IRWIN AND JOAN JACOBS
CENTER FOR COMMUNICATION AND INFORMATION TECHNOLOGIES

Normal Approximations of Geodesics on Smooth Triangulated Surfaces

Eli Appleboim, Emil Saucan and Jonathan Stern

CCIT Report #739
August 2009

Electronics
Computers
Communications

DEPARTMENT OF ELECTRICAL ENGINEERING
TECHNION - ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 32000, ISRAEL

CCIT REPORT #739 August 2009

NORMAL APPROXIMATIONS OF GEODESICS ON SMOOTH TRIANGULATED SURFACES

ELI APPLEBOIM[1] EMIL SAUCAN[2] AND JONATHAN STERN[3]

Abstract. We introduce a novel method for the approximation of shortest geodesics on smooth triangulated surfaces. The method relies on the theory of normal curves on triangulated surfaces and their relations with geodesics. We also relate in this work to normal surfaces and comment on the possible extension of the method for finding minimal surfaces inside 3-manifolds.

Contents

Date: August 9, 2009.

Second Author research partly supported by the Israel Science Foundation.

1. **Introduction**

In this paper the ambient space is some triangulated surface and we study relations between geodesics on that surface, and *shortest normal curves* with respect to the given triangulation and its subdivisions. In [App] ,two flavors of results where proved.

- *Convergence results*: There is a semi discrete variational method under which a sequence of locally shortest normal curve will converge to a geodesic curve, not necessarily shortest.
- *Approximation results*: Every geodesic on a surface is the limit curve, under the variational method mentioned above, of a sequence of short normal curves with respect to some triangulation and its subdivisions.

While the study of geodesics on surfaces has long history and occupies a huge amount of literature, we will focus in this paper on the relations between them and the very specific type of curves, the so called short normal curves which will be presented shortly. In general, normal curves can be regarded as a specific type of quasi-geodesics in the sense of [AZ] and the ones we will be interested in, form a subset of these that will be termed as "straight".

In order to achieve the above goals we will define *length* and *curvature* measures on normal curves according to which *least-weight, minimal, shortest* and *straight* curves will be defined. This will be done in section3. Straight curves are in interest also in the paper [PS], yet, since the curvature measure we will define herein is different from the one in [PS], (see also [Sto]), so does the meaning of being straight. In fact, in some sense the term we will use generalizes the one in [PS], and serves for a different analysis of the relations between geodesics and quasi geodesics.

In Section 4 we will describe the variational process, basically, a *curvature flow*, given in [App] for normal curves and show that we obtain a family of *straight* normal curves which converges to a limit curve that is a geodesic. We will have two types of flows where one is merely an extension of the standard curvature flow as studied in [GH], [Gra1], [Gra2] and others, to piecewise smooth curves, while the second flow is a more discrete version of the latter, where we restrict the flow only to the points at which the curve is non-smooth. In this section we will prove convergence results for both flows.

In 5, we will alter the direction of interest and briefly discuss the approximation theorems proven in [App] for geodesic curves. We will take into account a special type of triangulations, the so called *fat triangulations*. We will rely on works such as [SAZ1] for the ability to build a desirable fat triangulation for our needs. We will also consider *Pl-approximations* of the surface that will be based on this fat triangulation and its subdivisions, and show that for every geodesic curve on a smooth surface there is an *approximating sequence* of *PL* straight normal curves. We will also deduce an algorithmic way to find such a sequence in the case of *shortest geodesics*.

In section 6 we will present some very preliminary experimental results of the algorithm mentioned above of approximating shortest geodesics by straight normal curves.

Finally, in 7 we will hint about our current study in which we intend to prove some similar convergence and approximation results for *least area normal surface*, with respect to minimal surfaces.

In the following Section 2 we will give a brief introduction to normal surfaces and normal curves. We will start with surfaces for the sake of being historically compatible as normal surfaces where introduced first and basically, draw much more attention.

2. Preliminaries - Normal surfaces and curves

2.1. **Normal Surfaces.** Throughout the entire review $(M, \Sigma, \mathcal{T})$ will denote a 2 dimensional surface, Σ, embedded in 3 dimensional manifold, M, which admits a triangulation, $\mathcal{T}$. Let $\mathcal{T}^{(i)}$ denote the i-skeleton of $\mathcal{T}$, i.e.

$$\mathcal{T}^{(i)} = \bigcup \{\text{simplices of dimension} \leq i; \;\; i = 0, ..., 3\}$$

Definition 2.1.
(1) Let t be a tetrahedron of $\mathcal{T}$, then there are seven possible special disks properly embedded in t, called *normal disks*, as illustrated in Figure 2.1. Two disjoint disks of the same type inside some $t \in \mathcal{T}$, are called *parallel*.
(2) We say that Σ is a *normal surface* in M (with respect to $\mathcal{T}$) if its intersections of any of the tetrahedra of $\mathcal{T}$,for any tetrahedron, t of $\mathcal{T}$, is made of a collection of disjoint normal discs. Note that since we assume the surface to be embedded in M we can have only one type of a 4-gon at each tetrahedron.

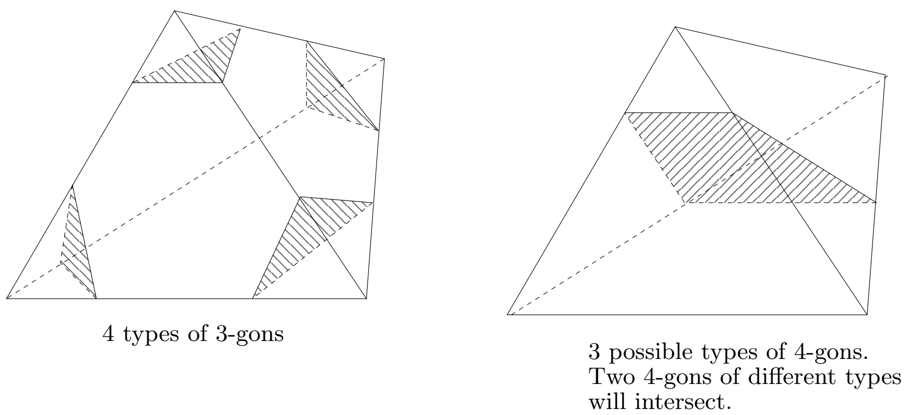


Figure 1. Normal disks

Definition 2.2. The *weight* of a normal surface is the number of intersection point of the surface with the 1-skeleton of the triangulation and is denoted by $w(\Sigma)$.

