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GEOMETRIC APPROACH TO SAMPLING AND COMMUNICATION

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ABSTRACT. Relationships that exists between the classical, Shannon-type, and geometric-based approach to sampling are investigated. Some aspects of coding and communication through a Gaussian channel problem are considered. In particular, a constructive method to determine the quantizing dimension in Zador's theorem is provided. A geometric version of Shannon's Second Theorem is introduced. Applications to Pulse Code Modulation and Vector Quantization of Images are provided. In addition, we sketch the geometerization of wavelets for Image Processing purposes. We also discuss the implications of the Uncertainty Principle on sampling and reconstruction of images. An extension of our sampling scheme to a certain class of infinite dimensional manifolds is considered.

1. General background

1.1. **Introduction.** We consider a geometric approach to Shannon's sampling theorem, i.e. one based on sampling the *graph* of the signal, considered as a *manifold*, rather than a sampling of the *domain* of the signal, as is customary in both theoretical and applied signal and image processing, motivated by the framework of harmonic analysis. In this context it is important to note that Shannon's original intuition was deeply rooted in the geometric approach, as exposed in his seminal work [60], and also in [61]. Indeed, it is this geometric viewpoint of the problem which distinguishes Shannon from Kotelnikov [35] and Nyquist [45], and allows him to transcend the restricted context of technical communication theory. We were also inspired in our endeavor by the "dictionary" of geometric to communication theory notions, and we strived to emulate it.[1]

Our approach is based upon the following sampling theorem for differentiable manifolds that was recently presented and applied in the context image processing ([54], [55])[2]:

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1991 *Mathematics Subject Classification.* AMS Classification. Primary: 94A24, 94A20, 9408, 94A40; Secondary: 94A34, 49Q15, 68P30, 53C21.

[1]Different paths towards the geometerization of Sampling Theory can be found, e.g. in [49] and [34].

[2]An approach similar to ours appeared in [37], however mathematically less rigorous and comprehensive. Unfortunately, we were not aware of the existence of this study upon

Theorem 1.1. *Let $\Sigma^n \subset \mathbb{R}^N, n \geq 2$ be a connected, not necessarily compact, smooth manifold, with finitely many compact boundary components. Then, there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\left(\frac{1}{k(p)}\right)$, where $k(p) = \max\{|k_1|, ..., |k_n|\}$, and where $k_1, ..., k_n$ are the principal curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Moreover, the following corollary is also applicable to this problem:

Corollary 1.2. *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling scheme of Σ^n of finite density everywhere. In particular, if Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

Note, however, that this is not necessarily the optimal scheme (see [55]).

The constructive proof of this theorem is based on the existence of the so-called fat triangulations (see [52]). The density of the vertices of the triangulation (i.e. of the sampling) is given by the inverse of the maximal principal curvature. An essential step in the construction of the said triangulations consists of isometrically embedding of Σ^n in some $\mathbb{R}^N$, for large enough N (see [48]), where the existence of such an embedding is guaranteed by Nash's Theorem ([44]). Resorting to such a powerful tool as Nash's Embedding Theorem appears to be an impediment of our method, since the provided embedding dimension N is excessively high (even after further refinements due to Gromov [27] and Günther [28]). Furthermore, even finding the precise embedding dimension (lower than the canonical N) is very difficult even for simple manifolds. However, as we shall indicate in the next section, this high embedding dimension actually becomes an advantage, at least from information theory of viewpoint.

The resultant sampling scheme is in accord with the classical Shannon theorem, at least for the large class of (bandlimited) signals that also satisfy the condition of being $\mathcal{C}^2$ curves. In our proposed geometric approach, the radius of curvature substitutes for the condition of the Nyquist rate. To be more precise, our approach parallels, in a geometric setting, the *local bandwidth* of [29] and [72]. In other words, manifolds with bounded curvature represent a generalization of the *locally band limited signals* considered in those papers. However, the 1-dimensional case is a limiting, degenerated case, from the geometric viewpoint. Specifically, the notion of *fatness* (of simplices), essential to the geometric sampling scheme (see [55]), reduces, in this case to the (rather week) condition that there exists $f_0 > 0$ such that $(s_{i+1} - s_i)/(s_{j+1} - s_j) \geq f_0$, for all pairs of distinct sampling points s_i, s_j, i.e such that the ratios of the distances between consecutive sampling points are bounded from below and above. (Indeed, no notion of angle exists on the real line.) This fact induces an apparently weakness of the geometric method (as compared to the classical Shannon theorem), since curvature is

the publication of our previous work [54], [55] and [2]. We use therefore this opportunity to rectify it.

