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Symmetric Hill Order – an Optimal Solution of a Linear Ordering Adjacency Problem

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Abstract

This paper solves a type of Linear Ordering Problem (LOP) which arises in VLSI interconnect design. Let $f(x,y):\mathbb{R}^2\rightarrow\mathbb{R}$ be symmetric and $f\in\mathrm{D}^2$, satisfying $\partial f(x,y)/\partial x\times\partial f(x,y)/\partial x>0$ and $\partial^2 f(x,y)/\partial x\partial y<0$. The LOP solved comprises n objects, each associated with a real value parameter r_i, $1\le i\le n$, and a cost associated any two objects defined by $f(r_i,r_j)$ if $|\pi(i)-\pi(j)|=1$, $1\le i,j\le n$, and zero otherwise. We show that the permutation π which minimizes the total cost $\sum_{i=1}^{n-1} f\left(r_{\pi(i)},r_{\pi(i)+1}\right)$ is determined upfront by the relations between the parameter values r_i. Such permutation is called "***symmetric hill***". It generalizes a family of well known problems arising in interconnect design of VLSI circuits where objects are parallel wires and the cost reflects various design metrics such as delay, power consumption or cross-coupling noise.

1. Introduction and motivation

Let $f(x,y):\mathbb{R}^2\rightarrow\mathbb{R}$ be symmetric and $f\in\mathrm{D}^2$, satisfying $\partial f(x,y)/\partial x\times\partial f(x,y)/\partial x>0$ and $\partial^2 f(x,y)/\partial x\partial y<0$. In the rest of the paper $f(x,y)$ will assume these properties without further mention. Let $(r_1,\ldots,r_n)$ be a sequence of n real non negative numbers associated with n objects, let Π be the set of all permutations $\pi:\{1,\ldots,n\}\rightarrow\{1,\ldots,n\}$, and we denote by $\left(r_{\pi(1)},\ldots,r_{\pi(n)}\right)$ the sequence obtained by applying π to the original set. This paper explores the problem of finding $\pi^*\in\Pi$ which minimizes the sum of costs defined for any two adjacent objects:

$$F(\pi) = \sum_{i=1}^{n-1} f\left(r_{\pi(i)}, r_{\pi(i)+1}\right). \tag{1}$$

The problem in (1) is a type of Linear Ordering Problem (LOP) which is well known and used in economy and other applications. It has been studied extensively in the literature [1][2][3][4][5]. Given a $n \times n$ matrix of weights $C = \left(c_{ij}\right)$, LOP aims at finding a permutation π which maximizes the expression $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_{\pi(i)\pi(j)}$. LOP is NP-complete. The cost in (1) is derived from a function of two variables satisfying $\partial f(x,y)/\partial x \times \partial f(x,y)/\partial x > 0$ and $\partial^2 f(x,y)/\partial x \partial y < 0$. It is defined for two adjacent objects of the permutation, namely $\left|\pi(i) - \pi(j)\right| = 1,\ \ 1 \le i, j \le n$, and is zero otherwise.

Equation (1) generalizes a family of well known VLSI interconnects design problems aiming at minimizing wire delay, dynamic power consumption and crosstalk noise. An interconnect model is shown in Fig 1. There, logic gates called drivers drive signals that propagate along interconnecting wires. These signals stimulate other logic circuits, called receivers, connected at the opposite end of the wires. Cross-coupling parasitic capacitance which is the dominant cause for delay, power consumption and crosstalk noise interference occurs only between adjacent wires. On both sides of the bus there are shielding wires connected to ground. The terms $f\left(r_i, r_{i+1}\right)$ in (1) result from the cross-coupling capacitance, and they take the form $\sqrt{r_i + r_{i+1}}$ in the above mentioned problems. The goal is therefore to determine the order of wires within the bus and obtain their adjacencies such that (1) is minimized.

