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On the Corner Points of the Capacity Region of a Two-User Gaussian Interference Channel

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Abstract

This work considers the corner points of the capacity region of a two-user Gaussian interference channel (GIC). In a two-user GIC, the rate pairs where one user transmits its data at the single-user capacity (without interference), and the other at the largest rate for which reliable communication is still possible are called corner points. This paper relies on existing outer bounds on the capacity region of a two-user GIC to derive informative bounds on these corner points for the case of two-sided weak interference (i.e., when both cross-link gains in standard form are positive and below 1). The informative bounds on the corner points are given in closed-form expressions, and a refinement of these bounds that refers to the possible increase in the rate of the second user when the first user operates at rate that is ε away from the single-user capacity is also considered. Upper and lower bounds on the gap between the sum-rate and the maximal achievable total rate at the two corner points are derived. This is followed by an asymptotic analysis analogous to the study of the generalized degrees of freedom (where the SNR and INR scalings are coupled), leading to asymptotic characterizations of this gap. The characterization is tight for the whole range of this scaling, and the simple upper and lower bounds on this gap are asymptotically tight in the sense that they achieve this asymptotic characterization. The upper and lower bounds on this gap are improved for finite SNR and INR, and these improvements are exemplified numerically.

1. Introduction

The two-user Gaussian interference channel (GIC) has been extensively treated in the literature during the last four decades (see, e.g., [8, Chapter 6] and references therein). For completeness and to set notation, the model of the two-user GIC in *standard form* is introduced shortly: this discrete-time, memoryless interference channel is characterized by the following relation between the two inputs (X_1, X_2) and corresponding two outputs (Y_1, Y_2):

$$Y_1 = X_1 + \sqrt{a_{12}}\, X_2 + Z_1 \tag{1}$$
$$Y_2 = \sqrt{a_{21}}\, X_1 + X_2 + Z_2 \tag{2}$$

where the cross-link gains a_{12} and a_{21} of the GIC are time-invariant, the inputs and outputs are real valued, and Z_1 and Z_2 denote additive Gaussian noise samples that are independent of the inputs. Let $X_1^n \triangleq (X_{1,1}, \ldots, X_{1,n})$ and $X_2^n \triangleq (X_{2,1}, \ldots, X_{2,n})$ be the two transmitted codewords across the channel where $X_{i,j}$ denotes the symbol that is transmitted by user i at time instant j (here, $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$). No cooperation between the transmitters is allowed (so X_1^n, X_2^n are independent), nor between the receivers; however, it is assumed that the receivers have full knowledge of the codebooks used by both users. The power constraints on the inputs are given by $\frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_{1,i}^2] \le P_1$ and $\frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_{2,i}^2] \le P_2$ where $P_1, P_2 > 0$. The additive Gaussian noise samples of Z_1^n and Z_2^n are i.i.d. with zero mean and unit variance, and they are also independent of the inputs X_1^n and X_2^n. Furthermore, Z_1^n and Z_2^n can be assumed to be independent since the capacity region only depends on the marginal conditional *pdf*s $p(y_i|x_1, x_2)$ for $i = 1, 2$ (due to the non-cooperation between the receivers). Perfect synchronization between the pairs of transmitters and receivers is assumed, which implies that the capacity region is convex (since time-sharing between the users is possible).

In spite of the simplicity of this model, the exact characterization of the capacity region of a GIC is yet unknown, except for strong ([11], [18]) or very strong interference [2]. Specifically, the corner points of the capacity region have not yet been determined for GICs with weak interference; for GICs with mixed or one-sided interference, only one corner point is known (see [14, Section 6.A], [16, Theorem 2] and [19, Section 2.C]).

The operational meaning of the study of the corner points of the capacity region for a two-user GIC is to explore the situation where one transmitter sends its information at the maximal achievable rate for a single-user (in the

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absence of interference), and the second transmitter maintains a data rate that enables reliable communication to the two non-cooperating receivers [4]. Two questions occur in this scenario:

Question 1: What is the maximal achievable rate of the second transmitter ?

Question 2: Does it enable the first receiver to reliably decode the messages of both transmitters ?

In his paper [4], Costa presented an approach suggesting that when one of the transmitters, say transmitter 1, sends its data over a two-user GIC at the maximal interference-free rate $R_1 = \frac{1}{2}\log(1+P_1)$ bits per channel use, then the maximal rate R_2 of transmitter 2 is the rate that enables receiver 1 to decode both messages. The corner points of the capacity region are therefore related to a multiple-access channel where one of the receivers decodes correctly both messages. However, [16, pp. 1354–1355] pointed out a gap in the proof of [4, Theorem 1], though it was conjectured that the main result holds. It therefore leads to the following conjecture:

Conjecture 1: For rate pairs (R_1, R_2) in the capacity region of a two-user GIC with arbitrary positive cross-link gains a_{12} and a_{21} and power constraints P_1 and P_2, let

$$C_1 \triangleq \frac{1}{2}\log(1+P_1) \tag{3}$$

$$C_2 \triangleq \frac{1}{2}\log(1+P_2) \tag{4}$$

be the capacities of the single-user AWGN channels (in the absence of interference), and let

$$R_1^* \triangleq \frac{1}{2}\log\left(1+\frac{a_{21}P_1}{1+P_2}\right) \tag{5}$$

$$R_2^* \triangleq \frac{1}{2}\log\left(1+\frac{a_{12}P_2}{1+P_1}\right). \tag{6}$$

Then, the following is conjectured to hold for achieving reliable communication at both receivers:

1) If $R_2 \geq C_2 - \varepsilon$, for an arbitrary $\varepsilon > 0$, then $R_1 \leq R_1^* + \delta_1(\varepsilon)$ where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.
2) If $R_1 \geq C_1 - \varepsilon$, then $R_2 \leq R_2^* + \delta_2(\varepsilon)$ where $\delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The discussion on Conjecture 1 is separated in the continuation to this introduction into GICs with mixed, strong or one-sided interference; it is done by restating some known results from [4], [11], [14], [16], [17], [18] and [19]. The focus of this paper is on GICs with weak interference (i.e., the channel model in (1) and (2) where $0 \leq a_{12}, a_{21} \leq 1$). For this class, the corner points of the capacity region are unknown yet, and they are studied in the converse part of this paper by relying on some existing outer bounds on the capacity region. Although these are not the tightest existing outer bounds on the capacity region of GICs, they provide informative and simple closed-form expressions that are proved to be asymptotically tight. The interested reader is referred to various existing outer bounds on the capacity region of GICs (see, e.g., [1], [3], [8], [9], [12], [14], [15], [17] and [19]–[21]).

A. On Conjecture 1 for a GIC with Mixed Interference

Conjecture 1 is considered in the following for a two-user GIC with mixed interference. Let $a_{12} \geq 1$ and $a_{21} < 1$ (i.e., it is assumed that there exists a strong interference between transmitter 2 and receiver 1, and a weak interference between transmitter 1 and receiver 2).

Proposition 1: Consider a two-user GIC with mixed interference where $a_{12} \geq 1$ and $a_{21} < 1$, and assume that transmitter 1 sends its message at rate $R_1 \geq C_1 - \varepsilon$ for an arbitrary $\varepsilon > 0$. Then, the following holds:

1) If $1 - a_{12} < (a_{12}a_{21} - 1)P_1$, then

$$R_2 \leq \frac{1}{2}\log\left(1+\frac{P_2}{1+a_{21}P_1}\right) + \varepsilon. \tag{7}$$

   This implies that the maximal rate R_2 is *strictly smaller* than the corresponding upper bound in Conjecture 1.
2) Otherwise, if $1 - a_{12} \geq (a_{12}a_{21} - 1)P_1$, then

$$R_2 \leq R_2^* + \varepsilon. \tag{8}$$

   This coincides with the upper bound in Conjecture 1.

The above two items refer to a corner point that achieves the sum-rate. On the other hand, if $R_2 \geq C_2 - \varepsilon$, then

$$R_1 \leq \frac{1}{2} \log\left(1 + \frac{P_1}{1+P_2}\right) + \delta(\varepsilon) \tag{9}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof: Eqs. (7) and (8) follow from [14, Theorem 10] or the earlier result in [12, Theorem 1]. Eq. (9) is a consequence of [12, Theorem 2]. ■

B. On Conjecture 1 for a GIC with Strong Interference

For strong interference, where $1 \leq a_{12} \leq 1 + P_1$ and $1 \leq a_{21} \leq 1 + P_2$, the capacity region of the GIC is known exactly: it is given by the intersection of the capacity regions of the two Gaussian multiple-access channels where each receiver decodes both messages (see [11, Theorem 5.2] and [18]). The resulting two corner points of this capacity region are consistent with Conjecture 1. Question 2 is answered in the affirmative for the case of strong interference because each receiver is able to decode the messages of both transmitters.

For very strong interference, where $a_{12} \geq 1 + P_1$ and $a_{21} \geq 1 + P_2$, the capacity region is not affected by the interference [2]. This is a trivial case where Conjecture 1 does not provide a tight upper bound on the maximal transmission rate (note that if $a_{12} > 1 + P_1$ and $a_{21} > 1 + P_2$, then $R_1^* > C_1$ and $R_2^* > C_2$).

C. On the Corner Points of a One-Sided GIC with Weak Interference

In [4], an interesting equivalence has been established between one-sided and degraded GICs: a one-sided GIC with power constraints P_1 and P_2, and cross-link gains $a_{12} = 0$ and $a_{21} = a \in (0, 1)$ in standard form has an identical capacity region to that of a degraded GIC whose standard form is given by

$$Y_1 = X_1 + \sqrt{\frac{1}{a}}\, X_2 + Z_1 \tag{10}$$

$$Y_2 = \sqrt{a} X_1 + X_2 + Z_2 \tag{11}$$

having the same power constraints (P_1 and P_2) on the inputs, and where $Z_1 Z_2$ are independent Gaussian random variables with zero mean and unit variance. The first part of Proposition 1 implies that one corner point of the one-sided GIC is determined exactly (the achievability part relies on considering the interference as noise), and the value of this corner point is

$$\left(\frac{1}{2}\, \log(1 + P_1),\, \frac{1}{2}\, \log\left(1 + \frac{P_2}{1 + aP_1}\right)\right). \tag{12}$$

In [16, Theorem 2], it is shown that this corner point achieves the sum-rate of the one-sided GIC. We proceed to a characterization of the second corner point of the capacity region: according to Proposition 1, the second corner point of the considered one-sided GIC is given by (R_1, C_2) where (see (9))

$$\frac{1}{2}\, \log\left(1 + \frac{aP_1}{1 + P_2}\right) \leq R_1 \leq \frac{1}{2}\, \log\left(1 + \frac{P_1}{1 + P_2}\right). \tag{13}$$

The lower bound on R_1 follows from the achievability of the point (R_1^*, C_2) for the degraded GIC in (10) and (11). The following statement summarizes this short discussion on the one-sided GIC with weak interference.

Proposition 2: Consider a two-user and one-sided GIC with weak interference, which in standard form has power constraints P_1 and P_2 for transmitters 1 and 2, respectively, and whose cross-link gains are $a_{12} = 0$ and $a_{21} = a$ for $0 < a < 1$. One of the two corner points of its capacity region is given in (12), and it achieves the sum-rate. The other corner point is (R_1, C_2) where R_1 satisfies the bounds in (13), and these bounds are tight when $a \to 1$.

The achievable rate region of Costa [5] for a one-sided GIC with weak interference coincides with the Han-Kobayashi achievable region for i.i.d. Gaussian codebooks (see [24, Section 2]). This region has a corner point (R_1, C_2) where R_1 is equal to the lower bound in (13). However, it remains unknown whether the capacity-achieving input distribution is Gaussian.

D. Organization of this paper

The structure of this paper is as follows: Conjecture 1 is considered in Section 2 for a two-user GIC with a two-sided weak interference. The excess rate for the sum-rate w.r.t. the corner points of the capacity region is considered in Section 3; the study of this measure is considered to be the main novelty of this paper. A summary is provided in Section 4 with some directions for further research.