scale dependent. However, for bandlimited signals (and certainly for any "real life" signals), the energy of the signal is bounded from above by some E_0, hence we can assume it is normalized to equal 1. Then it is easy to see that the curvature condition is actually equivalent to the Nyquist rate.[3,4]

Here we further investigate the extent and implications of this analogy, and of the geometric approach in general. We begin by making a few observations regarding the extent of our results, by finding the largest space of signals wherein our results may be applied effectively. Next, in Section 2, we establish the proper analogies considering concepts originating from classical sampling and from coding theory, considered in the context of Gaussian channels. In doing so, we attempt to construct a "dictionary" of geometric sampling and concepts originating from Shannon's fundamental approach [59].

In the last section we address a few implementation problems. In particular, we sketch a geometrical approach to wavelets in the context of Image Processing. We also discuss the implications of the Uncertainty Principle insofar as sampling and reconstruction of images are concerned.

Finally, we extend in the Appendix our investigation to (a class of) infinite dimensional manifolds. Such manifolds, and the need for a sampling theory for this class of geometrical objects naturally arise, for instance, in the context of continuous variations and deformations, of signals, e.g video, perceived as infinite series of trigonometric functions.

1.2. **General "geometric" signals.** We begin our investigation by noting that, by the Paley-Wiener Theorem (see, e.g., [50]), any bandlimited signal is of class $\mathcal{C}^\infty$. We have already shown in [54] that our geometric sampling method applies not only to bandlimited signals, but also to more general "blackboard signals", i.e. L^2 functions whose graphs are smooth $\mathcal{C}^2$ curves, not necessarily planar.In fact, the geometric sampling approach can be extended to a far larger class of manifolds. Indeed, every piecewise linear (PL) manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) *smoothing* (see for example [65]), and every topological manifold of dimension $n \leq 3$ admits a PL structure (cf., for example, [65]). In particular, for curves and surfaces, one can first consider a smoothing of class $\geq \mathcal{C}^2$ (so that curvature can be defined properly), which can then be sampled with sampling rate given by the maximal curvature radius. Since the given manifold and its smoothing are arbitrarily close [43], one obtains the desired sampling result. (This very scheme is developed and applied in [54] for gray-scale images.)

[3]In comparing the classical and geometric approaches, one should bear in mind that no algebraic structure is presumed in the geometric context , whereas it is, implicitly assumed in the (infinite) sum appearing in the classical version.

[4]In fact, Shannon already had the intuition of the role of curvature (using second partial derivatives) for sampling and begun to explore its geometry in[61].

While numerical schemes for practical implementation of smoothing exist, they are not necessarily computationally satisfactory. For practical applications, one can circumvent this problem by applying numerical schemes based on the finite element method ([56]). However, for the sake of mathematical correctness and in order to be able to tackle more general applications, one would like to consider more general curvature measures (see, e.g. [71]) and avoid smoothing altogether (see [56] for the full details of this approach and Section 2 below, for a brief discussion of this topic in a slightly different context).

It is worthwhile to highlight yet another aspect of our geometric sampling method: Shannon's Sampling theorem relates to bandlimited signals, that are, necessarily, unbounded in time (space). Obviously, unbounded signals are not encountered in Image Processing, nor in any other practical implementations. In contrast, geometric sampling presents no unboundedness restrictions, quite the opposite: it is far easier to apply for bounded manifolds. The importance of this fact is far from being purely theoretical. Indeed, by eliminating the need to produce periodic signals (surfaces) it drastically reduces *aliasing* effects. There is, of course, a fundamental constraint of uncertainty (see also [23]) in the background of Shannon's Sampling Theorem. As already stated, this will be addressed in the context of our geometrical approach in Section 3.

1.3. **Pulse Code Modulation for Images.** Our geometrical approach to sampling lends itself to consideration of a much broader range of topics in communications. In particular, it offers a new method for PCM (*pulse code modulation*) of images.[5] This approach is endowed with an inherent advantage in that the sampling points are associated with relevant geometric features (via curvature) and are not chosen randomly via the Nyquist rate. Moreover, the sampling is in this case adaptive and, indeed, compressive (see [8]), lending itself to interesting technological benefits.

