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Stable Image Enhancement by Approximated Forward-and-Backward Wave Equation

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Abstract The recently-proposed family of Forward-and-Backward (FAB) Telegraph-Diffusion (TeD) equations [9] is analyzed in the context of image enhancement. Stability of such schemes in terms of energy convergence is investigated. An approximated version of the enhancement operator that offers an increased stability is proposed and examined. This scheme is implemented in both stable and unstable regimes of the original FAB-TeD operator and shown to yield similar results to those obtained by the latter, without losing stability. The theoretical conclusions regarding stability of the approximated FAB-TeD are reinforced by simulations, exhibiting enhanced images with sharpened edges and yet very limited compromise on the quality of other image details.

Keywords Diffusion equations, FAB Telegraph-Diffusion, Stable image enhancement, Image restoration, Wave equations

1 Introduction

One of the most difficult, yet important, tasks in image processing is image enhancement, i.e. improving the visibility of some features in an image that has been affected by blurring and/or contaminated by noise. The difficulties in such enhancement are twofold. Firstly, enhancing meaningful features usually generates additional noise. Secondly, sharpening processes are often unstable and difficult to control, resulting, even in noiseless images, in spurious artifacts.

Recently proposed processing schemes, based on spatially-varying Partial Differential Equations (PDEs), offer an elegant solution to one of these problems. By locally varying the equation coefficients, these methods achieve image-dependent, anisotropic behavior, which allows differential treatments of meaningful features/details and noisy areas [8].

However, enhancement techniques based on PDE-based processing, such as the Forward-and-Backward (FAB) methods ([3], [9]), still remain ill-posed in the continuous settings. The most difficult problem encountered in such processing is that in some cases the energy of the solution drastically increases after arbitrarily short time (here, and in the rest of this paper, unless otherwise specified, 'time' means the independent variable of the evolution of the dynamic process).

This problem was addressed in the discrete space in [13]. There the authors proposed a spatial discretization scheme of the FAB equation that satisfies the minimum-maximum principle, thus preventing explosion of the solution.

Here we adopt a different approach. We do not aim to find a well-posed formulation of the enhancement problem. Instead, we consider a different enhancement problem which is still ill posed, and, as such, the energy of its solution still may explode after long time. However, in this case we require that the solution energy will remain bounded during finite time. This allows achieving enhancement of an image, without introducing artifacts that are characteristic of the unstable regime.

This paper expands the preliminary report presented by the authors in [10], providing more theoretical and experimental results.

2 Enhancement Methods

2.1 Background

The basic image enhancement problem is concerned with finding the signal u given the following smoothed noisy input:

$$\tilde{u} = S(u) + \hat{n},$$

where S is some smoothing kernel which may result, for example, from optical problems, light dispersion or low resolution imaging constraints. The noise $\hat{n}$ is usually assumed to be zero-mean *iid* Gaussian noise.

The above problem requires a model of an "ideal" image in order to derive the enhancement strategy and to evaluate the result. Several such models have been proposed over the years. This paper is concerned mainly with the piecewise-smooth model, where the "ideal" target image consists of smooth areas separated by edges. We first demonstrate this model on a simpler example of image denoising.

Most denoising problems assume $S(u) = u$, therefore, one of the solutions is anisotropic smoothing. There has been some confusion about the use of the term "anisotropic smoothing" in image processing. When the term was introduced in [8], it was meant to describe spatially-varying (but similar in all directions) smoothing, i.e. strong smoothing of flat areas and weak smoothing of edges. The more meaningful "anisotropic" was redefined and used later in [14], where it accounted for directional smoothing. In this paper we focus on the former, historical meaning, applied to image processing by PDEs.