Theorem 2.3. [JR1], [Ha1], [Ha2] *Any surface, embedded in a* 3*-manifold, can be made into a (possibly empty) set of normal surfaces, w.r.t any triangulation of the manifold, using a finitely many local isotopies and finitely many compressions. In particular, if the surface is incompressible then the surface is isotopic to a normal surface.*

2.2. **Normal Curves.** In a very similar way one can define normal curves on surfaces.

Definition 2.4. A curve C on a triangulated orientable surface Σ is called *normal curve* iff all intersections of C with 2-simplecies of the triangulation $\mathcal{T}$ are made of normal pieces. i.e. made of pathes which run from one edge to a different edge of the same triangle. We assume C is smooth in the interior of the 2-cells it pathes through and it may not be smooth along its intersections with the edges of the triangulation.

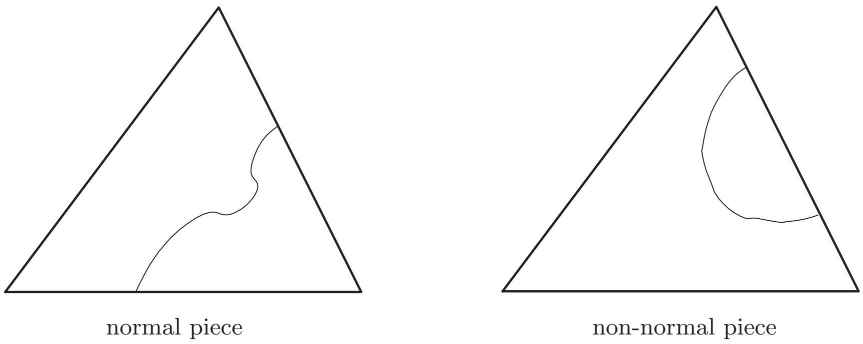
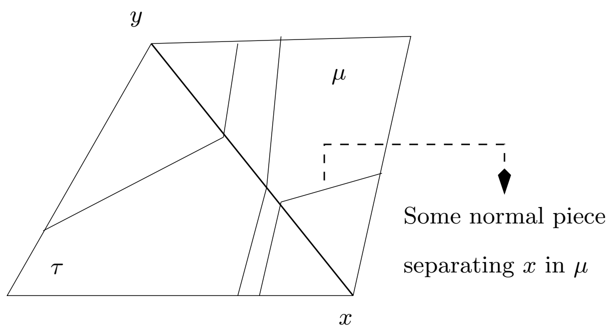


FIGURE 2. normal and not normal pieces.

2.2.1. **Matching equations.** Consider a normal curve γ and let t, r be two 2-cells, adjacent along an edge e. We may code each normal piece of γ in t say, according to the single vertex of t separated from the other two, by the normal piece. Since each normal piece in t which pathes through e into r will separate either of the vertices of e, we get an equation of the form,

$$x_t + y_t = x_r + y_r,$$

where x_t, x_r are the number of normal pieces separating the vertex x in t, r respectively.



Theorem 2.5 ([Hak], [JR1]). *Every normal curve on a triangulated surface induces a set of matching equations and vice versa every* integral solution *whose all values are* non negative*, of a system of matching equations can be realized as a normal curve.*

3. Length and curvature measures for normal curves

In the sequel we will define length and curvature measures for normal curves with respect to which we will have definitions of minimal (shortest) curves. These definitions lead in turn to curvature flows minimizing the length of normal curves. In this process we will go from fine to coarse in the sense that first we will have a subtle length function that will result in principally the curvature flow very known in the literature for few decades, [Gra1], [Gra2], [GH]. The second length function we will use will be purely combinatoric but we will show that with the definition of curvature of a normal curve at every point we can obtain a discrete version of the curvature flow that yields a sequence of minimal normal curves that converges to a smooth geodesic curve. All results herein are given without proofs. Detailed proofs can be found in [App]

3.1. Length and weight.

Let $(\Sigma, \mathcal{T})$ be a triangulated surface where $\mathcal{T} = (V, E, F)$ is a geodesic triangulation with respect to a given Riemannian structure on Σ. Using Rivin [Riv1], and Leibon, [Leib], one can assume $\mathcal{T}$ is a Euclidean triangulation. As such we have an induced metric on the edges and facets of $\mathcal{T}$.

Definition 3.1.
(1) The weight, ω, of an embedded curve on a triangulated surface is the total number of intersection points of the curve with $\mathcal{T}^1$.
(2) The length L, of a **normal** curve on a surface is the sum of lengths of all its normal pieces.
   (a) A curve will be called *minimal* iff its length is stationary with respect a small variation.

(b) A normal curve is called *least-weight* iff its weight is minimal in its isotopy class. For curves with *non empty fixed boundary*, we can also have curves with *non-zero* minimal weight amongst all isotopy classes rel-boundary points.

(c) A normal curve will be called *shortest* iff its length is minimal amongst all normal curves isotopic to it[1]. The same distinguish between closed curves and curves with boundary, as above, holds for the term *shortest*. For curves with fixed boundary we can have *non-zero* length, minimal with respect to all isotopy classes.

Unless mentioned otherwise, when dealing with curves with boundary, *least-weight/shortest* will always mean relative to *all isotopy classes*.

Remark 3.2. *Using the weight function one can easily show that any curve on a triangulated surface can always be isotoped to a curve which is either normal or completely contained in a single triangle. If the curve is closed and essential, i.e. not null-homotopic it is isotopic to a normal curve. Hence, a* least-weight *curve is always normal or has zero weight. Yet, all normally homotopic curves have the same weight so, there is no uniqueness of a least-weight normal curve.*

Remark 3.3. *Assuming the triangulation is convex, say, Euclidean, and the geodesic is compact, we can use general position argument in order to modify the triangulation a little to ensure that the geodesic misses the vertices while maintaining the convexity property. In particular, if we assume the Triangulation to be Euclidean, then the above holds.*

Lemma 3.4. *If σ is a hyperbolic surface and $\mathcal{T}$ is a* geodesic ideal triangulation *of σ, then we have a minimal length normal curve, in every normal class.*

3.2. **Graph reflections of normal curves.** Let γ be a normal curve on the triangulated surface $(\Sigma, \mathcal{T})$. Suppose $\partial\gamma = \{p, q\}$ are two fixed points on Σ inside some two 2-cells, τ_p, τ_q, respectively. It may be possible that $p = q$. We will interpret such a normal curve as a *path* in the *dual graph* of $\mathcal{T}$.

3.2.1. **Reflection in the dual graph.** γ naturally defines a weighted path γ^* in the dual graph $\mathcal{T}^*$ of $\mathcal{T}$, in the following way.