1.4. **Vector Quantization for Images.** A complementary byproduct of the constructive proof of Theorem 1.1 is a precise method of *vector quantization* (or *block coding*). Indeed, the proof of Theorem 1.1 consists in the construction of a Voronoi (Dirichlet) cell complex $\{\bar{\gamma}_k^n\}$ (whose vertices will provide the sampling points). The centers a_k of the cells (satisfying a certain geometric density condition) represent, as usual, the *decision vectors*. An advantage of this approach, besides its simplicity, is entailed by the possibility to estimate the error in terms of length and angle distortion when passing from the cell complex $\{\bar{\gamma}_k^n\}$ to the Euclidean cell complex $\{\bar{c}_k^n\}$ having the same set of vertices as $\{\bar{\gamma}_k^n\}$ (see [48]). Indeed, in contrast to other related studies, our method not only produces a piecewise-flat simplicial approximation of the given manifold, it also actually renders a simplicial complex on the manifold. Moreover, one can actually compute the local distortion

[5] considered as such and not as 1-dimensional signals

resulting by passing from the Euclidean geometry of the piecewise-flat approximation to the intrinsic geometry of its projection on the manifold. If $M = M^n$ is a manifold without boundary, then locally, for any triangulation patch[6], the following inequality holds [48]:

$$\frac{3}{4}d_M(x,y) \le d_{eucl}(\bar{x},\bar{y}) \le \frac{5}{3}d_M(x,y)\,;$$

where d_{eucl}, d_M denote the Euclidean and intrinsic metric (on M) respectively, and where $x, y \in M$ and $\bar{x}, \bar{y}$ are their preimages on the piecewise-flat complex. For manifolds with boundary, the same estimate holds (for the *intM* and ∂M), except for a (small) zone of "mashing" triangulations (see [52]), where the following weaker distortion formula is easily obtained:

$$\frac{3}{4}d_M(x,y) - f(\theta)\eta_\partial \le d_{eucl}(\bar{x},\bar{y}) \le \frac{5}{3}d_M(x,y) + f(\theta)\eta_\partial\,;$$

where $f(\theta)$ is a constant depending on the $\theta = \min\{\theta_\partial, \theta_{\mathrm{int}\,M}\}$ – the fatness of the triangulation of ∂M and int M, respectively, and η_∂ denotes the *mesh*[7] of the triangulation of certain neighbourhood of ∂M (see [52]). In other words, the (local) projection mapping π between the triangulated manifold M and its piecewise-flat approximation Σ is (locally) *bi-lipschitz* if M is open, but only a *quasi-isometry* (or *coarsely bi-lipschitz*) if the boundary of M is not empty.[8]

Yet, a more important advantage stems from Zador's Theorem [70], implying that we can turn into an advantage the inherent "curse of dimensionality". Indeed, by of Zador's Theorem, the *average mean squared error per dimension*:

$$\mathcal{E} = \frac{1}{N}\int_{\mathbb{R}^N} d_{eucl}(x,p_i)p(x)dx\,,$$

p_i being the *code point* closest to x and $p(x)$ denoting the *probability density function* of x, can be reduced by making avail of higher dimensional quantizers (see [14]). Since for embedded manifolds it obviously holds that $p(x) = p_1(x)\chi_M$, we obtain:

$$\mathcal{E} = \frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)p_1(x)dx\,,$$

It follows that, if the main issue is accuracy, not simplicity, then 1-dimensional coding algorithms (such as the classical Ziv-Lempel algorithm) perform far worse than higher dimensional ones. Of course, there exists an upper limit for the coding dimension.[9] The geometric coding method proposed here

[6] The building of these patches is essential for the control of the fatness of the triangulation. Their size essentially depends upon the (local) maximal curvature, (see [48])

[7] i.e. the supremum of the diameters of the simplices belonging to the triangulation

[8] In fact, as the two inequalities above show, π is sightly stronger, in both cases.

[9] Otherwise one could just code the whole data as one N-dimensional vector (albeit of unpractically high dimension)!...

provides a *natural* high dimension for the quantization of M^n – the embedding dimension N. Moreover, it closes (at least for images and any other data that can be represented as Riemannian manifolds) an open problem related to Zador's Theorem: finding a constructive method to determine the dimension of the quantizers (Zador's proof is nonconstructive). In fact, for a uniformly distributed input (as manifolds, hence noiseless images, can assumed to be, at least in first approximation) a better estimate of the average mean squared error per dimension can be obtained, namely:

$$\mathcal{E} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\int_{M^n} dx} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\mathcal{V}_n(M^n)dx}\,,$$

where $\mathcal{V}_n$ denotes the n-dimensional volume (area) of M. Whence, for compact manifolds one obtains the following expression for $\mathcal{E}$:

$$\mathcal{E} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\sum_i^m \int_{V_i} dx} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\sum_i^m \mathcal{V}_n(V_i)dx}\,,$$

where V_i represent the Voronoi cells of the partition.