The authors of [6] claimed by intuition that different orderings of the signal wires in Fig. 1 may yield different amounts of dynamic power consumption, and then proposed some monotonic order of the signals to reduce the power. Similarly, the authors of [7] proposed an intuitive monotonic order aiming at reducing the noise interference. It was shown in [8] and [9] that both the total delay and power consumption occurring in the bus are governed by an expression of the form:

$$F\left(r_1,\ldots,r_n\right)=\sqrt{r_1}+\sum_{i=1}^{i=n-1}\sqrt{r_i+r_{i+1}}+\sqrt{r_n}\ . \tag{2}$$

The parameters $r_1,\ldots,r_n$ are derived from the resistances of the drivers in Fig. 1 and the parameters of manufacturing process technology. Notice that (2) is a special case of (1), We could add two objects such that $r_0=r_{n+1}=0$ and then replace the first and last terms in right hand side of (2) by $\sqrt{r_0+r_1}$ and $\sqrt{r_n+r_{n+1}}$, respectively.

The problem of how to permute the signals in the bus such that the expression in (2) is minimized was addressed in [8]. It was shown that for the square root function a ***"symmetric hill"*** order is optimal. This paper proves that the optimality of *symmetric hill* order exists for a broad range of functions, where square root is just a particular case. The technique of this proof can be used to derive different orders of objects (permutations) for different optimization objectives of interest. We'll discuss this further in the concluding section.

2. Minimizing a general objective function

We assume without loss of generality that the original set of objects is ordered such that $r_1>r_2>\cdots>r_n$ and associate the identity permutation $\pi\left(i\right)=i$ with the original sequence $\left(r_1,\ldots,r_n\right)$. Let us modify the equation in (1) for the sake of proof convenience into a cyclical sum as follows:

$$F\left(\pi\right)=\sum_{i=1}^{n} f\left(r_{\pi(i)},r_{(\pi(i)+1)\bmod n}\right). \tag{3}$$

This doesn't change the original problem as we could add an artificial $\left(n+1\right)th$ zero object. Assume further that n is odd, since if it was even we could add another zero object. In the rest of the discussion we'll drop the modulo notation but keep in mind that summation is cyclical.

Lemma 1: Let a,b,c and d be nonnegative real numbers satisfying $a>b>c>d$. Then:

$$f\left(a,b\right)+f\left(c,d\right)<f\left(a,c\right)+f\left(b,d\right),\ \text{and} \tag{4}$$

$$f(a,c)+f(b,d)<f(a,d)+f(b,c). \tag{5}$$

Proof: Let the absolute value of $h<0$ and $k<0$ be sufficiently small. It follows from Taylor's formula that:

$$f(x-h,y-k)=f(x,y)-h\frac{\partial f(x,y)}{\partial x}-k\frac{\partial f(x,y)}{\partial y}+\frac{1}{2}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2}+2hk\frac{\partial^2 f(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 f(x,y)}{\partial y^2}\right], \tag{6}$$

$$f(x,y-k)=f(x,y)-k\frac{\partial f(x,y)}{\partial y}+\frac{1}{2}k^2\frac{\partial^2 f(x,y)}{\partial y^2}, \text{ and} \tag{7}$$

$$f(x-h,y)=f(x,y)-h\frac{\partial f(x,y)}{\partial x}+\frac{1}{2}h^2\frac{\partial^2 f(x,y)}{\partial x^2}. \tag{8}$$

Substitution of $a=x,\ b=y,\ c=x-h$ and $d=y-k$ obtains $f(a,b)=f(x,y)$, $f(c,d)=f(x-h,y-k)$, $f(a,d)=f(x,y-k)$ and $f(c,b)=f(x-h,y)$. Substitution of (6), (7) and (8) yields $[f(a,d)+f(c,b)]-[f(a,b)+f(c,d)]=2hk\frac{\partial^2 f(x,y)}{\partial x\partial y}<0$ which proves (4).

In order to prove (5) we substitute $a=x,\ b=x-h,\ c=y,\ d=y-k$ and obtain $f(a,c)=f(x,y)$, $f(b,d)=f(x-h,y-k)$, $f(a,d)=f(x,y-k)$ and $f(b,c)=f(x-h,y)$. Substitution of (6), (7) and (8) yields $[f(a,c)+f(b,d)]-[f(a,d)+f(b,c)]=2hk\frac{\partial^2 f(x,y)}{\partial x\partial y}<0$ which proves (5). ☻