- Each vertex of the dual path γ^* corresponds to a triangle intersected by γ.
- Each edge of γ^* is assigned to an edge of $\mathcal{T}$ at which γ crosses from one 2-cell to an adjacent one.
- We weight each edge of γ^* according to the number of times γ crosses through the corresponding edge of $\mathcal{T}$.
- We define the *length* of γ^* to be the sum of weights of all its edges.

---

[1] What is the analogue, if any of a **vertex linking sphere?** Looks like a **vertex linking circle.**

It is our intension to find efficient characteristics for *least-weight* normal curves. in this course we would like to view such a normal curve as a realization of a shortest weighted path in the dual graph. Yet, some caution must be taken since, generally we cannot expect a general weighted path in the dual graph to be realizable as a normal curve on the surface. Restrictions on this are due to the matching equations. We will show in the sequel that for a least-weight normal curve the situation is simpler and that, in fact, it can indeed be viewed as a realization of a shortest path in the dual graph.

Decoding of the matching equations in the dual setting is doable and should give a characterization of the weighted dual paths that can be realized as normal curves yet, it is cumbersome and tedious hence left out of the scope of this current paper.

Remark 3.5. *It is essential at this context to consider curves that are* least-weight *relatively to all isotopy classes of a given initial path for while proving some of the proceeding lemmas we may not be able to keep isotopy class fixed. If the curve is closed, we will assume it is an essential curve with some fixed starting point p, and consider* least weight *curves with respect to all isotopy classes of closed essential curves starting at p. In fact, it is enough of course that the curve pathes through p. In the case of a non-essential closed curve where the* least-weight *one weights zero, the results below are evident.*

Lemma 3.6. *Let γ be a normal curve from p to q, that has* least-weight *amongst all normal curves connecting p to q. Then, γ has at most one normal piece in every triangle of $\mathcal{T}$.*

Lemma 3.7. *A path in the dual graph for which the all weights along its edges equal to 1 is realizable as a normal curve with respect to $\mathcal{T}$.*

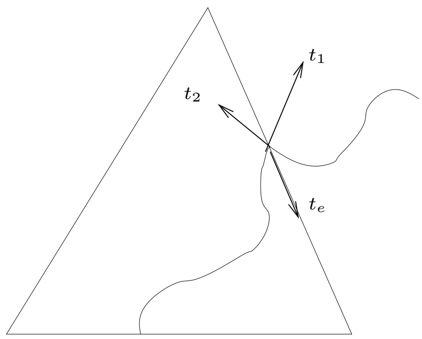
Theorem 3.8. *A normal curve is a* global*(w.r.t isotopy classes)*least-weight *curve if and only if its corresponding dual path is a shortest path according to the length function defined above.*

Corollary 3.9. *There is an algorithm which finds in finite time least-weight normal curve on a triangulated surface.*

Definition 3.10. Let Γ be a normal curve with respect to a triangulation $\mathcal{T}$ on a surface Σ. We will define the curvature of Γ at a point x accordingly.

(1) If x is an internal point on a segment of $\Gamma \cap \mathcal{T}^2$, we will take the curvature to be,
$$\mathfrak{K}(x) = k_g(\Gamma, x),$$
where $k_g(\Gamma, x)$ denotes the geodesic curvature of Γ at x.

(2) If x is a vertex formed by the intersection of Γ with $\mathcal{T}^1$, and let t_1, t_2, be the two vectors tangent to Γ at x. The curvature at x is defined to be
$$\mathfrak{K}(x) = \cos(\theta_1) + \cos(\theta_2),$$
where θ_1, θ_2 are the angles between t_1, t_2 and t_e the vector tangent to the edge e at x.



Remark 3.11. *There is a vast literature concerning curvature measures for curves which may not be smooth and it is usual to consider some angle defect at points where the curve is not smooth. The definition given herein is meant to be compatible with a variation of length formula which will be referred to soon, and also for being compatible with a definition of curvature for normal surfaces given in* [JR2] *(to be referred to in Section* 6*).*

4. **Curve shortening flows**

Following Cheeger and Ebin, [CE], we use first variation of arc length to find out conditions for which a normal curve is minimal. suppose $\Gamma_s(t)$ is a small variation of Γ with arc length parameter t. Let S be the variation vector field and T be the tangent vector field ($S = \frac{\partial}{\partial s}$, $T = \frac{\partial}{\partial t}$). Then the length of $\Gamma_s(t)$ is given by,

$$L_s = \sum_i L_i = \sum_i \int_0^{L_i} |\Gamma'_{s,\,i}(t)|dt = \sum_i \int_0^{L_i} < T, T >^{1/2} dt\ ,$$

where the indices i stands for the i^{th} normal piece of Γ.
For a curve to be minimal we demand

$$\frac{\partial L_s}{ds}|_{(s=0)} = 0.$$

$$\frac{\partial L_s}{ds} = \sum_i \int_0^{L_i} \frac{d}{ds} < T, T >^{1/2} dt = \sum_i \int_0^{L_i} S < T, T > dt =$$

$$= \sum_i \int_0^{L_i} (T < S, T > - < S, \nabla_T T >)dt = \sum_i < S, T > |_0^{L_i} - \sum_i \int_0^{L_i} < S, \nabla_T T >)dt =$$

$$\sum_i \int_0^{L_i} \nabla_T T dt + \sum_i (\cos(\theta_1^i) + \cos(\theta_2^i)).$$

This gives therefore, that in order for a curve to be a minimal normal curve it should both be a geodesic arc along the interior of each 2-cell it intersects, as well being “straight” along the intersections with the 1-skeleton. Straight means that along each edge, $(\cos(\theta_1)+\cos(\theta_2)) = 0$ which in turn generalizes the condition ($\theta_1 + \theta_2 = \pi$), traditionally obtained from other common curvature measures for polygonal curves, [**?**].

Definition 4.1. From the above we define the semi smooth curvature flow to be

$$\frac{\partial \Gamma}{\partial t} = \mathfrak{K}(\Gamma) \tag{1}$$

Where Γ is a normal curve and $\mathfrak{K}$ is its curvature as defined previously.

Lemma 4.2. *Let Γ be a piecewise smooth normal curve on a smooth triangulated surface $(\Sigma, \mathcal{T})$. Suppose $x \in \Gamma \cap \mathcal{T}^1$ is an intersection point of the curve on the interior of some edge e, so that Γ is* straight *at x in the sense defined above. Then the geodesic curvature of Γ at x exists and equals zero.i.e.*

$$\kappa_g(\Gamma, x) = 0$$

Corollary 4.3. *The curvature flow defined above transforms a normal curve embedded in a smooth surface to a geodesic curve.*

As said earlier in the text, the above is a rephrase of the curvature flow, extensively studied in the past, for piecewise smooth curves. In the following, we will further modify the flow by restricting it to $\Gamma \cap \mathcal{T}^1$.