Moreover, we have the following estimate for the *quantizer problem*, that is: Chose centers of cells such that the quantity

$$\mathcal{Q} = \frac{1}{N}\frac{\frac{1}{m}\int_{M^n} d_{eucl}(x,p_i)dx}{\left(\frac{1}{m}\sum_i^m \mathcal{V}_n\right)^{1+\frac{2}{N}}}\,.$$

is minimized. Here, again, the high embedding dimension N furnishes us with yet an additional advantage. Indeed, manifolds N increases dramatically, even for compact manifolds and even taking into consideration Gromov's and Günther's improvement of Nash's original method (see [27], resp. [28]). For instance, $n = 2$ requires embedding dimension $N = 10$ and $n = 3$ the necessitates $N = 14$. Hence, for large enough n one can write the following rough estimate:

$$\mathcal{Q} \approx \frac{1}{N}\frac{\int_{M^n} d_{eucl}(x,p_i)dx}{\sum_i^m \mathcal{V}_n}\,.$$

2. Sampling and Codes

2.1. **Packings, Coverings and Lattice codes.** In classical signal processing, the required signal bandwidth W and the *Nyquist sampling rate* are constrained by the condition $W = \eta/2$.

This admits an immediate generalization to periodic signals, or, in geometric terms, for signals represented over a *lattice*: $\Lambda = \{\lambda_i\}$. One can even interpret the sides (boundaries) of the lattice as the various coordinates in a multi-dimensional (warped) space or time (see, e.g. [63], [39]).[10] Note that such signals can be viewed as distributions on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. According to this interpretation, the (n-dimensional!) period is the *fundamental cell* λ of the lattice. Two scalars are naturally associated with this cell: its diameter $\mathrm{diam}(\lambda)$[11] and its volume $\mathrm{Vol}(\lambda)$. Either one may be used as a measure of the n-dimensional period. However, they are both interrelated and associated with one geometric feature, the so-called "*fatness*":

Definition 2.1. Let $\gamma = \gamma^k$ be an k-dimensional cell. The *fatness* (or *aspect-ratio*) of γ is defined as:

$$\varphi(\gamma) = \min_{\lambda} \frac{\mathrm{Vol}(\lambda)}{\mathrm{diam}^l(\lambda)}\,,$$

where the minimum is taken over all the l-dimensional faces of γ, $0 \leq k$. (If $\dim\lambda = 0$, then $\mathrm{Vol}(\lambda) = 1$, by convention.)

In the case of simplices (and of regular cells) this definition of fatness is equivalent to the following one (see [48])[12]:

Definition 2.2. Let $\gamma = \gamma^k$ be an k-dimensional cell. γ is called *φ-fat* if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$; where r denotes the radius of the inscribed sphere of γ (or *inradius*) and R denotes the radius of the circumscribed sphere of γ (or *circumradius*). A cell-complex $\Gamma = \{\gamma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $\varphi \geq 0$ such that all its cells are *φ-fat*.

Recall that the in- and circum-radius are important in lattice problems: given a lattice Λ with (dual) Voronoi cell Π (of volume 1), one has to minimize the inradius to solve the packing problem, and to minimize the circumradius for solving the covering problem (see [14]). Note that Λ and Π are simultaneously fat. It follows that fat cell-complexes and, in particular, fat triangulations, represent a mini-max optimization for both the packing and the covering problem. Moreover, since fat triangulations are essential for the sampling theorem for manifolds, it appears that there exists an intrinsic relation between the sampling problem for manifolds and the covering and packing problems.

2.2. **Average Power, Rate of Code and Channel Capacity.**

It is natural to extend the classical definitions of *average power* in the signal:

$$P = \frac{1}{T}\int_0^T f^2(t)dt\,,$$

[10] Alternatively, one can interpret the dimensions as representing wave length, or even as mixed fundamental quantities, e.g. space-time or even space-time-wave-length, as they arise in Medical Imaging (CT).