2. On the Corner Points of the Capacity Region of a Two-User GIC with Weak Interference

This section considers Conjecture 1 for a two-user GIC with weak interference. It is easy to verify that the points $(R_1, R_2) = (C_1, R_2^*)$ and (R_1^*, C_2) are both included in the capacity region of a GIC with weak interference, and that the corresponding receiver of the transmitter that operates at its maximal rate (C_1 or C_2) can be designed to decode the messages of both transmitters. We show, for example, the inclusion of the point (C_1, R_2^*) in the capacity region. Assume that two Gaussian codebooks are used by transmitters 1 and 2, under the power constraints P_1 or P_2, respectively. Let $R_1 = C_1 - \varepsilon_1$, and $R_2 = R_2^* - \varepsilon_2$ for arbitrary small $\varepsilon_1, \varepsilon_2 > 0$. At the input of receiver 1, the power of the signal sent by transmitter 2 is $a_{12}P_2$, so since $R_2 < R_2^*$, it is possible for receiver 1 to first decode the message sent by transmitter 2 by treating the message sent by transmitter 1 and the additive Gaussian noise of its channel (Z_1) as a Gaussian noise with zero mean and variance $1 + P_1$. After successfully decoding the message sent by transmitter 2, receiver 1 is able to decode its own message by subtracting the interfering signal of transmitter 2 from the received signal, where the problem at this stage is reduced to a point-to-point communication over an AWGN channel (note that $R_1 < C_1$). Furthermore, receiver 2 is able to decode its message from transmitter 2 by treating the interference from transmitter 1 as an additive Gaussian noise; this is made possible because $0 \leq a_{12}, a_{21} \leq 1$, so from (6) we have $R_2^* < \frac{1}{2} \log\left(1 + \frac{P_2}{1+a_{21}P_1}\right)$. By swapping the indices, the point (R_1^*, C_2) is also shown to be achievable, with the same additional property that enables receiver 2 to decode both messages. To conclude, both points are achievable when the interference is weak, and the considered decoding strategy also enables one of these receivers to decode both messages.

We proceed in the following to the converse part, which leads to the following statement:

Theorem 1: Consider a two-user GIC with weak interference, and let C_1, C_2, R_1^* and R_2^* be as defined in (3)–(6), respectively. If $R_1 \geq C_1 - \varepsilon$ for an arbitrary $\varepsilon > 0$, then reliable communication requires that

$$R_2 \leq \min\left\{R_2^* + \frac{1}{2} \log\left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right) + 2\varepsilon, \quad \frac{1}{2} \log\left(1 + \frac{P_2}{1+P_1}\right) + \left(1 + \frac{1+P_1}{a_{21}P_2}\right)\varepsilon\right\}. \tag{14}$$

Similarly, if $R_2 \geq C_2 - \varepsilon$, then

$$R_1 \leq \min\left\{R_1^* + \frac{1}{2} \log\left(1 + \frac{P_1}{(1+a_{21}P_1)(1+a_{12}P_2)}\right) + 2\varepsilon, \quad \frac{1}{2} \log\left(1 + \frac{P_1}{1+P_2}\right) + \left(1 + \frac{1+P_2}{a_{12}P_1}\right)\varepsilon\right\}. \tag{15}$$

Consequently, the corner points of the capacity region are (R_1, C_2) and (C_1, R_2) where

$$R_1^* \leq R_1 \leq \min\left\{R_1^* + \frac{1}{2} \log\left(1 + \frac{P_1}{(1+a_{21}P_1)(1+a_{12}P_2)}\right), \ \frac{1}{2} \log\left(1 + \frac{P_1}{1+P_2}\right)\right\} \tag{16}$$

$$R_2^* \leq R_2 \leq \min\left\{R_2^* + \frac{1}{2} \log\left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right), \ \frac{1}{2} \log\left(1 + \frac{P_2}{1+P_1}\right)\right\}. \tag{17}$$

In the limit where P_1 and P_2 tend to infinity, which makes it an interference-limited channel,

1) Conjecture 1 holds, and it gives an asymptotically tight bound.
2) The rate pairs (C_1, R_2^*) and (R_1^*, C_2) form the corner points of the capacity region.
3) The answer to Question 2 is affirmative.

Proof: The proof of this theorem relies on the two outer bounds on the capacity region that are given in [9, Theorem 3] and [12, Theorem 2].

Suppose that $R_1 \geq C_1 - \varepsilon$ bits per channel use. The outer bound by Etkin *et al.* in [9, Theorem 3] (it is also known as the ETW bound) yields that the rates R_1 and R_2 satisfy the inequality constraint

$$2R_1 + R_2 \leq \frac{1}{2}\log\bigl(1 + P_1 + a_{12}P_2\bigr) + \frac{1}{2}\log\left(\frac{1+P_1}{1+a_{21}P_1}\right) + \frac{1}{2}\log\left(1 + a_{21}P_1 + \frac{P_2}{1+a_{12}P_2}\right)$$

which therefore implies that

$$\begin{aligned} R_2 \leq & \frac{1}{2}\log\bigl(1 + P_1 + a_{12}P_2\bigr) + \frac{1}{2}\log\left(\frac{1+P_1}{1+a_{21}P_1}\right) + \frac{1}{2}\log\left(1 + a_{21}P_1 + \frac{P_2}{1+a_{12}P_2}\right) - \bigl(\log(1+P_1) - 2\varepsilon\bigr) \\ = & \frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{1+P_1}\right) + \frac{1}{2}\log\left(\frac{1}{1+a_{21}P_1}\right) + \frac{1}{2}\log\left(1 + a_{21}P_1 + \frac{P_2}{1+a_{12}P_2}\right) + 2\varepsilon \\ = & R_2^* + \frac{1}{2}\log\left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right) + 2\varepsilon. \end{aligned} \tag{18}$$

The outer bound by Kramer in [12, Theorem 2], formulated here in an equivalent form, states that the capacity region is included in the set $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ where

$$\mathcal{K}_1 = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2}\log\left(1 + \frac{(1-\beta)P'}{\beta P' + \frac{1}{a_{21}}}\right) \\ 0 \leq R_2 \leq \frac{1}{2}\log(1+\beta P') \end{array} \right\} \tag{19}$$

with $P' \triangleq P_2 + \frac{P_1}{a_{12}}$ and $\beta \in \left[\frac{P_2}{(1+P_1)P'}, \frac{P_2}{P'}\right]$ is a free parameter; the set $\mathcal{K}_2$ is obtained by switching the indices in $\mathcal{K}_1$. From the boundary of the outer bound in (19), the value of β that satisfies the equality

$$\frac{1}{2}\log\left(1 + \frac{(1-\beta)P'}{\beta P' + \frac{1}{a_{21}}}\right) = C_1 - \varepsilon$$

is given by

$$\beta = \frac{2^{2\varepsilon}P_2 + \frac{(2^{2\varepsilon}-1)(1+P_1)}{a_{21}}}{(1+P_1)\left(P_2 + \frac{P_1}{a_{21}}\right)}.$$

The substitution of this value of β into the upper bound on R_2 in (19) implies that if $R_1 \geq C_1 - \varepsilon$ then

$$\begin{aligned} R_2 &\leq \frac{1}{2}\log(1+\beta P') \\ &= \frac{1}{2}\log\left(1 + \frac{2^{2\varepsilon}P_2 + \frac{(2^{2\varepsilon}-1)(1+P_1)}{a_{21}}}{1+P_1}\right) \\ &= \frac{1}{2}\log\left(1 + \frac{P_2}{1+P_1}\right) + \delta(\varepsilon) \end{aligned} \tag{20}$$

where

$$\delta(\varepsilon) = \frac{1}{2}\log\left(1 + \frac{(2^{2\varepsilon}-1)\left(P_2 + \frac{1+P_1}{a_{21}}\right)}{1+P_1+P_2}\right).$$

The function δ satisfies that $\delta(0) = 0$, and straightforward calculus shows that

$$0 < \delta'(c) < 1 + \frac{1+P_1}{a_{21}P_2}, \quad \forall\, c \geq 0.$$

It therefore follows (from the mean-value theorem of calculus) that

$$0 < \delta(\varepsilon) < \left(1 + \frac{1+P_1}{a_{21}P_2}\right)\varepsilon. \tag{21}$$

A combination of (18), (20), (21) gives the upper bound on the rate R_2 in (14). Similarly, if $R_2 \geq C_2 - \varepsilon$, the upper bound on the rate R_1 in (15) follows by switching the indices in (14).

From the inclusion of the points (C_1, R_2^*) and (R_1^*, C_2) in the capacity region, together with the bounds in (14) and (15) in the limit where $\varepsilon \to 0$, it follows that the corner points of the capacity region satisfy the bounds in (16) and (17).

Since the point (C_1, R_2^*) is achievable, also is (R_1, R_2^*) for $R_1 < C_1$; hence, if $R_1 \geq C_1 - \varepsilon$, then the maximal rate R_2 of transmitter 2 satisfies

$$R_2^* \leq R_2 \leq R_2^* + \frac{1}{2}\log\left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right) + 2\varepsilon.$$

The uncertainty in the maximal achievable rate R_2 when $R_1 \geq C_1 - \varepsilon$ and $\varepsilon \to 0$ is therefore upper bounded by $\frac{1}{2}\log\left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right)$. The asymptotic case where P_1 and P_2 tend to infinity and $\frac{P_2}{P_1} \to k$ for an arbitrary $k > 0$ is examined in the following. In this case, $R_2^* \to \frac{1}{2}\log(1 + ka_{12})$ and $\Delta R_2 \to 0$ which proves that Conjecture 1 holds in this asymptotic case. Since the points (C_1, R_2^*) and (R_1^*, C_2) are included in the capacity region, it follows from this converse that they form corner points of this region. Finally, as is explained above, operating at the corner points (C_1, R_2^*) and (R_1^*, C_2) enables receiver 1 or receiver 2, respectively, to decode both messages. This answers Question 2 in the affirmative in the asymptotic case where $P_1, P_2 \to \infty$. ■

Remark 1: Consider a two-user symmetric GIC where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0, 1)$. The corner points of the capacity region of this symmetric GIC are given by (C, R_c) and (R_c, C) where $C = \frac{1}{2}\log(1+P)$ is the capacity of a single-user AWGN channel with input power constraint P, and an additive Gaussian noise with zero mean and unit variance. Theorem 1 gives that

$$R_c \leq \min\left\{\frac{1}{2}\log\left(1 + \frac{aP}{1+P}\right) + \frac{1}{2}\log\left(1 + \frac{P}{(1+aP)^2}\right), \; \frac{1}{2}\log\left(1 + \frac{P}{1+P}\right)\right\}. \quad (22)$$

In the following, the we compare the two terms inside the minimization in (22) where the first term follows from the ETW bound in [9, Theorem 3], and the second term follows from Kramer's bound in [12, Theorem 2]. Straightforward algebra reveals that, for an arbitrary $a \in (0, 1)$, the first term gives a better bound on R_c if and only if

$$P > \frac{2a^2 - a + 1 + \sqrt{5a^2 - 2a + 1}}{2a^2(1-a)}. \quad (23)$$

Hence, for an arbitrary cross-link gain $a \in (0, 1)$, there exists a critical SNR where above this critical value, the ETW bound provides a tighter upper bound on the corner point than Kramer's bound, and below this critical value Kramer's bound is better in this respect. The critical value of P (in decibels) as a function of the cross-link gain $a \in (0, 1)$ is depicted in Figure 1, and it tends to infinity in the two extreme cases where $a \to 0$ or $a \to 1$; this implies that in these two extreme cases, Kramer's bound is better for all values of P. The latter observation is further addressed in the following:

1) If $a \to 0$ then, for every $P > 0$, the first term on the right-hand side of (22) tends to the capacity C; this forms a trivial upper bound on the value R_c of the corner point. On the other hand, the second term on the right-hand side of (22) gives the upper bound of $\frac{1}{2}\log\left(1 + \frac{P}{1+P}\right)$ which is smaller than C for all values of P. Note that the second term in (22) implies that, for a symmetric GIC, $R_c \leq \frac{1}{2}$ bit per channel use for all values of P. In fact, the advantage of the second term in the extreme case where $a \to 0$ for a given P served as the initial motivation for incorporating it in Theorem 1.
2) If $a \to 1$ then, for every $P > 0$, the first term tends to $\frac{1}{2}\log\left(1 + \frac{P}{1+P}\right) + \frac{1}{2}\log\left(1 + \frac{P}{(1+P)^2}\right)$ which is larger than the second term. Hence, also in this case, the second term gives a better bound for all values of P.

Example 1: The condition in (23) is consistent with [14, Figs. 10 and 11], as explained in the following:

1) According to [14, Figs. 10], for $P = 7$ and $a = 0.2$, Kramer's outer bound gives a better upper bound on the corner point than the ETW bound. For $a = 0.2$, the complementary of the condition in (23) implies that Kramer's bound is indeed better in this respect for $P < 27.48$.
2) According to [14, Figs. 11], for $P = 100$ and $a = 0.1$, the ETW is nearly as tight as Kramer's bound in providing an upper bound on the corner point. For $a = 0.1$, the complementary of the condition in (23) implies that Kramer's outer bound gives a better upper bound on the corner point than the ETW bound for $P < 100.36$; hence, for $P = 100$, there is only a slight improvement to Kramer's bound that cannot be

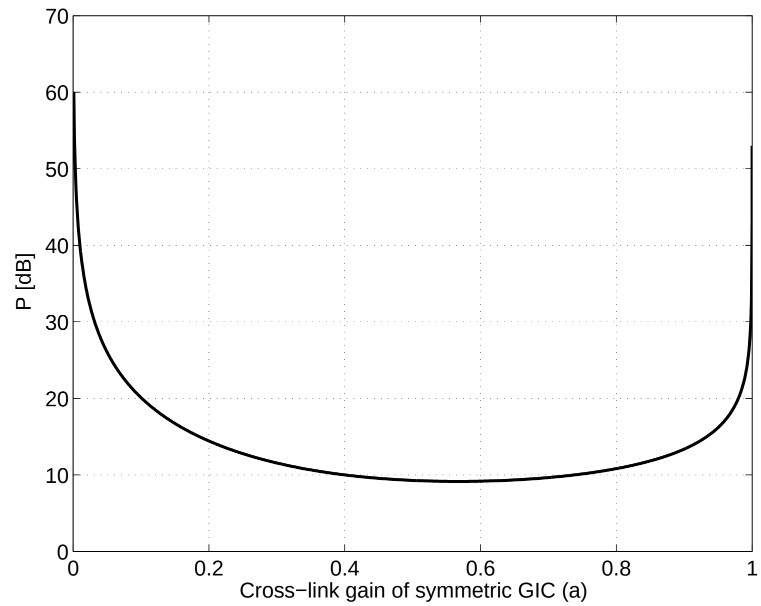


Fig. 1. The curve shows the critical value of P (in decibels) on the right-hand side of (23) as a function of the cross-link gain (a) for a two-user symmetric GIC with weak interference. For values of P above this curve, the ETW bound is better than Kramer's bound in providing an upper bound on the corner points of the capacity region; for values of P below this curve, Kramer's bound is better in this respect.

observed in the resolution of [14, Figs. 11]: Kramer's bound gives an upper bound on R_c that is equal to 0.4964 bits per channel use, and the ETW bound gives an upper bound of 0.5026 bits per channel use.