Lemma 2: In any permutation $\pi^*\in\Pi$ which minimizes (3) $\pi^*(2)$ and $\pi^*(3)$ must be adjacent to $\pi^*(1)$ on its two opposite sides, namely $|\pi^*(1)-\pi^*(2)|=|\pi^*(1)-\pi^*(3)|=1$, implying that the terms $f(r_1,r_2)$ and $f(r_1,r_3)$ must exist in the minimal sum:

$$F(\pi^*)=\min_{\pi\in\Pi}\{F(\pi)\}. \tag{9}$$

Proof: Let π^* satisfy (9) and assume on the contrary that $\pi^*(2)$ is not adjacent to $\pi^*(1)$, say $\pi^*(2) > \pi^*(1)+1$. Consider the non empty subsequence $\pi^*(1)+1,...,\pi^*(2)$. Let us transform π^* into π^{**} by reverting (flipping) the order of the elements in that subsequence into $\pi^*(2),...,\pi^*(1)+1$. The element adjacencies in π^{**} agree with π^* except those interacting with $\pi^*(2)$ and $\pi^*(1)+1$. Evaluating $F(\pi^{**})$ and then subtracting from $F(\pi^*)$, all identical terms are canceled out except terms involving $\pi^*(2)$ and $\pi^*(1)+1$, yielding:

$$0 > F(\pi^*) - F(\pi^{**}) = \sum_{i=1}^{n} f\left(r_{\pi^*(i)}, r_{\pi^*(i)+1}\right) - \sum_{i=1}^{n} f\left(r_{\pi^{**}(i)}, r_{\pi^{**}(i)+1}\right) =$$
$$f\left(r_{\pi^*(1)}, r_{\pi^*(1)+1}\right) + f\left(r_{\pi^*(2)}, r_{\pi^*(2)+1}\right) - \left(f\left(r_{\pi^*(1)+1}, r_{\pi^*(2)+1}\right) + f\left(r_{\pi^*(1)}, r_{\pi^*(2)}\right)\right).$$

It follows that

$$f\left(r_{\pi^*(1)}, r_{\pi^*(2)}\right) + f\left(r_{\pi^*(1)+1}, r_{\pi^*(2)+1}\right) > f\left(r_{\pi^*(1)}, r_{\pi^*(1)+1}\right) + f\left(r_{\pi^*(2)}, r_{\pi^*(2)+1}\right). \tag{10}$$

On the other hand the initial setting $r_1 > r_2 > \cdots > r_n$ implies $r_{\pi^*(1)} > r_{\pi^*(1)+1}$, $r_{\pi^*(1)} > r_{\pi^*(2)+1}$, $r_{\pi^*(2)} > r_{\pi^*(1)+1}$ and $r_{\pi^*(2)} > r_{\pi^*(2)+1}$. Setting $a = r_{\pi^*(1)}$, $b = r_{\pi^*(2)}$, $c = \max\left\{r_{\pi^*(1)+1}, r_{\pi^*(2)+1}\right\}$ and $d = \min\left\{r_{\pi^*(1)+1}, r_{\pi^*(2)+1}\right\}$, the conditions of Lemma 1 are satisfied and equation (4) yields $f\left(r_{\pi^*(1)}, r_{\pi^*(2)}\right) + f\left(r_{\pi^*(1)+1}, r_{\pi^*(2)+1}\right) < f\left(r_{\pi^*(1)}, r_{\pi^*(1)+1}\right) + f\left(r_{\pi^*(2)}, r_{\pi^*(2)+1}\right)$, which contradicts (10). Notice that the setting of *c* and *d* as max and min is allowed since *f* is symmetric. In conclusion there exists $\left|\pi^*(1) - \pi^*(2)\right| = 1$.

It can be shown similarly that $\left|\pi^*(1) - \pi^*(3)\right| = 1$, where r_3 and r_2 reside on the opposite sides of r_1. Notice that the cyclical order of the objects yields two subsequences in π^* between $\pi^*(1)$ and $\pi^*(2)$, and two between $\pi^*(1)$ and $\pi^*(3)$, thus is enabling the selection of two disjoint subsequences for inversion. ☻

Theorem 1: Let $f(x,y):\mathbb{R}^2 \to \mathbb{R}$ be symmetric and $f \in \mathrm{D}^2$, satisfying $\partial f(x,y)/\partial x \times \partial f(x,y)/\partial x > 0$ and $\partial^2 f(x,y)/\partial x \partial y < 0$. Given $(r_1,...,r_n)$ a sequence of n non negative real numbers satisfying $r_1 > r_2 > \cdots > r_n$, the permutation:

$$\pi^*(i) = \begin{cases} \dfrac{n+1+i}{2} & i \text{ is even} \\ \dfrac{n+2-i}{2} & i \text{ is odd} \end{cases}, \tag{11}$$

satisfies:

$$F(\pi^*) = \min_{\pi \in \Pi} \sum_{i=1}^{n} f\left(r_{\pi(i)}, r_{\pi(i)+1}\right). \tag{12}$$