[11] or, alternatively, the length of the longest edge

[12] Any cell can be canonically decomposed into simplices, hence the equivalence of the definition can be extended to general cells.

and the *rate* of the code:

$$R = \frac{1}{T}\log_2 N\,,$$

in the context of lattices with fundamental cell λ, where N represents the number of code points, in the following manner:

$$P = \frac{1}{\mathrm{Vol}(\Lambda)}\int_\lambda f^2(t)dt = \frac{1}{N_1\mathrm{Vol}(\lambda)}\int_\lambda f^2(t)dt\,,$$

and

$$R = \frac{1}{\mathrm{Vol}(\Lambda)}\log_2 N = \frac{1}{N_1\mathrm{Vol}(\lambda)}\log_2 N,$$

respectively, N_1 being the number of cells.

Similarly, one can adapt the classical definition of the *channel capacity*:

$$C = \lim_{T\to\infty} R = \lim_{T\to\infty}\frac{\log_2 N}{T}\,,$$

to become

$$C = \lim_{T\to\infty}\frac{\log_2 N}{\mathrm{Vol}(\Lambda)} = \lim_{T\to\infty}\frac{1}{N_1\mathrm{Vol}(\lambda)}\log_2 N.$$

Since N and N_1 are related by $N_1 = \alpha(N)$, where α is the *growth function* of the manifold, the expression of C becomes:

$$C = \lim_{T\to\infty}\frac{1}{\mathrm{Vol}(\lambda)}\frac{\log_2 N}{\alpha(N)}.$$

It follows that $C = \infty$ for non-compact Euclidean and Hyperbolic manifolds, and $C = 0$ for their Elliptic counterparts. Unfortunately, no such immediate estimates can be readily produced for manifolds of variable curvature.

Note that by putting $1/T = \mathrm{Vol}(M)$, the definitions above apply for any sampling scheme of any manifold of finite volume, not just for lattices. In this case N and N_1 represent the number of vertices, respective simplices, of the triangulation.[13]

The interpretation of frequency considered above does not extend, however, to general geometric signals. For a proper generalization we have to look into the geometric analogue of W. Based on [55], Theorem 5.2 for the case of curves, i.e. 1-dimensional (geometric) signals, W equals the *curvature rate* $k/2$, were k represents the maximal absolute curvature of the curve. This, and the sampling Theorem 4.11 of [55] naturally lead us to the following definition of W for general geometric signals:

Definition 2.3. Let $M = M^n$ be an n-dimensional manifold $n \geq 2$. $W = W_M = 1/k_M$, where $k_M = \max k_i$ and $k_i, i = 1,\dots,n$ are the principal curvatures of M.

[13]In the language of classical signal processing, this approach is known as '*recurrent nonuniform sampling* [19]. (See also [68].)

According to classical considerations, the energy E of the signal $f : \mathbb{R} \to \mathbb{R}$ is considered to be equal to its L^2 norm:

$$E = E(f) = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2W} \sum_{k=-\infty}^{+\infty} f^2\big(\frac{k}{2W}\big)\,.$$

One would like, of course, to find proper generalizations of the notion of energy for more general (geometric) signals. In view of the discussion above, it is clear that a first step is to replace $2W$ by its proper generalization. However, when considering more general function spaces of specific relevance (s.a. *bounded variation* (BV), *bounded oscillation* (BO), *bounded mean oscillation* (BMO)), one should consider energies fitting the specific norm of the space under consideration. This discussion is, of course, also valid with regard to the best way to define average power P, and rate R, of a geometric signal.

We can now look into the first definition of code efficiency: the *nominal coding gain* of a code c_1 over another code, say (c_2), is:

$$\mathrm{ncg}(C1, C2) = 10\log_{10}\left(\frac{\mu_1}{E_1}\Big/\frac{\mu_2}{E_2}\right),$$

where μ is the square of the minimal squared-distance between coding points. For geometric codes of bounded curvature (hence compact ones), the expression for μ is particularly simple: $\mu = 1/\min k$ (k denoting again principal curvature).

2.3. **The Channel Coding Problem.** It is most natural to approach the problems associated with the *Gaussian white noise channel* in the context of "geometric signals", i.e. in the context of manifolds. Recall that in the classical context, a *received signal* is represented by a vector $X = F + Y$, where $F = (f_1, \ldots, f_N)$ is the *transmitted signal*, and $Y = (y_1, \ldots, y_n)$ represents the noise, whose components y_i are independent Gaussian random variables, of *mean* 0 and *average power* σ^2. The main, classical result for the Gaussian channel is the following:

Theorem 2.4 (Shannon's Second Theorem, [60])**.** *For any rate R not exceeding the* capacity *C_0:*

$$C_0 = \frac{1}{T}\log_2\left(1 + \frac{P}{\sigma^2}\right),$$

there exists a sufficiently large T, such that there exists a code of rate R and average power $\leq P$, and such that the probability of a decoding error is arbitrarily small. Conversely, it is not possible to obtain arbitrarily small errors for rates $R > C_0$.