Remark 2: If $a_{12}a_{21}P_{1,2} \gg 1$, then it follows from (16) and (17) that the two corner points of the capacity region approximately coincide with the points (R_1^*, C_2) and (C_1, R_2^*) in Conjecture 1.

In the following example, we evaluate the bounds in Theorem 1 for finite values of transmitted powers (P_1 and P_2) to illustrate the asymptotic tightness of these bounds.

Example 2: Consider a two-user symmetric GIC with weak interference where $a = 0.5$ and $P = 100$. Assume that transmitter 1 sends its data at the maximal single-user rate of $C_1 = \frac{1}{2}\log(1+P) = 3.33$ bits per channel use. According to (17), the corresponding maximal rate R_2 of transmitter 2 is between 0.292 and 0.317 bits per channel use; the upper bound on R_2 in this case follows from the ETW bound. This gives high accuracy in the assessment of the two corner points of the capacity region (see Remark 2 where in this case $a^2P = 25 \gg 1$). If P is increased by 10 dB (to 1000), and transmitter 1 sends its data at rate $C_1 = \frac{1}{2}\log(1+P) = 5.0$ bits per channel use, then the corresponding maximal rate R_2 is between 0.292 and 0.295 bits per channel use. Hence, the precision of the assessment of the corner points is improved in the latter case. The improved accuracy of the latter assessment when the value of P is increased is consistent with Remark 2, and the asymptotic tightness of the bounds in Theorem 1 as P_1 and P_2 tend to infinity.

Figure 2 refers to a symmetric GIC where $P_1 = P_2 = 100$ and $a_{12} = a_{21} = 0.5$. The solid line in this figure corresponds to the boundary of the ETW outer bound on the capacity region (see [9, Theorem 3]) which is given (in units of bits per channel use) by

$$\mathcal{R}_{\text{o}} = \left\{ (R_1, R_2) : \begin{array}{c} 0 \le R_1 \le 3.3291 \\ 0 \le R_2 \le 3.3291 \\ R_1 + R_2 \le 4.1121 \\ 2R_1 + R_2 \le 6.9755 \\ R_1 + 2R_2 \le 6.9755 \end{array} \right\}. \tag{24}$$

The two circled points correspond to Conjecture 1; these points are achievable, and (as is verified numerically) they almost coincide with the boundary of the outer bound in [9, Theorem 3].

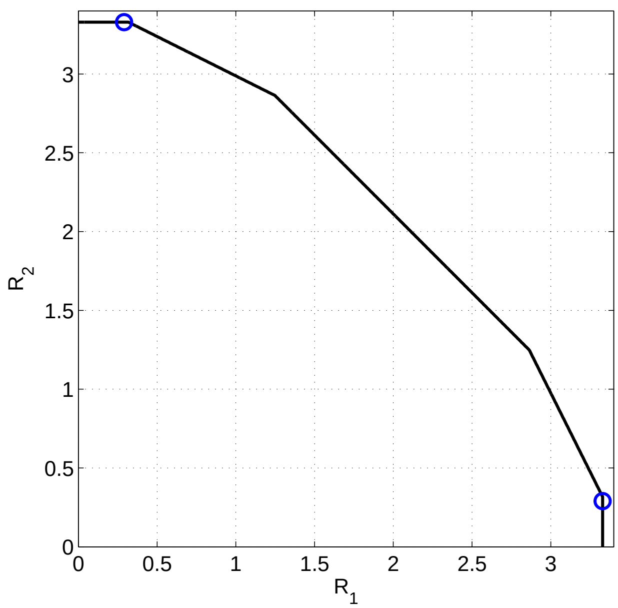


Fig. 2. The figure refers to a two-user symmetric GIC with cross-link gains $a_{12} = a_{21} = 0.5$ and a common transmitted power $P_1 = P_2 = 100$ in the standard form (see Example 2). The solid curve is the boundary of the outer bound in [9, Theorem 3] (the ETW bound in (24)), and the two circled points refer to Conjecture 1; these points are achievable, and they almost coincide with the boundary of the outer bound.

3. THE EXCESS RATE FOR THE SUM-RATE W.R.T. THE CORNER POINTS OF THE CAPACITY REGION

The sum-rate of a GIC is attained at one of the corner points of the capacity region for mixed, strong or one-sided interference, and this corner point is known exactly. This is in contrast to a GIC with two-sided weak interference whose sum-rate is not achieved at one of the corner points of its capacity region. It is therefore of interest to examine the excess rate for the sum-rate w.r.t. these corner points by measuring the gap (Δ) between the sum-rate (C_{sum}) and the maximal total rate $(R_1 + R_2)$ that is obtainable by a corner point of the capacity region:

$$\Delta \triangleq C_{\text{sum}} - \max\bigl\{R_1 + R_2 \colon (R_1, R_2) \text{ is a corner point}\bigr\}. \tag{25}$$

We therefore have $\Delta = 0$ for a GIC with mixed, strong or one-sided interference, so the focus in this section is on a GIC with two-sided weak interference. The purpose of this section is to derive bounds on Δ for the case of weak interference, and to provide a quantitative measure of the excess rate for the sum-rate w.r.t. the corner points. To this end, the bounds in Theorem 1 (see Section 2) as well as upper and lower bounds on the sum-rate are used to obtain upper and lower bounds on Δ; in general, bounds on the sum-rate are provided in [6], [9], [10], [19], [22] and [23]. The proposed study measures the excess rate for the sum-rate w.r.t. the scenario where one transmitter operates at its maximal possible single-user rate, and the other transmitter reduces its rate to the point where reliable communication is achievable.

A. An Upper Bound on the Excess Rate (Δ) for the Sum-Rate w.r.t. the Corner Points

For the derivation of an upper bound on Δ, we rely on an upper on the sum-rate and a lower bound on the maximal value of $R_1 + R_2$ that can be obtained by the two corner points of the capacity region of a GIC with weak interference. Since the points (R_1^*, C_2) and (C_1, R_2^*) are achievable, it follows that

$$\begin{aligned} &\max\bigl\{R_1 + R_2 \colon (R_1, R_2) \text{ is a corner point}\bigr\} \\ &\geq \max\{R_1^* + C_2, R_2^* + C_1\} \\ &= \frac{1}{2}\,\max\Bigl\{\log(1 + P_2 + a_{21} P_1), \log(1 + P_1 + a_{12} P_2)\Bigr\}. \end{aligned} \tag{26}$$

Furthermore, the outer bound on the capacity region of a GIC with weak interference in [9, Theorem 3] provides the following upper bound on the sum-rate:

$$C_{\text{sum}} \leq \frac{1}{2} \min\left\{\log(1+P_1) + \log\left(1 + \frac{P_2}{1+a_{21}P_1}\right),\ \log(1+P_2) + \log\left(1+\frac{P_1}{1+a_{12}P_2}\right),\right.$$
$$\left.\log\left(1 + a_{12}P_2 + \frac{P_1}{1+a_{21}P_1}\right) + \log\left(1 + a_{21}P_1 + \frac{P_2}{1+a_{12}P_2}\right)\right\}. \tag{27}$$

An upper bound on Δ follows from a combination of (25)–(27), which yields that

$$\Delta \leq \frac{1}{2}\left[\min\left\{\log(1+P_1) + \log\left(1 + \frac{P_2}{1+a_{21}P_1}\right),\ \log(1+P_2) + \log\left(1+\frac{P_1}{1+a_{12}P_2}\right),\right.\right.$$
$$\left.\log\left(1 + a_{12}P_2 + \frac{P_1}{1+a_{21}P_1}\right) + \log\left(1 + a_{21}P_1 + \frac{P_2}{1+a_{12}P_2}\right)\right\}$$
$$\left. - \max\Bigl\{\log(1+P_2+a_{21}P_1), \log(1+P_1+a_{12}P_2)\Bigr\}\right]. \tag{28}$$

For a symmetric GIC with weak interference, where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a$ $(0 < a < 1)$, the bound in (28) is simplified to

$$\begin{aligned}\Delta &= \Delta(P,a) \\ &\leq \frac{1}{2}\left[\min\left\{\log(1+P) + \log\left(1+\frac{P}{1+aP}\right),\ 2\log\left(1+aP+\frac{P}{1+aP}\right)\right\} - \log\bigl(1+(1+a)P\bigr)\right]\end{aligned} \tag{29}$$

and, consequently, in the limit where we let P tend to infinity,

$$\lim_{P\to\infty} \Delta(P,a) \leq \frac{1}{2}\log\left(\frac{1}{a}\right), \quad \forall\, a \in (0,1). \tag{30}$$

Note that, for $a = 1$, the capacity region is the polyhedron that is obtained by intersecting the capacity regions of two Gaussian multiple-access channels. There is therefore no excess rate for the sum-rate w.r.t. the corner points of this capacity region (i.e., $\Delta(P,1) = 0$); hence, the bound in (30) is tight and continuous at a left neighborhood of $a = 1$.

B. A Lower Bound on the Excess Rate (Δ) for the Sum-Rate w.r.t. the Corner Points

For the derivation of a lower bound on Δ, we rely on a lower bound on the sum-rate and an upper bound on the maximal value of $R_1 + R_2$ that can be obtained by the two corner points of the capacity region. From Theorem 1, it follows that for a two-user GIC with weak interference

$$\begin{aligned}&\max\bigl\{R_1+R_2\colon (R_1,R_2) \text{ is a corner point}\bigr\} \\ &\leq \min\left\{\max\left\{R_1^* + C_2 + \frac{1}{2}\log\left(1+\frac{P_1}{(1+a_{21}P_1)(1+a_{12}P_2)}\right),\right.\right. \\ &\qquad\left.\left. R_2^* + C_1 + \frac{1}{2}\log\left(1+\frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right)\right\},\ \frac{1}{2}\log(1+P_1+P_2)\right\} \\ &= \frac{1}{2}\min\left\{\max\left\{\log(1+P_2+a_{21}P_1) + \log\left(1+\frac{P_1}{(1+a_{21}P_1)(1+a_{12}P_2)}\right),\right.\right. \\ &\qquad\left.\left.\log(1+P_1+a_{12}P_2) + \log\left(1+\frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right)\right\},\ \log(1+P_1+P_2)\right\}.\end{aligned} \tag{31}$$

In order to get a lower bound on the sum-rate of the capacity region of a GIC with weak interference, we rely on [9] and [20]. The particularization of the outer bound in [20] for the GIC leads to the outer bound $\mathcal{R}_{\text{o}}$ in [8, Section 6.7.2] which is given by

$$\mathcal{R}_{\text{o}} = \left\{ (R_1, R_2) : \begin{array}{l} 0 \le R_1 \le \frac{1}{2}\log(1+P_1) \\ 0 \le R_2 \le \frac{1}{2}\log(1+P_2) \\ R_1 + R_2 \le \frac{1}{2}\log(1+P_1+a_{12}P_2) + \frac{1}{2}\log\left(1+\frac{P_2}{1+a_{12}P_2}\right) \\ R_1 + R_2 \le \frac{1}{2}\log(1+P_2+a_{21}P_1) + \frac{1}{2}\log\left(1+\frac{P_1}{1+a_{21}P_1}\right) \\ R_1 + R_2 \le \frac{1}{2}\log\left(1+a_{12}P_2+\frac{P_1}{1+a_{21}P_1}\right) \\ \qquad\qquad\quad +\frac{1}{2}\log\left(1+a_{21}P_1+\frac{P_2}{1+a_{12}P_2}\right) \\ 2R_1 + R_2 \le \frac{1}{2}\log(1+P_1+a_{12}P_2) + \frac{1}{2}\log\left(1+\frac{P_1}{1+a_{21}P_1}\right) \\ \qquad\qquad\quad +\frac{1}{2}\log\left(1+a_{21}P_1+\frac{P_2}{1+a_{12}P_2}\right) \\ R_1 + 2R_2 \le \frac{1}{2}\log(1+P_2+a_{21}P_1) + \frac{1}{2}\log\left(1+\frac{P_2}{1+a_{12}P_2}\right) \\ \qquad\qquad\quad +\frac{1}{2}\log\left(1+a_{12}P_2+\frac{P_1}{1+a_{21}P_1}\right) \end{array} \right\}. \tag{32}$$