Definition 4.4. Recall the definition of weight of a curve as defined in 3.1.

(1) The *normalized weight* of a normal curve is given by,

$$n\omega(\Gamma) = L(\Gamma) \cdot \lambda^2$$

where λ is the parameter of $\mathcal{T}$, i.e. the maximal edge length, where we range over all edges of $\mathcal{T}$.

Observation: In the context of the above definition shortest and least-weight are the same. Obviously, since all curves in the same normal homotopy class have the same weight, the is no uniqueness of shortest arcs in this respect

4.0.2. ***Minimizing through straightening.*** In this section we alter the minimization process while restricting it merely to points on $C \cap \mathcal{T}^1$. More precisely, consider the following procedure.

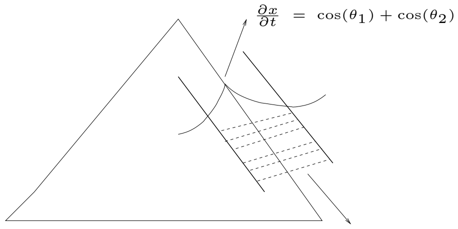
Let C be a rectifiable (i.e of finite total length) curve on a triangulated surface $(\Sigma, \mathcal{T})$.

(1) Normalize C with respect to $\mathcal{T}$.
(2) Take a least-weight normal curve $\widehat{C}$, isotopic to C.
(3) Straighten $\widehat{C}$ at all intersections with the edges of $\mathcal{T}$.
(4) Take a subdivision of $\mathcal{T}$ to obtain a new triangulation $\mathcal{T}_1$.
(5) Go to (1) while C is replaced by $\widehat{C}$.

Remark 4.5. *Step 3 is done by moving each point x of $\Gamma \cap \mathcal{T}^1$ according to,*

$$\frac{\partial x}{\partial t} = \cos(\theta_1) + \cos(\theta_2)\ , \tag{2}$$

where θ_1, θ_2 are as before. Using say, partition of unity we can smoothly extend this process inside some small collar neighborhood of $\mathcal{T}^1$.



Remark 4.6. *There is a whole variety of subdivisions that can be considered in the above procedure. The two that will be used herein are the barycentric subdivision and the "median" subdivision which is obtained by connecting the middle point of the edges of each triangle to each other. In each of these subdivision the length of an edge is half the length of its parent edge. If not necessary, we will not specify the actual subdivision taken.*

Proposition 4.7. *The obtained sequence of straight curves converges to a smooth geodesic on Σ*

Remark 4.8. *The limit geodesic curve may not be the a shortest geodesic. This is highly dependant on the geometry of the surface in a neighborhood of this limit curve being either hyperbolic, planar or spheric.*

Remark 4.9. *The fact that the curve is normal, is crucial and best illustrated by the lemma above for if the curve was not normal then we could not assume a bound on the arc length of its normal pieces, even when the division parameter λ gets very small, which makes the continuity of curvature useless. The normality assumption exactly prevents normal pieces to wind around the surface for a long period and then come back.*

5. PL Approximations of surfaces and geodesics

In this section we alter our attention in the opposite direction from the previous one. While in previous section we showed that we can have sequences of normal curves that converge to geodesics, which can be titled as *convergence results*, in this section we are interested in *approximation results*. We wish to show that every geodesic curve on a smooth surface is a limit of a sequence of normal curves with respect to some triangulation and its subdivisions. Opposed to previous section, we will have to consider a specific triangulation for our purpose or rather, a triangulation satisfying specific geometric constrains, and as a result we will also have to specify the subdivision scheme that can be used in order to fulfill these constrains.

Input: A surface Σ, two points p and q on it and a geodesic curve Γ from p to q.
Output: A sequence of curves Γ_n, each of which is a shortest/least-weight normal curve on a PL-surface Σ_n so that,

$$\Sigma_n \to \Sigma,$$

and,

$$\Gamma_n \to \Gamma.$$

5.0.3. **Fat triangulations and surface PL-approximations.** Before stating the main theorem of this section we will give a brief introduction on the technique we will use for PL-approximating of surfaces. The technique is based on existence results of fat triangulation of Riemannian manifolds in general, and surfaces in particular. In this subsection we will give basic definition notations and quote relevant results. All details can be found in the relevant cited literature.

Definition 5.1 (*Fat triangulations)*.
(1) A triangle in $\mathbb{R}^2$ is called *fat* (or φ-*fat*, to be more precise) iff all its angles are larger than a prescribed value φ.
(2) A k-simplex $\tau \subset \mathbb{R}^n$, $2 \leq k \leq n$, is φ-*fat* if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$. A triangulation of a submanifold of $\mathbb{R}^n$, $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is φ-*fat* if all its simplices are φ-*fat*.
(3) A triangulation $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $\varphi \geq 0$ such that all its simplices are φ-*fat*; for any $i \in \mathbf{I}$.

Proposition 5.2 ([CMS]). *There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma<\tau} \measuredangle(\tau,\sigma) \leq c(k)\cdot\varphi(\tau)\,, \tag{3}$$

and

$$\varphi(\tau) \leq \frac{Vol_j(\sigma)}{diam^j\,\sigma} \leq c(k)\cdot\varphi(\tau)\,, \tag{4}$$

*where φ denotes the fatness of the simplex τ, $\measuredangle(\tau,\sigma)$ denotes the (*internal*)* dihedral angle *of the face $\sigma < \tau$ and $Vol_j(\sigma)$; $diam\,\sigma$ stand for the Euclidian j-volume and the diameter of σ respectively. (If $dim\,\sigma = 0$, then $Vol_j(\sigma) = 1$, by convention.)*

Condition 3 is just the expression of fatness as a function of dihedral angles in all dimensions, while Condition 4 expresses fatness as given by "large area/diameter". Diameter is important since fatness is independent of scale.

Definition 5.3. (1) Let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}^*_+$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a δ-*approximation to* f iff:
(i) There exists a subdivision K' of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$, for any $x \in |K|$;
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \le \delta(a) \cdot d_{eucl}(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.

(2) Let K' be a subdivision of K, $U = \overset{\circ}{U}$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n)$, $g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a δ-*approximation* of f (on U) iff conditions (ii) and (iii) of Definition 2.6. hold for any $a \in U$.

Definition 5.4 (*Secant map*)**.** Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then the linear map: $L_s : s \to \mathbb{R}^n$, defined by $L_s(v) = f(v)$ where v is a vertex of s, is called the *secant map induced by* f.

The motivation for having fat triangulation for manifolds is stressed by the following theorem.