In the case of geometric signals, F is given by the sampling (code) points on the manifolds and, since the mean equals 0, the noisy transmitted signal $F + Y$ lies in the *tube* $\mathrm{Tub}_\sigma(M)$. Recall that $\mathrm{Tub}_\sigma(S) = \bigcup_{p\in S} I_{p,\sigma_p}$, where

I_{p,σ_p} is the open symmetric interval through p, in the direction of unit normal $\bar{N}_p$ of M at p, of length $2\sigma_p$, where σ_p is chosen to be small enough such that $I_{p,\sigma_p} \cap I_{q,\sigma_q} = \emptyset$, for any $p, q \in S$.

While not being seemingly evident, the existence of tubular neighborhoods is assured both locally, for any regular, orientable manifold, and globally for regular, compact, orientable manifold (see [26]). In addition, the regularity of the manifolds $\partial\mathrm{Tub}^-_\sigma(M) = M - \bigcup_{p\in M} \varepsilon\bar{N}_p$, $\partial\mathrm{Tub}^+_\sigma(M) = M + \bigcup_{p\in M} \varepsilon\bar{N}_p$, where $\partial\mathrm{Tub}^-_\sigma(M) \cup \partial\mathrm{Tub}^+_\sigma(M) = \partial\mathrm{Tub}_\sigma(M)$, is at least as high as that of M: If M is convex, then $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ are piecewise $\mathcal{C}^{1,1}$ manifolds (i.e. they admit parameterizations with continuous and bounded derivatives), for all $\sigma > 0$. Also, if M is a smooth enough manifold with a boundary, that is, at least piecewise $\mathcal{C}^2$, then $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ are piecewise $\mathcal{C}^2$ manifolds, for all small enough σ (see [20]).

In the geometric setting, σ can be taken, of course, to be the maximal Euclidean deviation. However, a better deviation measure is, at least for compact manifolds, the Haussdorf Distance (between M and $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$):

Definition 2.5. Let (X, d) be a metric space and let $A, B \subseteq (X, d)$. The *Hausdorff distance* between A and B is defined as:

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}\,.$$

For non-compact manifolds one has to consider the more general *Gromov-Hausdorff distance* (see, for example, [5]).

Since, according to the above arguments, both the distance between M and $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ and the deviations of their curvature measures are arbitrarily small for small enough σ, we can state a first geometric version of Shannon's Theorem for the Gaussian channel. While a perfect analogy is not available, we can nevertheless formulate the following theorem:

Theorem 2.6 (Qualitative geometric version of Shannon's Second Theorem)**.** *Let M^n be a smooth geometric signal (manifold) and let σ be small enough, such that $\mathrm{Tub}_\sigma(M)$ is a submanifold of $\mathbb{R}^{n+1}$. Then, given any noisy signal $M + Y$, such that the average noise power σ_Y is at most σ, there exists a sampling of $M + Y$ with an arbitrarily small probability of resultant decoding error.*

Remark 2.7. The analogue of the capacity in the context of the geometric approach to codes is $C_0 = C_0(k, \sigma, r)$, where r represents the differentiability class of M.

Remark 2.8. The existence of tube $\partial\mathrm{Tub}^+_\sigma(M)$ is, as noted, guaranteed globally in the case of compact manifolds. Hence it follows that the sampling scheme is also global and necessitates $O(N)$ points, $N = N_M$. However, for non-compact manifolds (in particular non-band limited geometric signals), the existence of $\partial\mathrm{Tub}^+_\sigma(M)$ is guaranteed only locally. Therefore "gluing "of patches is needed, an operation which requires the insertion of additional vertices (i.e. sampling points), their number being a function of the dimension of M. Hence, in this case, $N_{M+Y} = O(N_M^n)$.

It is important to note that, again, this result is not restricted to smooth manifolds, but rather extends to much more general signals: Indeed, for any compact set $M \in \mathbb{R}^n$, the $(n-1)$-dimensional sets $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$, are *Lipschitz manifolds* [14] for almost any ε (see [30]). Moreover, the generalized curvatures measures of $\partial\mathrm{Tub}^-_\sigma(M)$, $\partial\mathrm{Tub}^+_\sigma(M)$ are arbitrarily close to the curvature of M, for small enough σ ([12], [30]). It follows that the above generalization befits not only the case of the Gaussian noise, but to more general types of noise, as well (see, [60], [31], [36]).