Furthermore, if $(R_1, R_2) \in \mathcal{R}_{\text{o}}$ then $(R_1 - \frac{1}{2}, R_2 - \frac{1}{2}) \in \mathcal{R}_{\text{HK}}$ where $\mathcal{R}_{\text{HK}}$ denotes the Han-Kobayashi achievable region [11] (see [20, Remark 2] and also [8, Section 6.7.2]); the ”within one bit” result in [9] and [20] is per complex dimension, and here it is replaced by a half-bit per dimension since all the random variables involved in the calculations of the outer bound on the capacity region of a scalar GIC are real-valued [8, Theorem 6.6]. Consider the boundary of the outer bound $\mathcal{R}_{\text{o}}$ in (32). If one of the three constraints on $R_1 + R_2$ for this outer bound is active in (32) (this needs to be verified first), then a point on the boundary of $\mathcal{R}_{\text{o}}$ that is determined by one of these three constraints satisfies the equality

$$\begin{aligned} R_1 + R_2 = \frac{1}{2}\,\min\Bigg\{ & \log(1+P_1+a_{12}P_2) + \log\left(1+\frac{P_2}{1+a_{12}P_2}\right), \\ & \log(1+P_2+a_{21}P_1) + \log\left(1+\frac{P_1}{1+a_{21}P_1}\right), \\ & \log\left(1+a_{12}P_2+\frac{P_1}{1+a_{21}P_1}\right) + \log\left(1+a_{21}P_1+\frac{P_2}{1+a_{12}P_2}\right)\Bigg\}. \end{aligned} \tag{33}$$

Since $(R_1 - \frac{1}{2}, R_2 - \frac{1}{2})$ is a point that is included in the Han-Kobayashi achievable region, it follows that the maximal value of $R_1 + R_2$ over all the points of the achievable region $\mathcal{R}_{\text{HK}}$ is lower bounded by the value on the right-hand side of (33) minus 1. Consequently, under the above condition,

$$\begin{aligned} C_{\text{sum}} \ge \frac{1}{2}\,\min\Bigg\{ & \log(1+P_1+a_{12}P_2) + \log\left(1+\frac{P_2}{1+a_{12}P_2}\right), \\ & \log(1+P_2+a_{21}P_1) + \log\left(1+\frac{P_1}{1+a_{21}P_1}\right), \\ & \log\left(1+a_{12}P_2+\frac{P_1}{1+a_{21}P_1}\right) + \log\left(1+a_{21}P_1+\frac{P_2}{1+a_{12}P_2}\right)\Bigg\} - 1. \end{aligned} \tag{34}$$

From (25), (31) and (34), it follows that

$$\begin{aligned}\Delta \geq \frac{1}{2} \Bigg[\min\Bigg\{ & \log(1+P_1+a_{12}P_2) + \log\left(1+\frac{P_2}{1+a_{12}P_2}\right), \\ & \log(1+P_2+a_{21}P_1) + \log\left(1+\frac{P_1}{1+a_{21}P_1}\right), \\ & \log\left(1+a_{12}P_2+\frac{P_1}{1+a_{21}P_1}\right) + \log\left(1+a_{21}P_1+\frac{P_2}{1+a_{12}P_2}\right)\Bigg\} \\ - \min\Bigg\{ \max\Bigg\{ & \log(1+P_2+a_{21}P_1) + \log\left(1+\frac{P_1}{(1+a_{21}P_1)(1+a_{12}P_2)}\right), \\ & \log(1+P_1+a_{12}P_2) + \log\left(1+\frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)}\right)\Bigg\}, \\ & \log(1+P_1+P_2)\Bigg\}\Bigg] - 1 \end{aligned} \tag{35}$$

provided that the constraint on $R_1 + R_2$ that refers to the three inequalities in (32) is active; this needs to be verified as a condition for (35) to hold. In the following, the lower bound in (35) is particularized for a two-user symmetric GIC with weak interference, and a sufficient condition is stated for ensuring that (34) and (35) hold for this channel:

Lemma 1: For a two-user symmetric GIC with weak interference where $P_1 = P_2 = P \geq 2.550$ and $a_{12} = a_{21} = a \in (0,1)$, the constraint on $R_1 + R_2$ in the outer bound (32) is active.

Proof: Consider the straight lines that refer to the inequalities for $2R_1 + R_2$ and $R_1 + 2R_2$ in (32). For the symmetric GIC with weak interference ($a_{12} = a_{21} = a$ with $0 < a < 1$), it corresponds to the straight lines

$$\begin{aligned} 2R_1 + R_2 &= \frac{1}{2}\left[\log(1+P+aP) + \log\left(1+\frac{P}{1+aP}\right) + \log\left(1+aP+\frac{P}{1+aP}\right)\right], \\ R_1 + 2R_2 &= \frac{1}{2}\left[\log(1+P+aP) + \log\left(1+\frac{P}{1+aP}\right) + \log\left(1+aP+\frac{P}{1+aP}\right)\right] \end{aligned}$$

which intersect at a point (R_1, R_2) where $R_1 = R_2 \triangleq R$, and the corresponding value of $R_1 + R_2$ for this intersection point is equal to

$$R_1 + R_2 = \frac{1}{3}\left[\log(1+P+aP) + \log\left(1+\frac{P}{1+aP}\right) + \log\left(1+aP+\frac{P}{1+aP}\right)\right]. \tag{36}$$

The two inequalities for $R_1 + R_2$ in the outer bound (32), for a symmetric GIC, are given by

$$R_1 + R_2 \leq \frac{1}{2}\left[\log(1+P+aP) + \log\left(1+\frac{P}{1+aP}\right)\right], \tag{37}$$

$$R_1 + R_2 \leq \log\left(1+aP+\frac{P}{1+aP}\right). \tag{38}$$

The right-hand side of (36) is equal to the weighted average of the right-hand sides of (37) and (38) with weights $\frac{2}{3}$ and $\frac{1}{3}$, respectively; hence, it implies that one of the two inequalities for R_1+R_2 in (37) and (38) should be active in the determination of the boundary of the outer bound (32) if the above intersection point (R, R) also satisfies $R < \frac{1}{2}\log(1+P)$ for every $0 < a < 1$. To ensure the satisfiability of the latter condition, let

$$f_P(a) \triangleq \frac{1}{2}\log(1+P) - \frac{1}{6}\left[\log(1+P+aP) + \log\left(1+\frac{P}{1+aP}\right) + \log\left(1+aP+\frac{P}{1+aP}\right)\right], \quad \forall\, a \in [0,1] \tag{39}$$

where $P > 0$ is arbitrary; the satisfiability of the condition that $R < \frac{1}{2}\log(1+P)$ for every $0 < a < 1$ requires that f_P is non-negative over the interval [0,1]. The function f_P is concave over the interval [0,1] if and only if $P \geq 0.680$

(see the appendix for a proof), and also $f_P(0) = 0$. This implies that $f_P(a) \geq 0$ for all $a \in [0,1]$ if and only if $f_P(1) \geq 0$ and $P \geq 0.680$. Straightforward algebra shows that $f_P(1) \geq 0$ if and only if $P^4 + P^3 - 6P^2 - 7P - 2 \geq 0$, which is satisfied if and only if $P \geq 2.550$ (the other solutions of this inequality are infeasible for P). It therefore ensures that, if $P \geq 2.550$, then at least one of the inequalities for $R_1 + R_2$ in the outer bound (32) is active for a two-user symmetric GIC with weak interference; it is shown by proving that one of the two inequality constraints on $R_1 + R_2$ in (37) and (38) is relevant for the intersection point of the two straight lines that refer to the inequality constraints on $2R_1 + R_2$ and $2R_2 + R_1$ in (32), and this intersection point falls inside the square $[0, C] \times [0, C]$ with $C \triangleq \frac{1}{2} \log(1 + P)$. This completes the proof of the lemma. ■

Lemma 1 yields that the lower bound on the sum-rate in (34) is satisfied for a symmetric GIC with weak interference if $P \geq 2.550$. Consequently, also the lower bound on Δ in (35) holds for a symmetric GIC under the same condition on P. In this case, the lower bound in (35) is simplified to

$$\begin{aligned} \Delta &= \Delta(P,a) \\ &\geq \frac{1}{2} \Bigg[\min\left\{ \log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{1+aP}\right),\, 2\log\left(1+aP+\frac{P}{1+aP}\right) \right\} \\ &\quad - \min\left\{ \log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{(1+aP)^2}\right), \log(1+2P) \right\} \Bigg] - 1. \end{aligned} \tag{40}$$

In the following, we consider the limit of the lower bound on Δ in the asymptotic case where we let P tend to infinity. For large enough P

$$\begin{aligned} &\log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{1+aP}\right) \\ &\doteq \log\big((a+1)P\big) + \log\left(1+\frac{1}{a}\right) \\ &= \log\left(\frac{(a+1)^2 P}{a}\right), \end{aligned}$$

$$2\log\left(1+aP+\frac{P}{1+aP}\right) \doteq \log(a^2P^2)$$

so, we have

$$\min\left\{ \log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{1+aP}\right),\; 2\log\left(1+aP+\frac{P}{1+aP}\right) \right\} \doteq \log\left(\frac{(a+1)^2 P}{a}\right)$$

and, for $0 < a < 1$,

$$\min\left\{ \log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{(1+aP)^2}\right), \log(1+2P) \right\} \doteq \log\big((a+1)P\big).$$

Consequently, in the asymptotic case where P tends to infinity, it follows from (40) that

$$\lim_{P\to\infty} \Delta(P,a) \geq \frac{1}{2} \log\left(1+\frac{1}{a}\right) - 1, \quad \forall\, a \in (0,1). \tag{41}$$

For a symmetric GIC with weak interference, a comparison of the asymptotic upper and lower bounds on Δ in (30) and (41) yields that these two asymptotic bounds differ by at most 1 bit per channel use; this holds irrespectively of the cross-link gain $a \in (0,1)$. Note that the upper bound is tight for a close to 1, and also both asymptotic bounds scale like $\frac{1}{2} \log\left(\frac{1}{a}\right)$ for small value of a (so, they tend to infinity as $a \to 0$). This study provides a quantitative measure of the considered asymptotic excess rate (in the limit where $P \to \infty$).

C. An Analogous Measure to the Generalized Degrees of Freedom and its Implications

The following section is focused on the model of a two-user symmetric GIC, and it provides an asymptotic analysis of the excess rate for the sum-rate w.r.t. the corner points of its capacity region. The asymptotic analysis is analogous to the study of the generalized degrees of freedom (where the SNR and INR scalings are coupled), leading to asymptotic characterizations of this gap. The following analysis relies on the closed-form expressions for the upper and lower bounds on Δ derived in Sections 3-A and 3-B, leading to tight asymptotic characterizations in a certain range of this scaling.

Consider a two-user symmetric GIC whose cross-link gain a scales like $P^{\alpha-1}$ for some fixed value of $\alpha \geq 0$. For this GIC, the generalized degrees of freedom (GDOF) is defined as the asymptotic limit of the normalized sum-rate $\frac{C_{\text{sum}}(P, P^{\alpha-1})}{\log P}$ when $P \to \infty$. This GDOF refers to the case where the SNR (P) tends to infinity, and the interference to noise ratio $(\text{INR} = aP)$ scales such that $\frac{\log(\text{INR})}{\log(\text{SNR})} = \alpha$ is kept fixed for a non-negative α. The GDOF of a two-user symmetric GIC (without feedback) is defined as follows:

$$d(\alpha) \triangleq \lim_{P \to \infty} \frac{C_{\text{sum}}(P, P^{\alpha-1})}{\log P} \tag{42}$$

and this limit exists for every $\alpha \geq 0$ (see [9, Section 3.G]).

For large P, let us consider in an analogous way the asymptotic scaling of the normalized excess rate for the sum-rate w.r.t. the corner points of the capacity region. To this end, we study the asymptotic limit of the ratio $\frac{\Delta(P, P^{\alpha-1})}{\log P}$ for a fixed $\alpha \geq 0$ when P tends to infinity. Similarly to (42), the denominator of this ratio is equal to the asymptotic sum-rate of two parallel AWGN channels with no interference. However, in the latter expression, the excess rate for the sum-rate w.r.t. the corner points is replacing the sum-rate that appears in the numerator on the right-hand side of (42). Correspondingly, for an arbitrary $\alpha \geq 0$, let us define

$$\delta(\alpha) \triangleq \lim_{P \to \infty} \frac{\Delta(P, P^{\alpha-1})}{\log P}. \tag{43}$$

provided that this limit exists. In the following, we demonstrate the existence of this limit and provide a closed-form expression for δ.