Theorem 5.5 ([Mun])**.** *Let $f : \sigma \to \mathbb{R}^n$ be of class $\mathcal{C}^k$. Then, for $\delta > 0$, there exists $\varepsilon, \varphi_0 > 0$, such that, for any $\tau < \sigma$, fulfilling the following conditions:*
(i) diam$(\tau) < \varepsilon$ and,
(ii) τ is φ_0-fat,
then, the secant map L_τ is a δ-approximation of $f|\tau$.

Existence of fat triangulations of Riemannian manifolds is guaranteed by the studies given in [Ca], [Pel], [Sa1] and [Sa2]. These are summarized below.

Theorem 5.6 ([Ca])**.** *Every compact $\mathcal{C}^2$ Riemannian manifold admits a fat triangulation.*

Theorem 5.7 ([Pet])**.** *Every open (unbounded) $\mathcal{C}^\infty$ Riemannian manifold admits a fat triangulation.*

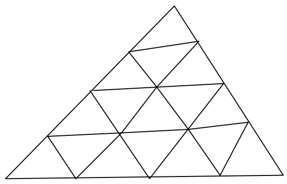
Theorem 5.8 ([Sa2])**.** *Let M^n be an n-dimensional $\mathcal{C}^1$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Following the above, a scheme for achieving fat triangulation for surfaces is given in,

Theorem 5.9 ([SAZ1], [SAZ2])**.** *Let Σ be a connected, non-necessarily compact, smooth of class $\mathcal{C}^k, k \ge 2$, n-Riemannian manifold, $n \ge 2$, with finitely many boundary components. Then, there exists a sampling scheme(i.e a way of building fat triangulation) of Σ, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\Big(\frac{1}{k(p)}\Big)$, where $k(p) = \max\{|k_1|, |k_2|\}$, and k_1, k_2 are the principal curvatures of Σ, at the point $p \in \Sigma$.*

Finally, we use the secant map as defined in order to reproduce a PL-surface as a δ-approximation for the sampled surface, [SAZ1], [SAZ2].

Definition 5.10. Let $\mathcal{T}$ be a triangulation of a surface Σ. The subdivision of $\mathcal{T}$ obtained by connecting the midpoints of edges of $\mathcal{T}$, will be noted as the *median subdivision*.



Median subdivision

We are now in a position to state the following approximation theorem. All proofs of the following results can be found in [App]

Theorem 5.11. *Let Γ be a geodesic curve between two points p and q on a triangulated surface $(\Sigma, \mathcal{T})$ where $\mathcal{T}$ is a fat triangulation of Σ, for which $\Gamma \cap \mathcal{T}^{(0)} = \emptyset$. Let $\mathcal{T}_{(n)}$ be the n^{th} median subdivision of $\mathcal{T}$. Let Σ_n be the PL-approximation of Σ built as the secant map on $\mathcal{T}_n$. Then, there exists a sequence of curves Γ_n embedded in Σ_n, each of which is a shortest/ least-weight normal curve (note that Σ_n is given as a triangulated polyhedral surface with a triangulation induced from $\mathcal{T}_n$), such that,*

$$\Gamma_n \to \Gamma.$$

While in the above theorem we showed that every *pre-given* geodesic curve can be *PL approximated* by *PL normal* curves, we will show that if only the end points are given, we can still find a sequences of *PL normal* curves that approximates the *shortest* geodesics between these points.

Theorem 5.12. *Let p and q be two given points on a smooth surface Σ. Then there is an algorithm to find a sequence of* PL-normal *curves, each of which embedded inside some* PL-approximation Σ_n *of Σ such that this sequence converges to a shortest geodesic curve from p to q.*

We will now define the projection of $\gamma_{(n)}$ onto Σ. First we will project its intersection points with the 1-skeleton of $\Sigma_{(n)}$ which was produced by the secant approximation[2]. This in turn will be done by defining a backprojection of the PL edge containing such intersection point onto the original edge on Σ it corresponds to. Recall that the two edges have common end points, two vertices of $\mathcal{T}$. Let these vertices be a and b and lat x be a point on the PL-edge $e_{(n)}$. We project $\tilde{x}$ onto $x \in e_{(n)}$ where $e_{(n)}$ is the appropriate edge belonging to $\mathcal{T}_{(n)}$ on Σ, in an arc length manner, i.e. x is the unique point

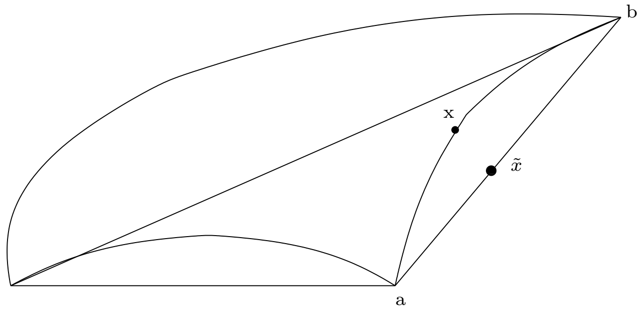
[2] All this parag. is way too cumbersome

satisfying,

$$\frac{\int_a^x |e'_{(n)}|d\xi}{L(e_{(n)})} = \frac{\int_a^x |e'_{(n)}|d\zeta}{L(e_{(n)})}\,;$$

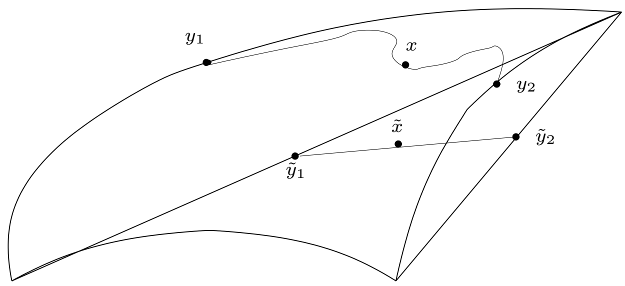
while $L(e), L(e)$ stands for the length of e and e respectively.

$$L(e) = \int_a^b |e'|dx \quad \text{resp. } L(e) = \int_a^b |e'|dx;$$



Now in a similar manner we project each normal piece of $\gamma_{(n)}$ on some 2-face of $\Sigma_{(n)}$, onto a normal piece on the corresponding 2-face of $\mathcal{T}_{(n)}$ on Σ. Let t, t[3] be the two 2-facets of $\mathcal{T}_{(n)}, \mathcal{T}_{(n)}$ corresponding to each other via the secant map. Let c be some normal arc on t having end points y_1, y_2 on two different edges of t. By the projection just defined we get two points y_1, y_2 on two edges of the face t, and we connect them by some normal arc on t. We have a freedom of choice here so we just pick some normal arc connecting y_1 to y_2. If one wants to have an explicit formulation for such projection use the same as above. For any point x on c, its corresponding projection is the unique point x, on the chosen normal piece such that

$$\frac{\int_{y_1}^x |c'|d\xi}{L(c)} = \frac{\int_{y_1}^x |c'|d\zeta}{L(c)}\,;$$



So far, doing so for every normal piece of $\gamma_{(n)}$ one obtains a normal curve on $\hat{\gamma}_{(n)}$, on σ connecting the points p and q. Mind that we can use the same as above to project the initial and final pieces of $\gamma_{(n)}$, the pieces that connect $p_{(n)}$ and $q_{(n)}$ to some points on edges. We have to make sure though that for all n, $p_{(n)}$ and $q_{(n)}$ are projected onto p and q respectively.