Proof: The proof follows by induction. We use the convention that in the cyclical order of the optimal permutation even indices are added one by one counterclockwise on one side of the maximal object r_1, while odd indices are added one by one clockwise on its opposite side. Since the sequence is cyclical, we set arbitrarily the value $\pi^*(1) = (n+1)/2$, which is consistent with (11). Lemma 2 proved that $\pi^*(2)$ and $\pi^*(3)$ must be adjacent to $\pi^*(1)$ on its two opposite sides. This also satisfies (11) since $\pi^*(2) = (n+3)/2$ and $\pi^*(3) = (n-1)/2$, obtaining the successiveness of $\pi^*(2)$, $\pi^*(1)$ and $\pi^*(3)$.

Let $p < n$ be the smallest index of the original sequence for which $\pi^*(p)$ does not satisfy (11). Assume without loss of generality that p is even. It follows from the initial setting that all even elements from r_2 to r_{p-2} satisfy $r_1 > r_2 > r_4 > \cdots > r_{p-2}$. In π^* they are positioned successively counterclockwise and satisfy $r_{\pi^*(1)} > r_{\pi^*(2)} > r_{\pi^*(4)} > \cdots > r_{\pi^*(p-2)}$ by induction. Similarly, all odd elements between r_3 and r_{p-1} satisfy $r_1 > r_3 > r_5 > \cdots > r_{p-1}$ in the initial setting, and are positioned by induction successively clockwise such that $r_{\pi^*(1)} > r_{\pi^*(3)} > r_{\pi^*(5)} > \cdots > r_{\pi^*(p-1)}$.

Consider the non empty subsequence $\pi^*(p-2)+1,...,\pi^*(p)$. Let us transform π^* into π^{**} by reverting (flipping) the order of the elements in the subsequence into $\pi^*(p),...,\pi^*(p-2)+1$. As in the proof of Lemma 2 we'll look at $F(\pi^*)-F(\pi^{**})$, which from the optimality of π^* must be non positive, and then obtain a contradiction.

There exist two cases. In the first case $\pi^*(p)$ and $\pi^*(p-1)$ are not adjacent. This implies equation (4) of Lemma 1. Usage of same proof as in Lemma 2 yields $F(\pi^*)>F(\pi^{**})$, which contradicts π^* being optimal. In the second case $\pi^*(p)$ and $\pi^*(p-1)$ are adjacent, namely $\pi^*(p)=\pi^*(p-1)-1$ ($p-1$ is odd), so conditions of Lemma 2 are not satisfied. It follows however from the induction hypothesis that $r_{\pi^*(p-2)}>r_{\pi^*(p-1)}>r_{\pi^*(p-1)-1}\left(=r_{\pi^*(p)}\right)>r_{\pi^*(p-2)+1}$. Setting $a=r_{\pi^*(p-2)}$, $b=r_{\pi^*(p-1)}$, $c=r_{\pi^*(p-1)-1}$ and $d=r_{\pi^*(p-2)+1}$, yields $a>b>c>d$, so the situation of Lemma 2 is not satisfied and equation (5) holds. We conclude again that $F(\pi^*)>F(\pi^{**})$ which is a contradiction. ☻

Figure 2 illustrates the *symmetric hill* optimal permutation proved by Theorem 1 to minimize the functional sum of cyclically ordered adjacent elements. It has one peak (maximum) and one valley (minimum) located oppositely to each other, while all elements are evenly distributed on both sides.

3. Conclusions

This existence of a solution for a family of LOP has been proven. In these problems the cost associated with any two objects obeys a very general functional form that was found useful in VLSI interconnect design. Only adjacent objects contribute to the total cost and their contribution depends on some function $f(x,y)$ where x and y are values of a parameter of the objects. For a broad range of problems the optimal solution is obtained by a unique permutation of the objects called *symmetric hill*. This order can be derived directly from the problem setting since it depends only on the given set of parameter values for the objects.