Theorem 2: The limit in (43) is well defined for every $\alpha \geq 0$, and the function δ admits the following closed-form expression:

$$\delta(\alpha) = \begin{cases} \left|\frac{1}{2} - \alpha\right|, & \text{if } 0 \leq \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2}, & \text{if } \frac{2}{3} \leq \alpha < 1 \\ 0, & \text{if } \alpha \geq 1 \end{cases}. \tag{44}$$

Proof: The capacity region of a two-user GIC with strong interference is equal to the intersection of the capacity regions of the two underlying Gaussian multiple-access channels (see [11, Theorem 5.1]), and therefore its sum-rate is also equal to the total rate at each of the corner points of its capacity region. Consequently, if $\alpha \geq 1$ and $P \geq 1$ then the cross-link gain is $a = P^{\alpha-1} \geq 1$, which implies that the excess-rate for the sum-rate w.r.t. the corner points is zero. Hence, $\delta(\alpha) = 0$ for every $\alpha \geq 1$.

Consider a two-user symmetric GIC with weak interference. The bounds on the corner points of the capacity region in Theorem 1 (see (16) and (17)) imply that

$$\frac{1}{2} \log(1+P) \leq \max\bigl\{R_1 + R_2 \colon (R_1, R_2) \text{ is a corner point}\bigr\} \leq \frac{1}{2} \log(1+2P). \tag{45}$$

For an input power constraint $P \geq 1$ and an interference level $\alpha \in (0, 1)$, the cross-link gain is $a = P^{\alpha-1} < 1$; this corresponds to a two-user symmetric GIC with weak interference. From the definition of the excess-rate for the sum-rate w.r.t. the corner points, and the bounds on the corner points of the capacity region in (45), it follows that for $P \geq 1$ and $0 < \alpha < 1$

$$C_{\text{sum}}(P, P^{\alpha-1}) - \frac{1}{2} \log(1+2P) \leq \Delta(P, P^{\alpha-1}) \leq C_{\text{sum}}(P, P^{\alpha-1}) - \frac{1}{2} \log(1+P). \tag{46}$$

Consequently, for $\alpha \in (0,1)$, a division by $\log P$ of the three sides of the inequality in (46) and a calculation of the limit as $P \to \infty$ gives that (see (42) and (43))

$$\delta(\alpha) = d(\alpha) - \frac{1}{2}, \quad \forall\, \alpha \in (0,1). \tag{47}$$

The limit in (42) for the GDOF of a two-user symmetric GIC (without feedback) is well-defined, and it gets the following closed-form expression (see [9, Theorem 2]):

$$\begin{aligned} d(\alpha) &= \min\Big\{1, \max\left\{\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right\}, \max\left\{\alpha, 1-\alpha\right\}\Big\} \\ &= \begin{cases} 1-\alpha, & \text{if } 0 \le \alpha < \frac{1}{2} \\ \alpha, & \text{if } \frac{1}{2} \le \alpha < \frac{2}{3} \\ 1 - \frac{\alpha}{2}, & \text{if } \frac{2}{3} \le \alpha < 1 \\ \frac{\alpha}{2}, & \text{if } 1 \le \alpha < 2 \\ 1, & \text{if } \alpha \ge 2 \end{cases}. \end{aligned} \tag{48}$$

A combination of (47) and (48) completes the proof of the closed-form expression for δ in (44). ■

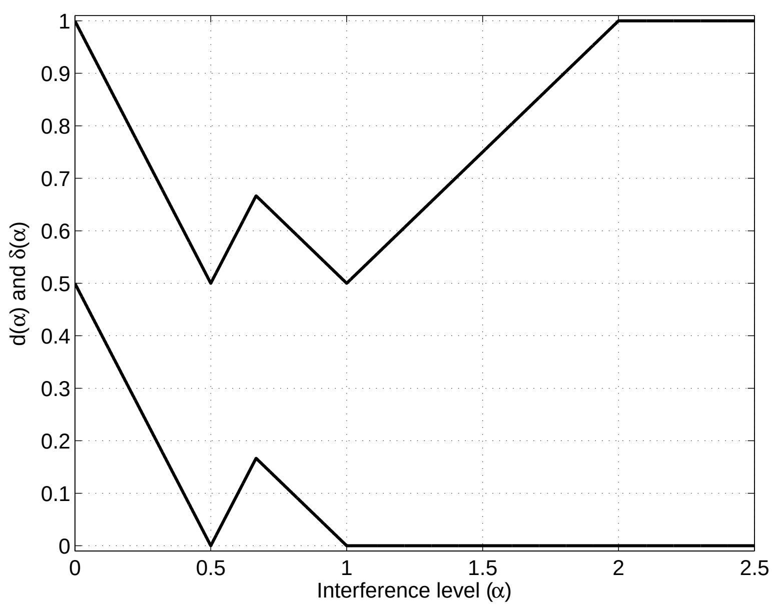


Fig. 3. A comparison of the generalized degrees of freedom (GDOF) in (48), and the exact asymptotic characterization of δ in (43) (see (44)) as a function of the non-negative interference level α.

Figure 3 provides a comparison of the GDOF with the function δ for an interference level α (i.e., the cross-link gain is $a = P^{\alpha-1}$). Equation (47) provides a connection between these two functions; the former is larger than the latter by half-bit per channel use (see Figure 3).

In light of the closed-form expression of δ in (44), the asymptotic tightness of the bounds in (29) and (40) is demonstrated in the following.

Theorem 3: Under the considered scaling of the SNR and INR for a two-user symmetric GIC with weak interference (where $\alpha \in (0,1)$), the bounds in (29) and (40), are asymptotically tight in the sense that the limit (as $P \to \infty$) of their normalization by $\log P$ coincides with $\delta(\alpha)$ in (44).

Proof: Substituting $a = P^{\alpha-1}$ into the bound in (29) gives that, for $P > 1$ and $\alpha \in [0,1)$,

$$\begin{aligned} &\Delta(P, P^{\alpha-1}) \\ &\le \frac{1}{2}\left[\min\Big\{\log(1+P) + \log\left(1 + \frac{P}{1+P^{\alpha}}\right),\ 2\log\left(1 + P^{\alpha} + \frac{P}{1+P^{\alpha}}\right)\Big\} - \log\big(1 + P + P^{\alpha}\big)\right] \\ &\triangleq \overline{\Delta}(P, P^{\alpha-1}). \end{aligned} \tag{49}$$

Consequently, one can verify that for large P and an arbitrary fixed value of $\alpha \in [0, 1)$,

$$\begin{aligned}
&\overline{\Delta}(P, P^{\alpha-1}) \\
&\doteq \frac{1}{2}\Big[\min\Big\{2-\alpha,\ 2\max\{\alpha, 1-\alpha\}\Big\} - 1\Big] \log P \\
&= \min\left\{\frac{1-\alpha}{2}, \left|\frac{1}{2}-\alpha\right|\right\} \log P.
\end{aligned}$$

Hence, it implies that by normalizing the bound in (49) by $\log P$ and taking its limit as $P \to \infty$ gives that for $\alpha \in (0, 1]$

$$\begin{aligned}
&\lim_{P\to\infty} \frac{\overline{\Delta}(P, P^{\alpha-1})}{\log P} \\
&= \max\left\{\min\left\{\frac{1-\alpha}{2}, \left|\frac{1}{2}-\alpha\right|\right\}, 0\right\} \\
&= \begin{cases} \left|\frac{1}{2}-\alpha\right|, & \text{if } 0 \le \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2}, & \text{if } \frac{2}{3} \le \alpha < 1 \end{cases} \\
&= \delta(\alpha).
\end{aligned}$$

This demonstrates the asymptotic tightness of the upper bound in (29) for the considered scaling of the SNR and INR. Analogously, the substitution of the equality $a = P^{\alpha-1}$ into the lower bound in (40) gives that for $P \geq 2.550$ and $\alpha \in (0, 1)$

$$\begin{aligned}
\Delta(P, P^{\alpha-1}) \geq \frac{1}{2}\Bigg[&\min\Bigg\{\log\big(1+P+P^{\alpha}\big) + \log\left(1+\frac{P}{1+P^{\alpha}}\right),\ 2\log\left(1+P^{\alpha}+\frac{P}{1+P^{\alpha}}\right)\Bigg\} \\
&- \min\left\{\log\big(1+P+P^{\alpha}\big) + \log\left(1+\frac{P}{(1+P^{\alpha})^2}\right), \log(1+2P)\right\}\Bigg] - 1 \\
\triangleq\ & \underline{\Delta}(P, P^{\alpha-1})
\end{aligned} \tag{50}$$

Consequently, for large P and $\alpha \in (0, 1)$,

$$\underline{\Delta}(P, P^{\alpha-1}) \doteq \frac{1}{2}\left[\min\Big\{2-\alpha,\ 2\max\{\alpha, 1-\alpha\}\Big\} - 1\right] \log P.$$

This implies that by normalizing the bound in (50) by $\log P$ and taking its limit as $P \to \infty$ gives that, for $\alpha \in [0, 1)$,

$$\begin{aligned}
&\lim_{P\to\infty} \frac{\underline{\Delta}(P, P^{\alpha-1})}{\log P} \\
&= \frac{1}{2}\left[\min\Big\{2-\alpha,\ 2\max\{\alpha, 1-\alpha\}\Big\} - 1\right] \\
&= \begin{cases} \left|\frac{1}{2}-\alpha\right|, & \text{if } 0 \le \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2}, & \text{if } \frac{2}{3} \le \alpha < 1 \end{cases} \\
&= \delta(\alpha).
\end{aligned}$$

This demonstrates the asymptotic tightness of the lower bound in (40) for the considered scaling, and it therefore completes the proof of the theorem. ■

Remark 3: The following remark provides an interpretation for the asymptotic tightness of the upper and lower bounds in (29) and (40), respectively, as is assured by Theorem 3.

The fact that these upper and lower bounds are asymptotically tight is first attributed to the ability of the ETW bound to give the exact asymptotic pre-factor of $\log P$ in the expression for the sum-rate (see [9, Theorem 2]). Furthermore, the asymptotic tightness of the bounds in (29) and (40) for the considered setting where $a = P^{\alpha-1}$ is also caused by the fact that, for a two-user symmetric GIC with weak interference, the achievable total rate at the two corner points taken into consideration bounded between $\frac{1}{2}\log(1+P)$ and $\frac{1}{2}\log(1+2P)$ (see (16) and (17)) which have the same pre-factor of $\log P$ (i.e., 1). One can verify that a removal of the second term in each minimization in (16) and (17) (i.e., by removing the effect of Kramer's bound while only keeping the effect of the ETW bound in the derivation of these upper and lower bounds) would have caused a significant asymptotic gap between the corresponding upper and lower bounds on Δ for $\alpha \in \left[0, \frac{1}{2}\right]$. More explicitly, the removal of this second term inside the minimization on the right-hand side of (40) (i.e., a removal of the term $\frac{1}{2}\log(1+2P)$) loosens the pre-factor of the resulting lower bound on Δ to zero, whereas the pre-factor of $\log P$ in the upper bound is $\frac{1}{2}-\alpha$; this gap is circumvented by using Kramer's bound for the derivation of the second term in the two minimizations that are used in (16) and (17).

As a consequence of asymptotic analysis in this sub-section, some implications are provided in the following:

1) From (42), (43), (44) and (48), it follows that for $\alpha \geq 0$

$$\lim_{P\to\infty} \frac{\Delta(P, P^{\alpha-1})}{C_{\text{sum}}(P, P^{\alpha-1})} = \frac{\delta(\alpha)}{d(\alpha)} = \begin{cases} \frac{1-2\alpha}{2(1-\alpha)}, & \text{if } 0 \leq \alpha < \frac{1}{2} \\ 1 - \frac{1}{2\alpha}, & \text{if } \frac{1}{2} \leq \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2-\alpha}, & \text{if } \frac{2}{3} \leq \alpha < 1 \\ 0, & \text{if } \alpha \geq 1 \end{cases} \tag{51}$$

is the asymptotic loss in the total rate, expressed as a fraction of the sum-rate, when the two users operate at one of the corner points of the capacity region; for the two-user symmetric GIC, it is assumed that the cross-link gain scales, for large P, like $P^{\alpha-1}$.

2) Analogously to the GDOF that refers to the asymptotic normalized limit of the sum-rate, the function δ is introduced in (43) by replacing the sum-rate with the excess rate for the sum-rate w.r.t. the corner points where it is assumed that $a = P^{\alpha-1}$ for some $\alpha \geq 0$. While the GDOF is known to be a non-monotonic function of α in the interval (0,1) (see [9, pp. 5542–5543] and (48)), it follows from (44), that also δ is a non-monotonic function over the interval $\left[\frac{1}{2}, 1\right]$. For $P > 1$, the cross-link gain $a = P^{\alpha-1}$ forms a monotonic increasing function of $\alpha \in [0,1)$, and it is a one-to-one mapping from the interval $(0,1)$ to itself. This implies that, for large P, the excess rate for the sum-rate w.r.t. the corner points is a non-monotonic function of $a \in (0,1)$, as is indeed supported by numerical results in Section 3-E.