The full details of a *quantitative* version, including the general case, are laborious and warrant a separate discussion (see [56]).

Before we conclude this section, we wish to emphasize that the importance of tubes is not necessarily limited to Differential Geometry. It is just as important in Statistics (see, e.g. [38]). However, its relevance to sampling theory is in particular evident. Indeed, Shannon's ideas, as exposed in [59] and [60] are very similar to our approach (even if lacking the specific geometric nomenclature)[15]. For instance, his equivalent of the tube radius is used in [59] as a measure of the uncertainty of the reconstruction (not to confused with the Heisenberg Uncertainty Principle – see Section 3). For the development of Shannon's approach see [62].

3. Implementation Considerations: Geometric Wavelets and Uncertainty

The question is whether the implementation of the geometric sampling scheme is feasible. We do not address the purely geometric aspects, that would be highly relevant in Computer Graphics implementation (these were partly addressed in [55] and [2]).

Instead, we focus on the far more important and popular Image Processing tool of wavelets. The versatility and adaptability of wavelets for a variety of tasks in Image Processing and related fields is too well established in the scientific community, and the bibliography pertaining to it is far to extensive, to even begin to review it here.

We do, however, stress the fact that the multiresolution property of wavelets has been already applied in determining the curvature of (planar) curves [1] and to the intelligence and reconstruction of meshed surfaces (see, e.g. [18], [25], [40], [67]). Moreover, the intimate relation between scale and differentiability in natural images has also been stressed [21].

[14] i.e. topological manifolds equipped with a maximal atlas for which the changes of coordinates are Lipschitz functions.

[15] In particular, convergence of the measure of entropy of the noise introduced in [60] is easily obtained in our setting.

Unfortunately, for many investigators concerned with wavelets applications, PL approximations are not necessarily among their most familiar tools. It is, therefore, a challenge to consider the integration of tools practiced by both communities. Although it may appear to be a surprising result to those primarily familiar with classical wavelets, the *Strömberg wavelets* [64], that are based on PL functions.

Another, more intriguing issue is whether one can replace the intuitive trade-off between scale and curvature, by a formal concept of *wavelet curvature*, in particular in cases such as those of the Strömberg wavelets, or, in the more difficult case of Haar wavelets that are not even piecewise linear[16]. Interestingly enough, this can be done by using *metric curvatures* [4] (and [53] for a short presentation); a detailed discussion of the technical aspects of these issues will appear elsewhere [57]). We should just mention here that for the Strömberg wavelets, the ideal metric candidate is the *Menger curvature* [17], which is by now well established as a tool both in Pure (e.g. [47]) and Applied (e.g. [24]) Mathematics. For the Haar wavelets, the desired metric curvature is the *Finsler-Haantjes curvature* [18].

A more suitable approach to surface reconstruction could, for example, implement *ridgelets* [7], or the more generalized, *curvelets* [9]. For these too metric curvatures are natural and computable, see [57].

The application of wavelet functions in the reconstruction of the signal brings up the issue of Uncertainty Principle with regard to the structure of, and constraints existing in the design of, such functions (see, e.g. [66]).

The Uncertainty Principle has an impact on the geometric sampling approach at a much more fundamental level than its direct effect on the structure of the approximation functions. It is valid in each and every of the parameterizations $\varphi : \mathbb{R}^n \to M^n$, that define the manifold M^n ([65]). Since these parameterizations represent a locally finite family, it follows that, at least for compact manifolds, a type of global Uncertainty Principle is valid. Of course, for manifolds that are graphs of functions $f : \mathbb{R}^n \to \mathbb{R}$ (or even $f : \mathbb{R}^n \to \mathbb{R}^m$) a proper global Uncertainty Principle holds. This includes the extension to images and, in particular, color and fully-textured images that have become so important in the field of image processing. On the other hand, such global parameterizations for images are not really attainable, hence the need for approximations and the relevance of the problems addressed in [66]. Moreover, different parameterizations can be obtained for

[16] and for any classical concept of curvature is not even definable.

[17] The Menger curvature of a triangle is defined to be the radius of its circumscribed circle.