[3]Check notations.

Next, we go for straightening. The normal curve $\hat{\gamma}_{(n)}$ can be assumed smooth along the interior of all 2-facets it goes through and at each point it intersects an edge of $\mathcal{T}_{(n)}$, we straighten it in a small neighborhood of that point keeping the normal class of $\hat{\gamma}_{(n)}$ fixed. As a result we have produced a straight normal curve $\gamma_{(n)}$.

As $n \to \infty$, we have that the sequence of PL-curves $\gamma_{(n)}$ converges to its straight projections $\gamma_{(n)}$ which in turn, according to Proposition 4.7 converges to a smooth geodesic arc from p to q. Combining these two facts completes the proof[4].

6. Experimental Results

In this section we give some preliminary experimental results obtained from testing the algorithm described in Section 3 on both analytical surfaces as well as a surface which is part of a CT-Colonoscopy.

6.1. **Analytic surfaces.**

We bring here results obtained on a sphere with different resolutions. Similar results where obtained also for the torus and in the final version of this paper will be referred to in the authors web site.

The tests where run on a sphere of radius 1. On such a sphere, the angle between two radii ending at the start and end points is equal to the length of the shortest path between these 2 points on the face of the sphere. The sphere was approximated to a mesh 5 times, using 80, 320, 1280, 5120 and 20480 triangles, respectively. The following figure depicts the algorithm results. For each mesh, several pairs of start and end points were picked, according to the angle between them, and the algorithm was applied.

It is important to notice that a mesh of a sphere is bounded by the sphere, and a path on its surface can be shorter than the minimum path on the sphere itself. The higher the accuracy of the mesh, the smaller this possible difference is.

6.2. **Colon Surface.**

The following figure shows results of computing shortest normal curve with respect a triangulated mesh of some resolution obtained from a CT-Colonoscopy. Even in this fairly low resolution one can see a reasonable approximation of the shortest geodesic between the start and end points by the computed normal geodesic.

6.3. **Convergence.**

In the following example we see an initial normal approximation of a geodesic on the sphere and its convergence to the actual geodesic while subdivisions of the triangulation is taken.

7. Further study

7.0.1. ***Geometric Measures on Normal Surfaces.*** We adopt the definitions given in [JR2] of length and area of a normal surface and the definitions of minimal and of least area normal surfaces.

[4] Consider the following. We project straight lines hence, it may be possible to show that the projection is straight.

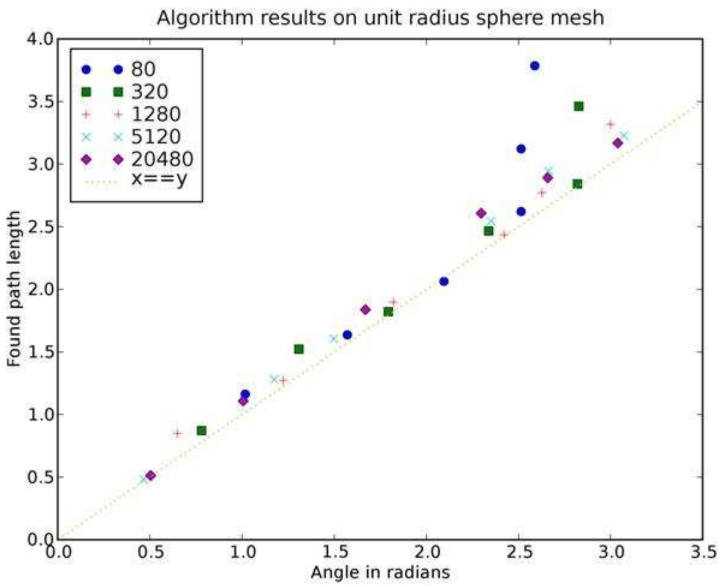


FIGURE 3. The dotted line is describes the length of the true shortest path on the face of the sphere.

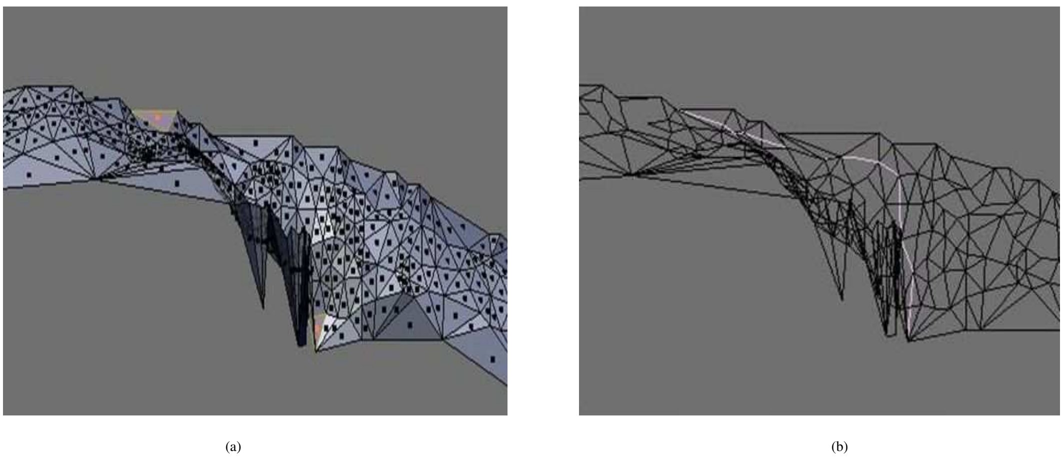


FIGURE 4. Shortest Normal curve on a Colon mesh. (a) The mesh with indicating start and end points. (b) Computed shortest normal curve

Definition 7.1. (1) Since every face of $\mathcal{T}^2$ can be identified with an ideal hyperbolic triangle in the hyperbolic plane $\mathbb{H}^2$, we can measure the length of each arc in the intersection of Σ with the 2-faces of $\mathcal{T}$.