3) From the closed-form expression for the GDOF of a two-user symmetric GIC (see (48)), if $a = P^{\alpha-1}$ for some $\alpha \in [0,1)$ then the worst case of interference for large P is obtained when $a \approx \frac{1}{\sqrt{P}}$ (i.e., if $\alpha \approx \frac{1}{2}$); for $\alpha = \frac{1}{2}$, the GDOF in (48) achieves its minimal value, and it is equal to

$$d\Big(\frac{1}{2}\Big) = \lim_{P\to\infty} \frac{C_{\text{sum}}\big(P, \frac{1}{\sqrt{P}}\big)}{\log P} = \frac{1}{2}. \tag{52}$$

On the other hand, from (51), it follows that

$$\lim_{P\to\infty} \frac{\Delta\big(P, \frac{1}{\sqrt{P}}\big)}{C_{\text{sum}}\big(P, \frac{1}{\sqrt{P}}\big)} = 0 \tag{53}$$

so, in the worst case of interference for large P, there is no harm in operating at one of the corner points.

4) The limit on the left-hand side of (51) is bounded between zero and one-half for $a = P^{\alpha-1}$ with $\alpha \geq 0$, and it gets a local maximal value at $\alpha = \frac{2}{3}$ (which is global maximum for $\alpha \geq \frac{1}{2}$). From (48) and (51) for this value of α, we have

$$\lim_{P\to\infty} \frac{C_{\text{sum}}\big(P, \frac{1}{\sqrt[3]{P}}\big)}{\log P} = \frac{2}{3}, \quad \lim_{P\to\infty} \frac{\Delta\big(P, \frac{1}{\sqrt[3]{P}}\big)}{C_{\text{sum}}\big(P, \frac{1}{\sqrt[3]{P}}\big)} = \frac{1}{4} \quad \Rightarrow \quad \lim_{P\to\infty} \frac{\Delta\big(P, \frac{1}{\sqrt[3]{P}}\big)}{\log P} = \frac{1}{6}. \tag{54}$$

5) The fact that $\delta(1) = 0$ is supported by the asymptotic upper and lower bounds on $\Delta(P,a)$ for large P and a fixed $a \in (0,1)$ (see (30) and (41)) which imply that

$$\frac{1}{2}\log\left(1+\frac{1}{a}\right) - 1 \leq \lim_{P\to\infty} \Delta(P,a) \leq \frac{1}{2}\log\left(\frac{1}{a}\right), \quad \forall\, a \in (0,1).$$

Since also $\Delta(P,a) = 0$ for every $a \geq 1$, it follows that for every fixed $a > 0$

$$\lim_{P\to\infty} \frac{\Delta(P,a)}{\log P} = 0.$$

6) The bounds on the excess rate for the sum-rate w.r.t. the corner points (see Sections 3-A and 3-B) refer to the case of an operation at one of the corner points of the capacity region of a two-user symmetric GIC. In this case, one of the users transmits its data at a rate that is equal to the single-user capacity of the respective AWGN channel. Consider now the case where the transmission rate of this user is within $\varepsilon > 0$ of the single-user capacity. Then, from Theorem 1, it follows that the upper bound on the transmission rate of the other user cannot increase by more than $\max\{2, 1+\frac{1+P_1}{a_{21}P_2}\}$. As a result of this, the lower bound on the excess rate for the sum-rate is decreased by no more than $\max\{2, 1+\frac{1+P_1}{a_{21}P_2}\}\,\varepsilon$. Furthermore, the upper bound on this excess rate can increase by no more than ε (note that if the first user reduces its transmission rate by no more than ε, then the other user can stay at the same transmission rate; overall, the total transmission rate it decreased by no more than ε, and consequently the excess rate for the sum-rate can increase by no more than ε). Revisiting the analysis in this sub-section by introducing a positive ε to the calculations, before taking the limit of P to infinity, leads to the conclusion that the corresponding characterization of δ in (44) stays un-affected as long as $\frac{\varepsilon}{\log P}$ tends to zero as we let P tend to infinity. For example, if $\varepsilon \triangleq \epsilon(P)$ scales like $(\log P)^{\beta}$ for some $\beta \in (0,1)$ then $\varepsilon(P) \to \infty$ but $\frac{\varepsilon(P)}{\log P} \to 0$ in the limit where $P \to \infty$, so the corresponding bounds in this subsection are not affected by ε in this case.

Consider a two-user symmetric GIC with weak interference where, in standard form, $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0,1)$. Let Δ denote the excess rate for the sum-rate w.r.t. the corner points of the capacity region, as it is defined in (25). The following summarizes the results that are introduced in this section so far for this channel model:

- The excess rate Δ satisfies the upper bound in (29).
- If $P \geq 2.550$, then it also satisfies the lower bound in (40).
- For large enough P, $\Delta = \Delta(P, a)$ is a non-monotonic function of a over the interval $(0, 1]$.
- The asymptotic pre-factor of $\log P$ in $\Delta(P, P^{\alpha-1})$ for $\alpha \geq 0$ is given by $\delta(\alpha)$ in (44). Furthermore, a connection between the function δ and the symmetric GDOF is given in (47) (see Fig. 3).
- Both upper and lower bounds on $\Delta(P, P^{\alpha-1})$ yield the exact pre-factor of $\log P$ for large P and for every non-negative α.
- In the asymptotic case of an interference-limited channel (i.e., when $P \to \infty$), and when the cross-link gain is kept fixed between 0 and 1, the excess rate Δ satisfies the upper and lower bounds in (30) and (41), respectively. These asymptotic bounds on Δ scale like $\frac{1}{2}\log\left(\frac{1}{a}\right)$, and they differ by at most 1 bit per channel use, irrespectively of the value of the $a \in (0,1]$ which is assumed not to scale with P.
- Let $a = P^{\alpha-1}$ for some $\alpha \geq 0$ and $P > 1$. Consider the loss in the total rate, expressed as a fraction of the sum-rate, when the users operate at one of the corner points of the capacity region. Then, this asymptotic normalized loss is provided in (51), and it is bounded between 0 and $\frac{1}{2}$. For large enough P, it roughly changes from 0 to $\frac{1}{4}$ by letting a grow (only slightly) from $\frac{1}{\sqrt{P}}$ to $\frac{1}{\sqrt[3]{P}}$.

The following remark refers to the third item above.

Remark 4: The excess rate for the sum-rate w.r.t. the corner points is the difference between the sum-rate and the maximal total achievable rate by any corner point. According to Theorem 1, for large P and two-sided weak interference, the total achievable rate at a corner point is an increasing function of $a \in (0,1]$. Although it is known that, for large P, the sum-rate of the capacity region is not monotonic decreasing in a, a priori, there was a possibility that by subtracting from it a monotonic increasing function in a, the difference (which is the excess rate) will be monotonic decreasing in a. However, it is shown not to be the case, so the fact that for large P, the excess rate Δ is not a monotonic decreasing function of a is a stronger property than the non-monotonicity of the sum-rate.

D. A Tightening of the Bounds on the Excess Rate (Δ) for Symmetric GICs with Weak Interference

In Sections 3-A and 3-B, closed-form expressions for upper and lower bounds on Δ are derived for GICs with weak interference. These expressions are used in Section 3-C for an asymptotic analysis in the case where $P \to \infty$. In the following, the bounds on the excess rate Δ are improved for finite P at the cost of introducing bounds that are subject to numerical optimizations. For simplicity, we focus on symmetric GICs with weak interference. In light of Theorem 3, a use of improved bounds does not imply any asymptotic improvement as compared to the bounds in Section 3-C that are expressed in closed form. Nevertheless, the new bounds are improved for finite SNR and INR, as is illustrated in Section 3-E.

1) An improved lower bound on Δ*:* An improvement of the lower bound on the excess rate for the sum-rate w.r.t. the corner points (Δ) is obtained by relying on an improved lower bound on the sum-rate (as compared to (34)). For tightening the lower bound on the sum-rate, it is suggested to combine (34) with the lower bound in [16, Eq. (32)] (it follows from the Han-Kobayashi achievable region, see [16, Table 1]):

$$C_{\text{sum}} \geq \max_{\alpha,\beta,\delta} \rho(P,a,\alpha,\beta,\delta) \tag{55}$$

where

$$\begin{aligned}
\rho(P,a,\alpha,\beta,\delta) \triangleq\, & \delta \log\left(1+\frac{2\alpha\delta P}{1+2a\beta\delta P}\right) + \delta \log\left(1+\frac{2\beta\delta P}{1+2a\alpha\delta P}\right) \\
& + \left(\frac{1-2\delta}{2}\right) \log\bigl(1+2(1+2\delta)P\bigr) \\
& + \min\left\{\frac{\delta}{2}\cdot\log\left(1+\frac{2\overline{\alpha}\delta P + 2a\overline{\beta}\delta P}{1+2\alpha\delta P+2a\beta\delta P}\right) + \frac{\delta}{2}\cdot\log\left(1+\frac{2\overline{\beta}\delta P + 2a\overline{\alpha}\delta P}{1+2\beta\delta P+2a\alpha\delta P}\right),\right. \\
& \qquad \delta \log\left(1+\frac{2\overline{\alpha}\delta P}{1+2\alpha\delta P+2a\beta\delta P}\right) + \delta \log\left(1+\frac{2\overline{\beta}\delta P}{1+2\beta\delta P+2a\alpha\delta P}\right), \\
& \qquad \left.\delta \log\left(1+\frac{2a\overline{\alpha}\delta P}{1+2a\alpha\delta P+2\beta\delta P}\right) + \delta \log\left(1+\frac{2a\overline{\beta}\delta P}{1+2\alpha\delta P+2a\beta\delta P}\right)\right\} \\
& \forall\,(\alpha,\beta,\delta) \quad \text{s.t.} \quad 0\leq\alpha\leq 1, \quad 0\leq\beta\leq 1, \quad 0\leq\delta\leq\frac{1}{2}\,.
\end{aligned}$$

A combination of (25), (31) and (55) gives the following lower bound on Δ for symmetric GICs with weak interference:

$$\Delta = \Delta(P,a) \geq \max_{\alpha,\beta,\delta}\Bigl\{\rho(P,a,\alpha,\beta,\delta)\Bigr\} - \frac{1}{2}\,\min\left\{\log\bigl(1+(a+1)P\bigr) + \log\left(1+\frac{P}{(1+aP)^2}\right),\, \log(1+2P)\right\}.$$

Furthermore, it follows from Lemma 1 that if $P \geq 2.550$, a combination of (25) and (31) with the two lower bounds on the sum-rate in (34) and (55) gives the following tightened lower bound on Δ (as compared to (40)):

$$\begin{aligned}
\Delta \geq \max\Biggl\{ & \max_{\alpha,\beta,\delta}\Bigl\{\rho(P,a,\alpha,\beta,\delta)\Bigr\}, \\
& \frac{1}{2}\,\min\left\{\log\bigl(1+(a+1)P\bigr)+\log\left(1+\frac{P}{1+aP}\right),\, 2\log\left(1+aP+\frac{P}{1+aP}\right)\right\} - 1\Biggr\} \\
& - \frac{1}{2}\,\min\left\{\log\bigl(1+(a+1)P\bigr)+\log\left(1+\frac{P}{(1+aP)^2}\right),\, \log(1+2P)\right\}.
\end{aligned} \tag{56}$$

2) An improved upper bound on Δ*:* An improvement of the upper bound on the excess rate for the sum-rate w.r.t. the corner points (Δ) is obtained by relying on an improved upper bound on the sum-rate (as compared to (27)). This is obtained by taking the minimum between Etkin's bound in [10] and Kramer's bound in [12, Theorem 2]. Following the discussion in [10], Etkin's bound outperforms the upper bounds on the sum-rate in [1], [9], [14], [19]; nevertheless, for values of a that are close to 1, Kramer's bound in [12, Theorem 2] outperforms the other known bounds on the sum-rate (see [10, Fig. 1]). Consequently, the minimum between Etkin's and Kramer's bounds in [10] and [12, Theorem 2] is calculated as an upper bound on the sum-rate. Combining [10, Eqs. (14)-(16)] (while adapting notation, and dividing the bound by 2 for a real-valued GIC), the simplified version of Etkin's upper bound on the sum-rate for symmetric real-valued GICs with weak interference gets the form

$$C_{\text{sum}} \leq \min_{\alpha,\sigma,\rho} \left\{ \min\left\{ \frac{1}{2} \log\left(1 + \frac{P(1+\alpha^2)\gamma}{(1-\rho^2)\sigma^2}\right),\ \log\left(1 + \frac{\alpha^2 P\gamma}{(1-\rho^2)\sigma^2}\right) \right\} + \log\left(\frac{\big(1+P(1+a)\big)\big(P(1+\alpha^2)+\sigma^2\big) - \big(P(1+\alpha\sqrt{a})+\rho\sigma\big)^2}{P(1+\alpha^2)\gamma + (1-\rho^2)\sigma^2} \right) \right\} \tag{57}$$

where

$$\gamma = \alpha^2 - 2\alpha\rho\sigma\sqrt{a} + \sigma^2 a, \quad \rho = \alpha\sigma\sqrt{a} \pm \sqrt{(1-\alpha^2)(1-\sigma^2 a)}, \quad \sigma \in \Big[0, \frac{1}{\sqrt{a}}\Big], \quad \alpha \in [-1,1]. \tag{58}$$

The two possible values of ρ in (58) need to be checked in the optimization of the parameters. For symmetric GICs with weak interference, Kramer's upper bound on the sum-rate (see [12, Eqs. (44) and (45)]) is simplified to

$$C_{\text{sum}} \leq \frac{1}{2} \log\left(1 + 2P + \frac{B}{2} - \frac{1}{2}\sqrt{B^2 - 4P^2\left(\frac{1}{a} - 1\right)^2}\right) \tag{59}$$

where $B = \frac{1}{a^2} + 2P\left(\frac{1}{a} - 1\right) - 1$. An improvement of the upper bound on the sum-rate in (27) follows by taking the minimal value of the bounds in (57) and (59); consequently, a combination of (25) and (26) with this improved upper bound on the sum-rate provides an improved upper bound on Δ (as compared to the bound in (29)).