[18] The Finsler-Haantjes curvature of an arc of curve basically measures the scaled difference between the length of the arc and the length of the chord it spans. Note that for curves in the plane (and space) the Finsler-Haantjes and Menger curvatures a a point can be defined by passing to the limit. In this context, the two notions coincide and, moreover, they reduce to the classical notion of curvature.

the same manifold, therefore bounds for the manifold[19] can not be attained in this manner.

In addition, we should note that one can use the Local Uncertainty Principle to prove the global one (see [13]) and that the Local Uncertainty Principle is, in fact, qualitatively stronger (see [22]).

It appears as though there is a discrepancy between the harmonic analysis-based approach and the geometric-based-approach (adopted herein and elsewhere [54], [55] and [2]). However, a direct relation between the geometry of the manifold M^n and the proper Uncertainty Principle has been recently revealed. To highlight it, one first proves that a generalized form of the Uncertainty Principle (the so called *Weak Uncertainty Principle*,[20] [46]) implies a number of classical inequalities, in particular Nash-type and Poincare-type inequalities. Then, one shows that these inequalities are strongly influenced by the geometry of the manifold, in particular by the Ricci curvature – see [17], [69], [51]. The connection between these facts is completed by the fact that the Weak Uncertainty Principle implies the classical one (see [46]).

A somewhat different approach to a geometric viewpoint of the Uncertainty Principle applies to the Pitt inequality and logarithmic Sobolev inequality (see [3]). See also [16] for an extended treatment, including a Uncertainty Principle for discrete-time signals.

Appendix – Geometric Sampling of Infinite Dimensional Signals

Since in the classical context band-limited signals are viewed as elements f of $L^2(\mathbb{R})$, such that $supp\,(\hat{f}) \subseteq [-\pi, \pi]$, where $\hat{f}$ denotes the Fourier transform of f, it is interesting to ask the following question: Can one extend the sampling theorem proven in [54] to infinite dimensional manifolds? Using an example introduced in [41], it may be concluded that this question is not only far from being naive, but rather the answer is positive in that our geometric sampling method translates directly into the context of infinite dimensional manifolds, at least for a class of functions that naturally arise in the the context of [41]. However, since the full proofs required in the example below are rather technical, we refer the reader to the original paper [41], and limit ourselves here to a brief presentation.

Consider the following spaces:

$$C_1^\infty = \left\{ e \in \mathcal{C}^\infty(\mathbb{R}) \,\middle|\, e(x+1) = e(x) \right\},$$

$$C_+^\infty = \left\{ e \in \mathcal{C}^\infty(\mathbb{R}) \,\middle|\, e(x+1) = e(x), \int_0^1 e^2 = 1 \right\},$$

$$M \subset C_+^\infty, M = \left\{ \lambda_0 = 0 \,|\, \lambda_0 \text{ first eigenvalue of} Q \right\},$$

where Q denotes the Hill operator: $Q = -D^2 + q\,, q(x+1) = q(x)$, and where $D = \partial/\partial x, 0 \le x < 1$.

[19] and not merely for the parameterizing functions

[20] We do not discuss here in detail this generalized inequality, because of the technicalities involved, for these the reader is referred to [46].

Since 1 an 0 are regular values for their respective functions, M is a smooth, co-dimension one hyper-surface in C_1^∞, that is endowed with the topology inherited from that of C^∞, endowed with the *sup* norm. Further, exactly as in the finite-dimensional case (see [55]), for any 2-dimensional section determined by unit tangent vectors to M at q, one can define (and compute) the maximal principal curvature (of the section).

Moreover, since the co-dimension of M in C_1^∞ is 1, it follows that, together with a tangent plane, a normal to M at q is also defined. It follows that one can use the same method as in the finite dimensional case (see [55]) to find a sampling of M.

It follows that a sampling scheme identical to that developed for the finite dimensional case can be applied for the manifold M, as well. Unfortunately, no uniform sampling is possible for the entire manifold: the maximal curvature, associated with functions approximating "saw-tooth" functions can be made as large as desired (see [41]).

One would like to extend these considerations, in a systematic manner, to general infinite dimensional manifolds (e.g. l_2 and Hilbert cube manifolds). However, even if the appropriate geometric differential notions are defined and computed, the fundamental problem of constructing fat triangulations for infinite dimensional manifolds still has to be dealt with. It is therefore important to note that triangulations of such manifolds exist (see [10]). However, even finding a notion analogous to that of fatness in the ∞-dimensional case represents a challenge. (For a different approach to a differential geometry of some infinite dimensional spaces see for example [42].)

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