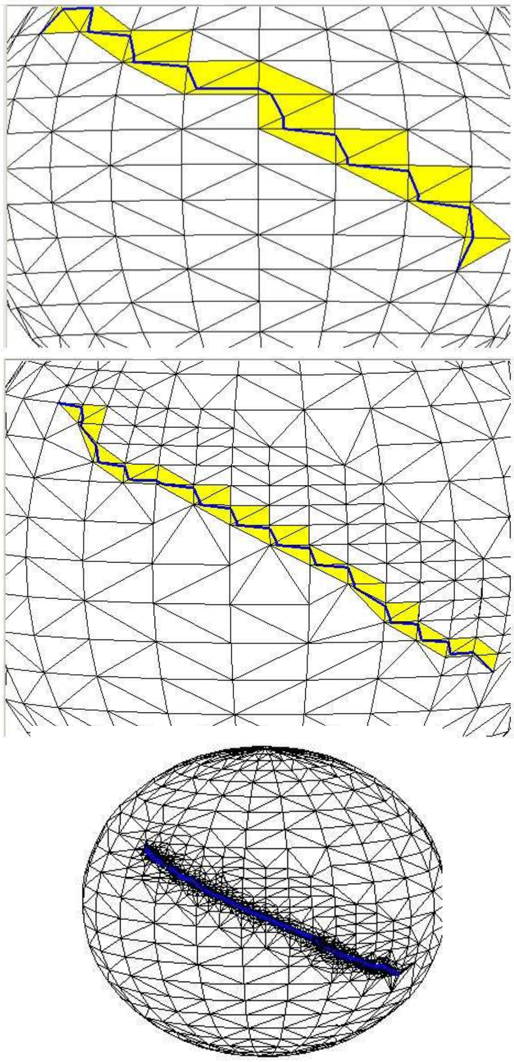


FIGURE 5. Initial normal curve and the curve after 1 and 9 iterations

The *length* of Σ, $l(\Sigma)$, is defined as the total sum of lengths of all such arcs.

(2) The *area* $a(\Sigma)$, is defined to be the pair (w, l), lexicographically ordered.

Remark 7.2. *One can think of this in an analogous way as a "blow up" of some Riemannian metric given on the* 3*-manifold* M*, first along the edges of* $\mathcal{T}$ *and afterwards by a smaller factor along its faces.*

A *normal homotopy* of a surface Σ is a smooth map $\Phi : \Sigma \times [0\ 1] \mapsto M$ so that for each $t \in [0\ 1]$, $\Sigma_t = \Phi(\Sigma \times t)$, is a normal surface. In particular it is a homotopy that does not meet the vertices of $\mathcal{T}$ at all times.

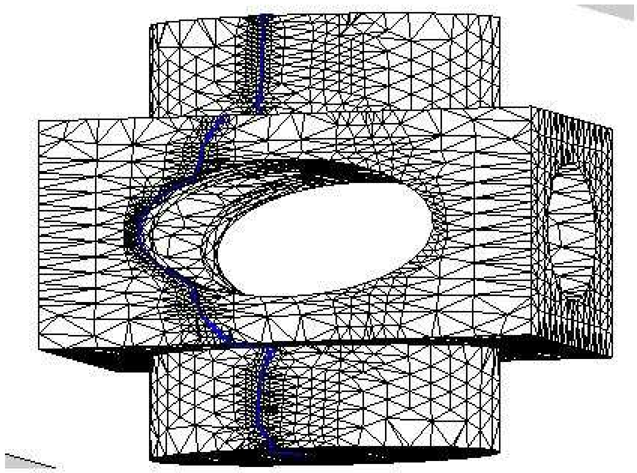


FIGURE 6. Normal approximation of a geodesic curve on a surface of non zero genus. The result shown is obtained after 12 iterations.

Definition 7.3.
- A normal surface Σ is called *PL minimal* if its *length* is *stationary* with respect to a small variation of it. i.e. if $F : \Sigma \times (\delta\ \delta) \mapsto M$, is a small variation of Σ, and $\sigma_s = F(\Sigma \times s)$, and suppose $\Sigma = \Sigma_0$. Then Σ is PL-minimal if $l'(0) = 0$ where, $l(s) = l(\Sigma_s)$.
- A normal surface is *PL least area* if its area is minimal with respect to its homotopy class.

It is possible to define a piecewise smooth *mean curvature* vector field, $\mathcal{H}$, on a normal surface in the following way.

Suppose α is some arc of $\Sigma \bigcap T^2$ and x is a point on α.

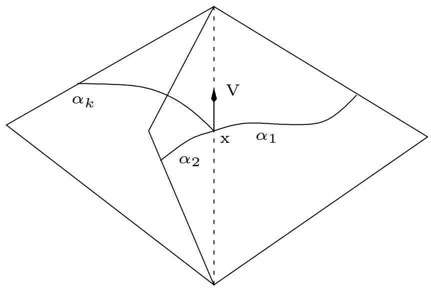
If x is internal in α then define $\mathcal{H}(x) = \nabla_{\bar{t}}(\bar{t}(x))$

Where $\bar{T}$ here stands for the tangent vector of α at x and ∇ is the contra-variant derivative of the tangent vector field. Hence, $|\bar{\mathcal{H}}|$ at an internal point is the geodesic curvature of α considered as an arc embedded in a triangular face of the triangulation (assume α is arc-length parameterized).

If $x \in \alpha \bigcap T^1$ is an end point of α, let $\{\alpha_1, \alpha_2, ...\alpha_k\}$ be all arcs of $\Sigma \bigcap T^2$ meeting at x, and define

$\mathcal{H}(x) = \Sigma_{i=1}^{k} \langle \bar{t}_i(x), \bar{V} \rangle \bar{V}$

Where $\bar{t}_i$ is the tangent vector of α_i at x and $\bar{V}$ is a vector tangent to the edge e of T where x is included. $|\bar{\mathcal{H}}|$ here is the sum of all co-sines of the angles between the t_is and $\bar{V}$.



Using first and second variations, it is shown in [JR2] that a *PL minimal* surface is a *stationary point* of an appropriate *mean curvature flow.* More precisely we have that Σ is a *PL minimal* surface iff $\mathcal{H}(\Sigma) \equiv 0$.