Smoothing properties of PDEs have been used in image processing for the purposes of denoising ([13], [3], [4], [11]) since the introduction of the Perona-Malik (PM) diffusion in [8]. Several variants of these schemes resulted from mathematical analysis of the Total-Variation minimization [11], whereas others were inspired by

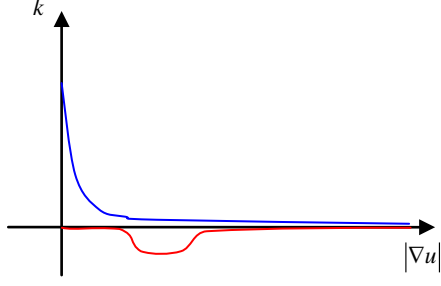


Fig. 1: The elasticity function, k (blue curve), of forward TeD (1) and the inverse smoothing component (5), μ (red curve).

physical processes [4]. However, the basic underlying diffusion equation remained the same.

The damped wave or Telegraph-Diffusion equation (TeD),

$$u_{tt} - \nabla \cdot \left(\left(k(|\nabla u|) \right) \nabla u \right) + cu_t = 0, \quad (1)$$

was introduced in the context of image processing by Ratner and Zeevi [9]. Inspired by the properties of the physical process of damped elastic deformation, its characteristic behavior of smoothing has been exploited in the processing of an input function (e.g. grayscale image) u . The TeD equation is considered under the following initial conditions:

$$\begin{aligned} u(t=0) &= \tilde{u} \\ u_t(t=0) &= \tilde{u}_t, \end{aligned} \quad (2)$$

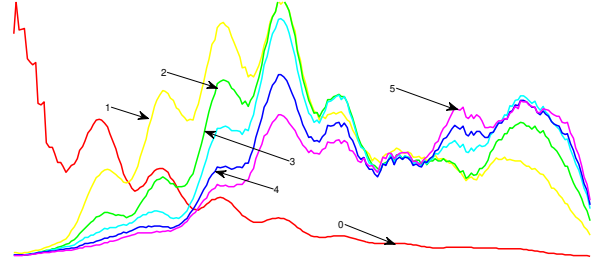
along with zero Neumann boundary conditions, where $\tilde{u}$ is an initial, noisy image, $\tilde{u}_t$ is usually equal to zero, and the damping c is a positive constant. The elasticity coefficient k is a monotonic decreasing function which locally controls the degree of smoothing, similarly to the Perona-Malik coefficients ([8]):

$$k(s) = (1 + s/\kappa)^{-1}; \quad 0 < \kappa = \text{constant}. \quad (3)$$

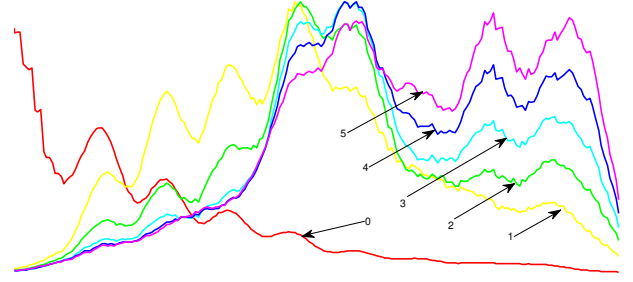
The usual choice for s is the absolute value of the gradient of u , as in (1). The function k guarantees, in this case, a decrease of elasticity near edges. This diminishes the effect of the smoothing process near regions containing important image structural information.

The advantages in processing that come along with the use of TeD, similarly to other PDE-based methods, are due to the feasibility to locally control the degree of smoothing by adjusting the elasticity coefficient. The benefits of using the TeD over the previously proposed diffusion-based methods ([8], [3]) were addressed in [9]. These include, in short, better edge preservation and faster convergence rate of the explicit discretization scheme.

The above method denoises an image while preserving meaningful features such as edges. It has been shown to minimize the total-variation ([9]), similarly to the diffusion-based methods. In the context of image enhancement, which is the main goal of the present study, the same framework that allowed content-dependent denoising also supports adaptive enhancement.



(a)



(b)

Fig. 2: Normalized frequency decompositions of different components u_i , showing that higher frequencies increase with i for both implementation methods. (a) – Theoretical nonlinear RHS; (b) – Heuristic nonlinear RHS.