3) A simplification of the improved upper bound on Δ for a sub-class of symmetric GICs with weak interference: The following simplifies the improved upper bound on the excess rate (Δ) for a sub-class of symmetric GICs with weak interference. It has been independently demonstrated in [1], [14] and [19] that if

$$0 < a < \frac{1}{4}, \quad 0 < P \leq \frac{\sqrt{a} - 2a}{2a^2} \tag{60}$$

then the sum-rate of the corresponding symmetric GIC is equal to

$$C_{\text{sum}} = \log\left(1 + \frac{P}{1+aP}\right). \tag{61}$$

This sum-rate is achievable by using single-user Gaussian codebooks, and treating the interference as noise. Under the conditions in (60), the exact sum-rate coincides with the upper bound given in (57). Hence, a replacement of the upper bound on the sum-rate in (27) with the exact sum-rate in (61), followed by a combination of (25) and (26) gives that

$$\Delta \leq \frac{1}{2} \log\left(\frac{1}{1+aP} + \frac{P}{(1+aP)^2}\right). \tag{62}$$

It is easy to verify that, under the conditions in (60), the upper bound on Δ in (62) is positive.

E. Numerical Results

The following section presents numerical results for the bounds on the excess rate for the sum-rate w.r.t. the corner points (denoted by Δ) while focusing on two-user symmetric GICs with weak interference.

Figure 4 compares upper and lower bounds on Δ as a function of the cross-link gain for a two-user symmetric GIC with weak interference. The upper and lower plots of this figure correspond to $P = 50$ and $P = 500$, respectively. The upper and lower bounds on Δ rely on (29) and (40), respectively, and the improved upper and lower bounds on Δ are based on Section 3-D. For $P = 50$ (see the upper plot of Figure 4), the advantage of the improved bounds on Δ is exemplified; the lower bound on Δ for the case where $P = 50$ is almost useless (it is zero unless the interference is very weak). The improved upper and lower bounds on Δ for $P = 50$ do not enable to conclude whether the function of Δ is monotonic decreasing as a function of a (for weak interference where $a \in [0,1]$). For $P = 500$ (see the lower plot of Figure 4), the improved bounds on Δ indicate that it is not a monotonic decreasing function of a (since the improved upper bound on Δ at $a = 0.045$ is equal to 0.578 bits per channel use, and its improved lower bound at $a = 0.110$ is equal to 0.620 bits per channel use). The observation that, for large P, the function of Δ is not monotonic decreasing in $a \in (0,1)$ is supported by the asymptotic

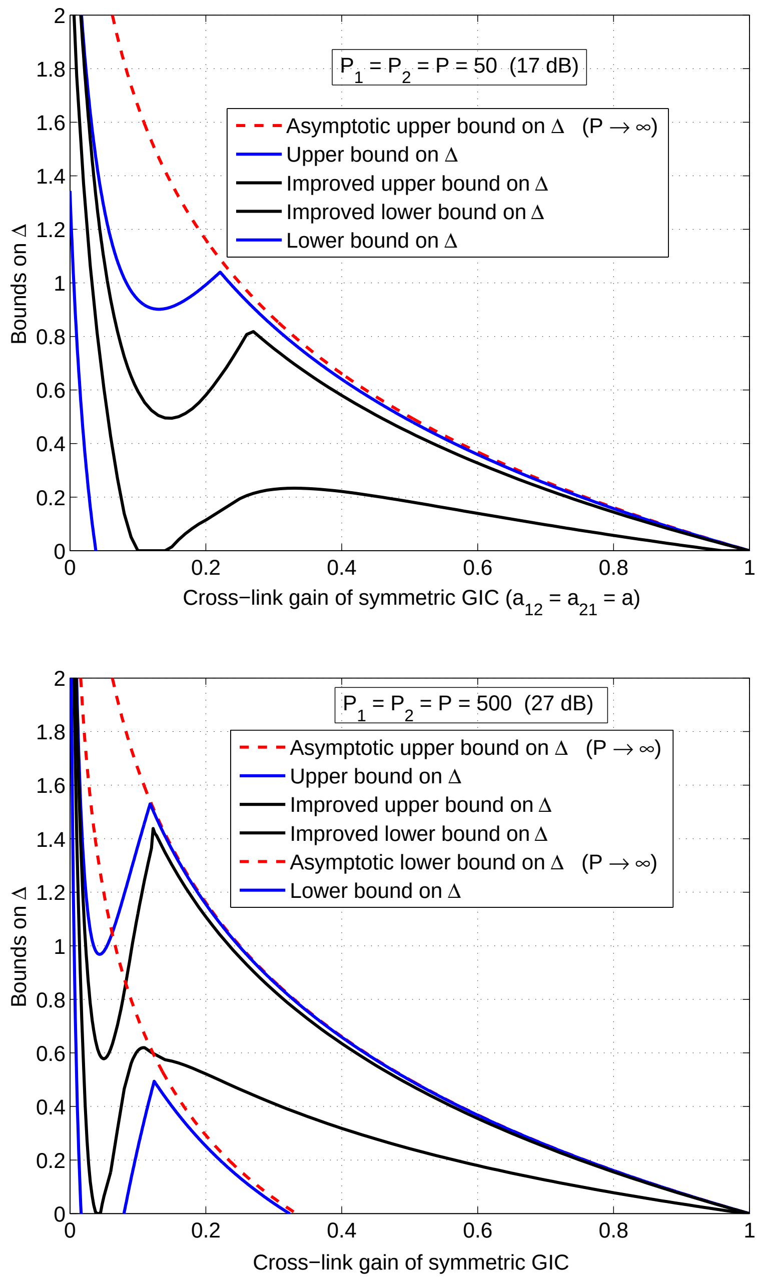
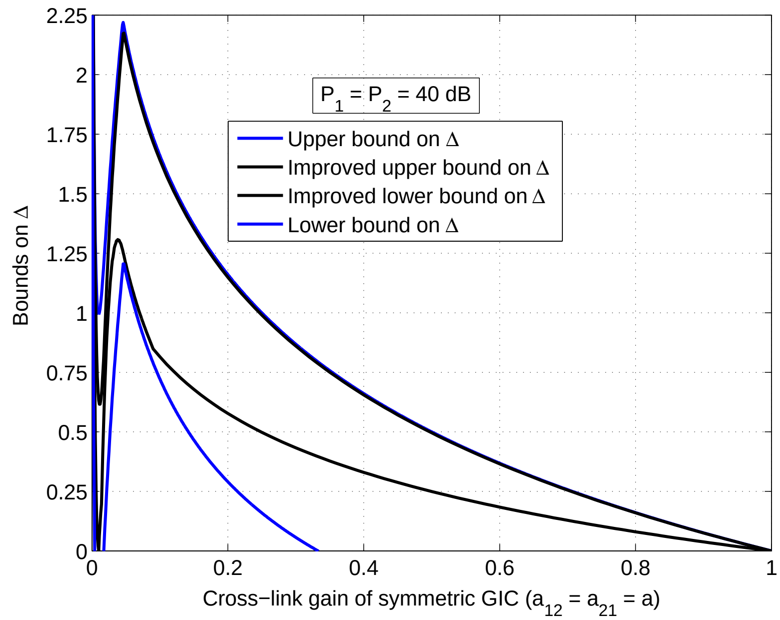


Fig. 4. Upper and lower bounds on the excess rate for the sum-rate w.r.t. the corner points (Δ) as a function of the cross-link gain (a). The plots refer to a two-user symmetric GIC with weak interference where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in [0, 1]$ in standard form. The upper and lower plots correspond to $P = 50$ and $P = 500$, respectively. The upper and lower bounds on Δ rely on (29) and (40), respectively, and the improved bounds on Δ rely on Section 3-D. The dashed lines refer to the asymptotic upper and lower bounds on Δ in (30) and (41), respectively.

analysis in Section 3-C. This conclusion is stronger than the observation that, for large enough P, the sum-rate is not a monotonic decreasing function of $a \in [0, 1]$ (see [9, pp. 5542–5543]), as it is discussed in Remark 4 (see Section 3-C). Figures 4 and 5 show that the phenomenon of the non-monotonicity of Δ as a function of a is more dominant when the value of P is increased. These figures also illustrate the advantage of the improved upper and lower bounds on Δ in Section 3-D as compared to the simple bounds on Δ in (29) and (40) (however, note that the simple bounds on Δ that are given in closed-form expressions are asymptotically tight, as is demonstrated in Theorem 3).



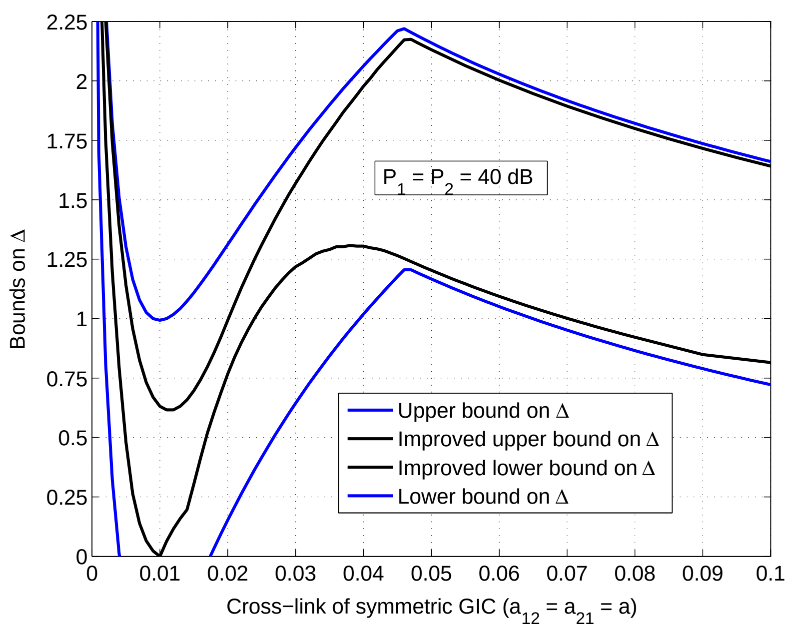


Fig. 5. Upper and lower bounds on the excess rate for the sum-rate w.r.t. the corner points (Δ) as a function of the cross-link gain (a). This figure refers to a two-user symmetric GIC with weak interference where $P_1 = P_2 = P = 40$ dB and $a_{12} = a_{21} = a \in [0, 1]$ in standard form. The upper and lower bounds on Δ are given in (29) and (40), respectively, and the improved bounds on Δ rely on Section 3-D. The upper plot shows upper and lower bounds on Δ over the range of weak interference $(0 \leq a \leq 1)$, and the lower plot zooms in the upper plot for $a \in [0, 0.1]$; it shows that Δ is a non-monotonic function of a in the weak interference regime.

Table I compares the asymptotic approximation of Δ with its improved upper bound in Section 3-D2. It verifies that, for large P, the minimal value of Δ is obtained at $a \approx \frac{1}{\sqrt{P}}$; it also verifies that, for large P, the maximal value of Δ for $a \geq \frac{1}{\sqrt{P}}$ is obtained at $a \approx \frac{1}{\sqrt[3]{P}}$. Table I also supports the asymptotic limits in (53) and (54), showing how close are the numerical results for large P to the corresponding asymptotic limits: specifically, for large P, at $a = \frac{1}{\sqrt{P}}$ and $\frac{1}{\sqrt[3]{P}}$, the ratio $\frac{\Delta}{\log P}$ tends to zero or $\frac{1}{6}$, respectively; this is supported by the numerical results in the 5th and 9th columns of Table I. The asymptotic approximations shown in Table I are consistent with the overshoots observed in the plots of Δ when the cross-link gain a varies between $\frac{1}{\sqrt{P}}$ and $\frac{1}{\sqrt[3]{P}}$; this interval is

TABLE I
COMPARISON OF THE ASYMPTOTIC APPROXIMATION OF THE EXCESS RATE FOR THE SUM-RATE W.R.T. THE CORNER POINTS (Δ) WITH ITS IMPROVED UPPER BOUND ON Δ IN SECTION 3-D2.