The technique used in this paper can be applied to other objective functions, thus yielding different permutations. The authors believe that under the same functional requirements, LOP aiming at maximization will yield a monotonic jigsaw permutation, which is open for a proof. Furthermore, the idea that the cost associated with two objects satisfies some function may be applicable to other permutation problems such as quadratic assignment.

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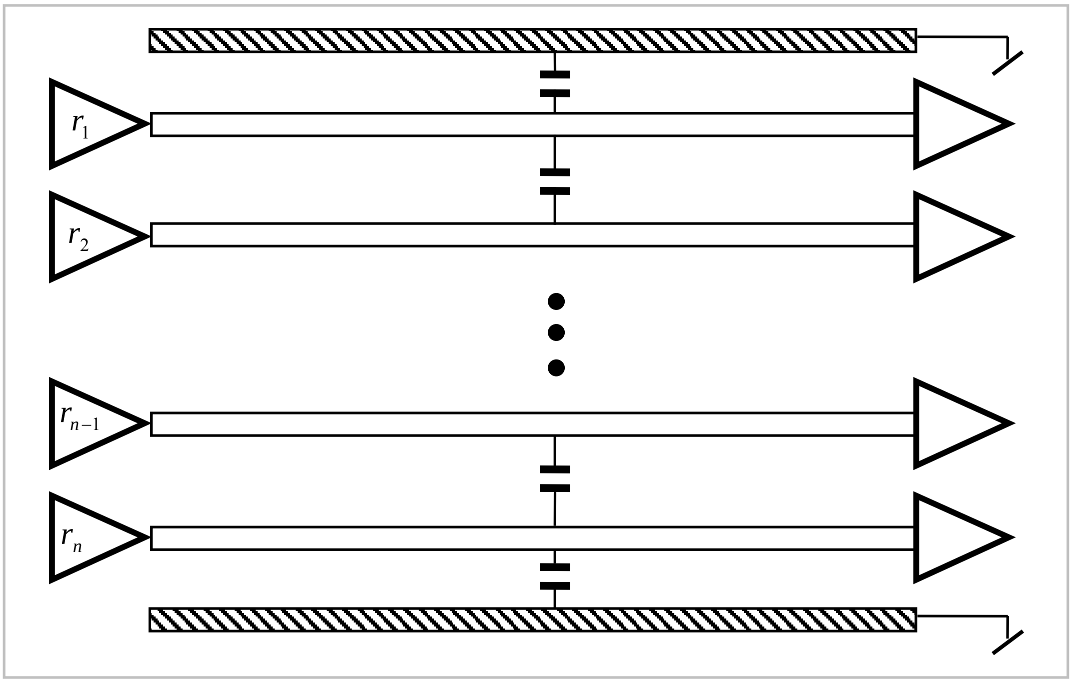

Figure 1: Typical VLSI interconnect bus. Drivers are shown on the left side and receivers on the right side. The bus is shielded on its two sides. A parasitic cross-coupling capacitance is incurring between any adjacent signals.

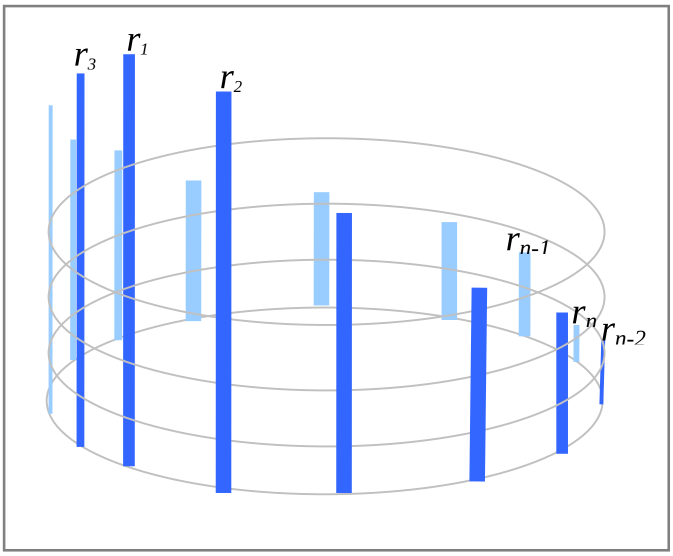

Figure 2: Symmetric Hill Order. The largest elements reside at the hill, while the smallest ones reside at the valley.