation constructed above. The fact that the density is a function solely of $k = max\{|k_1|, ..., |k_n|\}$ follows from the basic construction of Cairns and from the fact that the osculatory radius $\omega_\gamma(p)$ at a point p of a curve γ equals $1/k_\gamma(p)$, where $k_\gamma(p)$ is the curvature of γ at p; hence the maximal osculatory radius (of Σ) at p is: $\omega(p) = \max\{|k_1|, ..., |k_n|\} = \max\{\frac{1}{\omega_1}, ..., \frac{1}{\omega_n}\}$. (Here $\omega_i\,, i = 1, ..., n$ denote the minimal, respective maximal sectional osculatory radii at p.)

Appendix 2 – Band-limited signals and bounded curvature

Given the geometric sampling algorithm one is naturally conducted to pose the following question: "Are 1-dimensional band-limited signals of bounded curvature?" The answer to this question is both "Yes" and "No".

On the positive side, we can state the following proposition:

Proposition 3.5. *Let $f \in L^2(\mathbb{R})$ be a band-limited function. Than $f'' \in L^\infty(\mathbb{R})$.*

Proof. Since f is band-limited we have $\text{supp}\hat{f} \subset [-B, B]$. Since $f \in L^2(\mathbb{R})$ it is clear that $\hat{f} \in L^2([-B, B])$. The interval $[-B, B]$ is bounded and this implies $\hat{f} \in L^1([-B, B])$.

The mapping properties of the Fourier transform show that $f \in L^\infty(\mathbb{R})$. This just shows that band-limited functions are bounded.

The second derivative of a band-limited function is also band-limited (because $\widehat{f''}(w) = (-2\pi i w)^2 \hat{f}(w)$). Hence, $f'' \in L^\infty(\mathbb{R})$. □

On the other hand, the second derivative of a band-limited function can be arbitrarily large just because λf is again band-limited for every $\lambda > 0$. This shows, that the maximal frequency of a function does not imply a bound on the second derivative of the function.

The classical formula (see, e.g. [7]) for the curvature of the function f is given by:

$$\kappa_f(x) = \frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}} ,$$

while the scaling $g_\lambda(x) = \lambda f(x)$, has curvature

$$\kappa_{g_\lambda}(x) = \frac{\lambda f''(x)}{(1 + \lambda^2 f'(x)^2)^{\frac{3}{2}}} \xrightarrow{\lambda \to \infty} \begin{cases} 0 & , \ f'(x) \neq 0 \\ \infty & , \ f'(x) = 0 \end{cases} .$$

In other words, the curvature goes to infinity at the maximum points of f and to 0 everywhere else. Hence, the neither the second derivative nor the curvature of a bandlimited function can be bounded in terms of the bandwidth alone.

However, one can give bounds on the derivatives of f:

Proposition 3.6. *For f with $\text{supp}\hat{f} \subset [-B, B]$, we have the following estimates on the derivative of f:*

$$|f^{(n)}(x)| \leq (2\pi B)^n \|\hat{f}\|_{L^1}. \tag{3.6}$$

If we assume $\hat{f} \in L^\infty$, we have

$$|f^{(n)}(x)| \leq \frac{2(2\pi)^n B^{n+1}}{n+1} \|\hat{f}\|_{L^\infty} \leq \frac{2(2\pi)^n B^{n+1}}{n+1} \|f\|_{L^1} \tag{3.7}$$

(while the second inequality only holds if $f \in L^1$, of course). Another estimate is

$$|f^{(n)}(x)| \leq (\frac{2(2\pi)^{pn} B^{pn+1}}{pn+1})^{1/p} \|\hat{f}\|_{L^q} , \tag{3.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. All inequalities are based on the formula

$$|f^{(n)}(x)| = |\int_{-B}^{B}(-2\pi i\omega)^n e^{2\pi i\omega x}\hat{f}(\omega)d\omega| \le \int_{-B}^{B}|2\pi\omega|^n|\hat{f}(\omega)|d\omega.$$

Equation (3.6) follows from

$$\sup_{\omega\in[-B,B]}|2\pi\omega|^n = (2\pi B)^n\,.$$

Equation (3.7) follows from

$$\int_{-B}^{B}|2\pi\omega|^n|\hat{f}(\omega)|d\omega \le \|\hat{f}\|_{L^\infty}\int_{-B}^{B}|2\pi\omega|^n d\omega\,,$$

and the fact that $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}$. Finally equation (3.8) follows by applying Hölders inequality:

$$\int_{-B}^{B}|2\pi\omega|^n|\hat{f}(\omega)|d\omega \le \left(\int_{-B}^{B}|2\pi\omega|^{pn}d\omega\right)^{1/p}\|\hat{f}\|_q.$$

□

For certain special cases, the above proposition yields

Corollary 3.7. *Let f be a band-limited signal with bandwidth B. Then:*

(1) *If $\|f\|_1 \le K$, then:*

$$|f''(x)| \le \frac{8\pi^2 B^3 K}{3}\,;$$

(2) *If $\|f\|_2 \le K$, then:*

$$|f''(x)| \le \frac{32\pi^4 B^5 K}{5}\,.$$

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