Power constraint in standard form	Value of a achieving minimum of Δ		Normalized Δ by $\log P$		Value of a achieving maximum of Δ for $a \geq \frac{1}{\sqrt{P}}$		Normalized Δ by $\log P$	
	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value
(P)	$\left(a = \frac{1}{\sqrt{P}}\right)$		Eq. (53)		$\left(a = \frac{1}{\sqrt[3]{P}}\right)$		Eq. (54)	
27 dB	0.045	0.050	0	0.065	0.126	0.140	0.167	0.154
40 dB	0.010	0.011	0	0.046	0.046	0.042	0.167	0.164
60 dB	0.001	0.001	0	0.032	0.010	0.010	0.167	0.166

narrowed as the value of P is increased (see Figures 4 and 5). Finally, it is also shown in Figures 4 and 5 that the curves of the upper and lower bounds on Δ, as a function of the cross-link gain a, do not converge uniformly to their asymptotic upper and lower bounds in (30) and (41), respectively. This non-uniform convergence is noticed by the large deviation of the bounds for finite P from the asymptotic bounds where this deviation takes place over an interval of small values of a; however, this interval of a shrinks when the value of P is increased, and its length is approximately $\frac{1}{\sqrt[3]{P}}$ for large P. This conclusion is consistent with the asymptotic analysis in Section 3-C (see the items that correspond to Eqs. (51) and (54)), and it is also supported by the numerical results in Table I.

4. SUMMARY AND OUTLOOK

This paper considers the corner points of the capacity region of a two-user Gaussian interference channel (GIC). The operational meaning of the corner points is a study of the situation where one user sends its information at the single-user capacity (in the absence of interference), and the other user transmits its data at the largest rate for which reliable communication is possible at the two non-cooperating receivers. The approach used in this work for the study of the corner points relies on some existing outer bounds on the capacity region of a two-user GIC.

In contrast to GICs with strong, mixed or one-sided interference, both corner points of a two-user GIC with weak interference have not been determined yet. This paper is focused on the latter model that refers to a two-user GIC in standard form whose cross-link gains are positive and below 1. Theorem 1 provides rigorous bounds on the corner points of the capacity region, whose tightness is especially pronounced for high SNR and INR.

The sum-rate of a GIC with either strong, mixed or one-sided interference is attained at one of the corner points of the capacity region, and this corner point is known exactly (see [11], [14], [16], [18] and [19]). This is in contrast to a GIC with weak interference whose sum-rate is not attained at any of the corner points of its capacity region. This motivates the study in Section 3 which introduces and analyzes the *excess rate for the sum-rate w.r.t. the corner points*. This measure, denoted by Δ, is defined to be the gap between the sum-rate and the maximal total rate obtained by the two corner points of the capacity region. Simple upper and lower bounds on Δ are derived in Section 3, which are expressed in closed form, and the asymptotic characterization of these bounds is analyzed. In the asymptotic case where the channel is interference limited (i.e., $P \to \infty$) and symmetric, the corresponding upper and lower bounds on Δ differ by at most 1 bit per channel use (irrespectively of the value of the cross-link gain a); in this case, both asymptotic bounds on Δ scale like $\frac{1}{2} \log\left(\frac{1}{a}\right)$ for small a.

Analogously to the study of the generalized degrees of freedom (GDOF), an asymptotic characterization of Δ is provided in this paper. More explicitly, under the setting where the SNR and INR scalings are coupled such that $\frac{\log(\text{SNR})}{\log(\text{INR})} = \alpha$ for an arbitrary non-negative α, the exact asymptotic characterization of Δ is provided in Theorem 2. Interestingly, the upper and lower bounds on Δ are demonstrated to be *asymptotically tight for the whole range of this scaling* (see Theorem 3).

For high SNR, the non-monotonicity of Δ as a function of the cross-link gain follows from the asymptotic analysis, and it is shown to be a stronger result than the non-monotonicity of the sum-rate in [9, Section 3].

Improved upper and lower bounds on Δ are introduced for finite SNR and INR, and numerical results of these bounds are exemplified. The numerical results in Section 3-E verify the effectiveness of the approximations for high SNR that follow from the asymptotic analysis of Δ.

This paper supports in general Conjecture 1 whose interpretation is that if one user transmits at its single-user capacity, then the other user should decrease its rate such that both decoders can reliably decode its message.

We list in the following some directions for further research that are currently pursued by the author:

1) A possible tightening of the bound in (9) for a two-user GIC with mixed interference is of interest. It is motivated by the fact that the upper bound for the corresponding corner point is above the one in Conjecture 1.
2) The unknown corner point of a one-sided GIC with weak interference satisfies the bounds in Proposition 2; it is given by (R_1, C_2) where the gap between the upper and lower bounds on R_1 in (13) is large for small values of a. An improvement of these bounds is of interest (see the last paragraph in Section 1-C).
3) A possible extension of this work to the class of semi-deterministic interference channels in [20], which includes the two-user GICs and the deterministic interference channels in [7].

APPENDIX: ON THE CONCAVITY/ CONVEXITY OF THE FUNCTION f_P IN (39)

The following appendix is related to the proof of Lemma 1 that makes use of the concavity of the function f_P in (39) over the interval [0,1] for all $P \geq 0.680$. This concavity property is proved in the following, and it is also shown that the function f_P is convex over this interval for $P \approx 0$.

For an arbitrary $P > 0$, the function f_P in (39) can be expressed in the form

$$\begin{aligned} f_P(a) &= \frac{1}{2}\log(1+P) - \frac{1}{6}\left[2\log(1+P+aP) - \log(1+aP) + \log\left(1+aP+\frac{P}{1+aP}\right)\right] \\ &= \frac{1}{2}\log(1+P) - \frac{1}{6}\left[2\log(1+P+aP) + \log\left(1+\frac{P}{(1+aP)^2}\right)\right], \qquad \forall\, a \in [0,1]. \end{aligned}$$

Calculation of the second derivative gives

$$f''_P(a) = \frac{P^2}{3}\left[\frac{1}{(1+P+aP)^2} - \frac{1}{(1+aP)^2} - \frac{P}{\left[P+(1+aP)^2\right]^2} + \frac{(1+aP)^2}{\left[P+(1+aP)^2\right]^2}\right]. \tag{63}$$

For an arbitrary $P > 0$, the function f_P is concave over the interval [0,1] if and only if $f''_P(a) \leq 0$ for all $a \in [0,1]$. In particular, it is required that

$$f''_P(1) = \frac{1}{(1+2P)^2} - \frac{1}{(1+P)^2} - \frac{P-(1+P)^2}{\left[P+(1+P)^2\right]^2} \leq 0$$

and this inequality holds if and only if $P \geq 0.680$. One can verify that a (necessary and) sufficient condition for the second derivative f''_P to be monotonic increasing over the interval [0,1] is that it is increasing over a neighborhood of the right endpoint of this interval. This condition is satisfied if and only if

$$f_P^{(3)}(1) = -\frac{1}{(1+2P)^3} + \frac{1}{(1+P)^3} - \frac{1+2P^2+P^3}{\left[P+(1+P)^2\right]^3} \geq 0$$

which holds for $P \geq 0.226$. This implies that, for $P \geq 0.680$, $f''_P \leq 0$ over the interval [0,1] (since f''_P is monotonic increasing over [0,1], and it is non-positive at the right endpoint of this interval). It therefore yields that, for $P \geq 0.680$, the function f_P is concave over [0,1].

As a side note, if on the other hand $P \approx 0$, then it follows from (63) that $f''_P(a) \approx \frac{P^2}{3}$ for $a \in [0,1]$; hence, for small enough values of P, the function f_P is convex over the interval [0,1] (but its second derivative is close to zero, so the curve of f_P forms approximately a straight line).

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REFERENCES

[1] V. S. Annapureddy and V. V. Veeravalli, "Gaussian interference networks: sum capacity in the low-interference regime and new outer bounds on the capacity region," *IEEE Trans. on Information Theory*, vol. 55, no. 7, pp. 3032–3050, July 2009.

[2] A. B. Carleial, "A case where interference does not reduce capacity," *IEEE Trans. on Information Theory*, vol. 21, no. 5, pp. 569–570, September 1975.

[3] A. B. Carleial, "Interference channels," *IEEE Trans. on Information Theory*, vol. 24, no. 1, pp. 60–70, January 1978.

[4] M. H. M. Costa, "On the Gaussian interference channel," *IEEE Trans. on Information Theory*, vol. 31, no. 5, pp. 607–615, September 1985.

[5] M. H. M. Costa, "Noisebergs in Z- Gaussian interference channels," *Proceedings of the 2011 Information Theory and Applications Workshop (ITA)*, p. 1–6, San-Diego, California, USA, February 2011.

[6] M. H. M. Costa and C. Nair, "On the achievable rate sum for symmetric Gaussian interference channels," presented at the *2012 Information Theory and Applications Workshop*, San-Diego, California, USA, February 2012. [Online]. Available: http://ita.ucsd.edu/workshop/12/files/paper/late_paper.html.

[7] A. El Gamal and M. H. M. Costa, "The capacity region of a class of deterministic interference channels," *IEEE Trans. on Information Theory*, vol. 28, no. 2, pp. 343–346, March 1982.

[8] A. El Gamal and Y. H. Kim, *Network Information Theory*, Cambridge university press, 2011.

[9] R. Etkin, D. Tse and H. Wang, "Gaussian interference channel capacity to within one bit" *IEEE Trans. on Information Theory*, vol. 54, no. 12, pp. 5534–5562, December 2008.

[10] R. Etkin, "New sum-rate upper bound for the two-user Gaussian interference channel," *Proceedings of the 2009 IEEE International Symposium on Information Theory*, pp. 2582–2586, Seoul, Korea, June 2009.

[11] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. on Information Theory*, vol. 27, no. 1, pp. 49–60, January 1981.

[12] G. Kramer, "Outer bounds on the capacity of Gaussian interference channels," *IEEE Trans. on Information Theory*, vol. 50, no. 3, pp. 581–586, March 2004.

[13] G. Kramer, "Review of rate regions for interference channels," *Proceedings of the International Zurich Seminar on Communications*, pp. 162–165, Zurich, Switzerland, February 2006.

[14] A. S. Motahari and A. K. Khandani, "Capacity bounds for the Gaussian interference channel," *IEEE Trans. on Information Theory*, vol. 55, no. 2, pp. 620–643, February 2009.

[15] J. Nam and G. Caire, "A new outer bound on the capacity region of Gaussian interference channels," *Proceedings of the 2012 IEEE International Symposium on Information Theory*, pp. 2816–2820, MIT, Boston, MA, USA, July 2012.

[16] I. Sason, "On achievable rate regions for the Gaussian interference channels," *IEEE Trans. on Information Theory*, vol. 50, no. 6, pp. 1345–1356, June 2004.

[17] H. Sato, "On degraded Gaussian two-user channels," *IEEE Trans. on Information Theory*, vol. 24, no. 5, pp. 637–640, September 1978.

[18] H. Sato, "The capacity of the Gaussian interference channel under strong interference," *IEEE Trans. on Information Theory*, vol. 27, no. 6, pp. 786–788, November 1981.

[19] X. Shang, G. Kramer and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for Gaussian interference channels," *IEEE Trans. on Information Theory*, vol. 55, no. 2, pp. 689–699, February 2009.

[20] E. Telatar and D. Tse, "Bounds on the capacity region of a class of interference channels," *Proceedings of the 2007 IEEE International Symposium on Information Theory*, pp. 2871–2874, Nice, France, June 2007.

[21] D. Tuninetti, "K-user interference channels: General outer bound and sum-capacity for certain Gaussian channels," *Proceedings of the 2011 IEEE International Symposium on Information Theory*, pp. 1168–1172, Saint Petersburg, Russia, August 2011.

[22] D. Tuninetti and Y. Weng, "On the Han-Kobayashi achievable region for Gaussian interference channel," *Proceedings of the 2008 IEEE International Symposium on Information Theory*, pp. 240–244, Toronto, Canada, July 2008.

[23] Y. Zhao, C. W. Tan, A. S. Avestimehr, S. N. Diggavi and G. J. Pottie, "On the maximum achievable sum-rate with successive decoding in interference channels," *IEEE Trans. on Information Theory*, vol. 58, no. 6, pp. 3798–3820, June 2012.

[24] Y. Zhao, F. Zhu and B. Chen, "The Han-Kobayashi region for a class of Gaussian interference channels with mixed interference," *Proceedings of the 2012 IEEE International Symposium on Information Theory*, pp. 2801–2805, Boston, MIT, USA, August 2012.