Existence and uniqueness of *PL minimal* surfaces is shown in,

Theorem 7.4. [JR2]
In the normal homotopy class of any normal surface Σ there exists a unique PL-minimal *surface which is not a multiple of a linking* 2*-sphere.*

Remark 7.5. *Since the definition of a PL-minimal as well as PL-least area surfaces does not account for the interior of the normal disks, the uniqueness argument in* [JR2] *holds only with respect to the boundary of any such normal disks while the interior of a disk may vary. In* [Ni] *the definition of PL-minimal surface is altered by putting an additional constrain that the interior of each normal disk in $\Sigma \cap \mathcal{T}^3$, is a minimal surface in the usual differential geometric sense.*

With this altered definition of PL-minimality it is shown,

Theorem 7.6 ([Ni])**.** *There is exactly one PL-minimal surface in a normal homotopy class of any normal surface Σwhich is not a multiple of a linking* 2*-sphere.*

References

[AZ] A. D. Alexandrov, V. A. Zalgaller, *Intrinsic geometry of surfaces,* A. M. S. Trans., 15,1967.

[App] E. Appleboim, *Normal geodesics on surfaces,* preprint.

[Ca] S. S. Cairns, *A simple triangulation method for smooth manifolds*, Bull. Amer. Math. Soc. 67, 1961.

[CE] J. Cheeger, D. Ebin,*Comparison theorems in Riemannian geometry,*

[CMS] J. Cheeger, W. Müller and R. Schrader, *On the Curvature of Piecewise Flat Spaces*, Comm. Math. Phys., **92**, 1984

[Fed] H. Federer, *Geometric Measure Theory,* Springer, New York, 1969

[FHS] M. Freedman and J. Hass and P. Scott *Least Area Incompressible Surfaces in 3-Manifolds,* Invent. Math. **71** (1983)

[GH] Gage. , Hamilton. R,*The heat equation,*

[Gor] C.Mca. Gordon, *The Theory of Normal Surfaces,* Lecture Notes, (Typeset by R.P. Kent IV) (2001)

[Gra1] Grayson. M, *Shortening embedded curves,* Ann. Math.

[Gra2] Grayson. M, *Shortening embedded curves,* Jour. Diff. Geom.

[Hak] W. Haken, *Theorie der Normalflachen,* Acta. Math. **105** (1961)

[HASTK] S. Haker, S. Angenet, A.Tannenbaum, and R. Kikinis, *Laplace-Beltrami Operator and Brain Flattening,* IEEE Trans. Medical Imaging **18** (1999)

[Ha1] J. Hass, *Algorithms for Recognizing Knots and 3-Manifolds,* J. Differntial Geometry **27** (1997)

[Ha2] J. Hass, *Minimal and Normal Surfaces,* Lecture Given at MSRI-Berkeley (1997)

[HST] J. Hass, J. Snoeyink and W.P. Thurston, *The Size of Spanning Disc for Polygonal Curve,* Descrete Comput. Geom. **29** (2003)

[JO] W. Jaco and U. Oertel, *An Algorithm to Decide if a* 3*-Manifold is a Haken Manifold,* Topology **23** (1984)

[JR1] W. Jaco and J. H. Rubinstein, *PL-Equivariant Surgery and Invariant Decompositions of 3-Manifolds,* Advances in Mathematics **73** (1988)

[JR2] W. Jaco and J. H. Rubinstein, *PL-Minimal Surfaces in 3-Manifolds,* J. Differntial Geometry **27** (1988)

[Leib] Leibon, G., *Characterizing the Delauney decomposition of compact hyperbolic surfaces*, Geometry ∂ Topology, 6, (2002)

[KMS1] R. Kimmel, R. Malladi, and N. Sochen, *Image Processing via the Beltrami Operator,*

[Kne] H. Keser, *Geschlossene flachen in drei-dimesnionalen Mannigfaltigkeiten,* Jahresber.Deutsch. Math. Verbein 1929.

[MY1] W.H. Meeks. III and S.T. Yau, *Topology of Three Dimensional Manifolds and the Embedding Problem in Minimal Surface Theory,* Ann. Math **112** (1980)

[MY2] W.H. Meeks. III and S.T. Yau, *The Classical Plateau Problem and The Topology of Three Dimensional Manifolds ,* Topology **21** (1982)

[MY3] W.H. Meeks. III and S.T. Yau, *The Existence of Embedded Minimal Surfaces and The Problem of Uniqueness,* Math. Z. **179** (1982)

[Mun] J. Munkres, *Elementary Differential Topology,* Princeton University Press, Princeton, N.J., 1966.

[Mor] Y. Moriah *Topology of* 3 *Manifolds,* Lecture Notes. Technion, (1999)

[Pel] K. Peltonen, *On the existence of quasiregular mappings*, Ann. Acad. Sci. Fenn., Series I Math., Dissertationes, 1992.

[Pet] S. Peters, *Convergence of Riemannian Manifolds,* Comp. Math. **62** (1987)

[Petsn] P. Petersen, *Gromov-Hausdorff Convergence of Metric Spaces,* Proc. Symposia in Pure Math. 54 (1993).

[PS] K. Polthier, M. Schmies, *Streightest geodesics on polyhedral surfaces,* Preprint, 1998.

[Riv1] Rivin, I., *Euclidean structures on simplcial surfaces and hyperbolic volume*, Ann. Math., 139 (1994)

[Sa1] E. Saucan, *The Existence of Quasimeromorphic Mappings in Dimension 3*, Conform. Geom. Dyn., Conform. Geom. Dyn. **10** (2006), 21-40.

[Sa2] E. Saucan, *Note on a theorem of Munkres*, Mediterr. j. math., **2**(2), 2005, 215 - 229.

[SAZ1] Saucan, E., Appleboim, E. and Zeevi, Y. Y. *Sampling and Reconstruction of Surfaces and Higher Dimensional Manifolds*, J. Math. Img. and Vis., 30, 2008.

[SAZ2] Saucan, E., Appleboim, E. and Zeevi, Y. Y. *Image Projection and Representation on* $\mathbb{S}^n$, Journal of Fourier Analysis and Applications, 13, 2007.

[Sto] D. Stone, *Geodesics in piecewise linear manifolds,* Trans. Amer. Math. soc. 215, 1976.

[Thu] W.P. Thurston, *Three-Dimesional Geometry and Topology,* S. Levi, ed., Princeton Univ. Press (1997)

[Ni] Y. Ni, *Uniqueness of PL minimal surfaces,* Acta. Math. Sinica, 6, 2007.

1 Department of Electrical Engineering, Technion, Technion City, Haifa 32000
E-mail address: eliap@ee.technion.ac.il

2 Department of Mathematics, Technion, Technion City, Haifa 32000
E-mail address: semil@tx.technion.ac.il

3 Department of Physics, Technion, Technion City, Haifa 32000
E-mail address: stern@tx.technion.ac.il