2.2 Forward-and-Backward (FAB) Wave Equation

Let us assume a piecewise-smooth image model, i.e. that an image is a collection of smooth areas separated by edges. One possible way to enhance such an image that was degraded by blurring is to increase higher gradients, and thereby sharpen the edges. This can be achieved by allowing the elasticity coefficients to locally become negative ([9]), by introducing a backward component μ as follows:

$$u_{tt} - \nabla \cdot \left(\left(k(|\nabla u|) + \varepsilon \mu(|\nabla u|) \right) \nabla u \right) + cu_t = 0, \quad (4)$$

where ε is a small positive constant and μ is a non-positive function that is zero everywhere except in a small area:

$$\mu(s) = - \left(1 + \left((s - \kappa_f)/w \right)^8 \right)^{-1}; \quad 0 < \kappa_f, w, \quad (5)$$

where κ_f defines the median value of the gradients to be enhanced, and w determines the size of its neighborhood. The time parameter, t , is usually bounded by some positive value T ($0 < t < T$), since, in practice, we are interested in short-time evolution of the processed signal.

Backward TeD is similar to TeD that moves backwards in time. Both sharpen the image (an analogy addressed in [9]); a behavior achieved by reversing the time. However, as is the case in some other inverse problems, the backward TeD is ill-posed. For example, linear backward TeD (i.e. TeD with constant negative elasticity) is equivalent to a high-frequency enhancing filter ([9]), or an inverse of a low-pass filter problem, which is a prototype of an ill-posed problem.

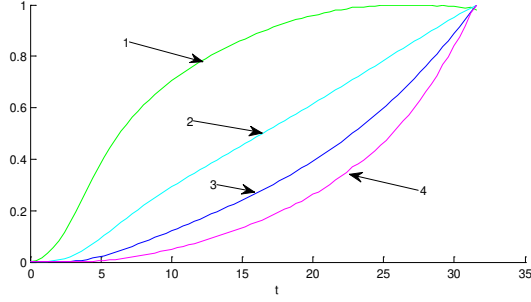


Figure 3: Empirical results of normalized energy functions of various components of the expansion, u_i (see text). As expected, the rate of increment of the energy increases with the index i .

We attempt to solve this trade-off between enhancement and well-posedness by proposing a processing scheme that both remains stable (in terms of bounded energy) for a desirable long enough time, and achieves (during this time) enhancement of the input image.

2.3 Force Term

Another way to enhance an image using the wave equation is inspired by an approach presented by Honigman and Zeevi in [4]. The authors proposed using an anisotropic inhomogeneous diffusion equation by incorporating the Schrödinger's potential. In case of elastic deformation, it is equivalent to applying a spatially-varying external force, g , to the system:

$$u_{tt} - \nabla \cdot \left(\left(k(|\nabla u|) \right) \nabla u \right) + cu_t = g(u, x, t), \quad (6)$$

with k defined by (3) and c being positive. There are many possible choices for g , depending on the features to be enhanced. One is a wavelet shrinkage of u (as was proposed in [4]), which can be used to enhance textures. Here we examine an edge enhancing force term similar to FAB-TeD, derived in the next section. Other feature/detail-specific forces will also be discussed.

3 Mathematical Considerations

In this section we take a closer look at the equations governing the methods described above, (4) and (6), wherein, for convenience, we limit our discussion to one spatial dimension.

3.1 The Relation Between FAB-TeD and Forward TeD Incorporating Force

Let us examine the FAB-TeD equation (4). It models a process of damped elastic deformation with varying elasticity that can become negative. There are no natural physical phenomena, that we are aware of, which exhibit negative elasticity. Further, in most cases of nonzero ε , the solution explodes after a very short time. This we must avoid in image processing.

In this section we attempt to represent the FAB-TeD problem in terms of the more intuitive forced damped deformation (6), using perturbation methods ([15]). We show, in fact, that a FAB-TeD equation with a linear μ term (backward component) is equivalent to a system of PM-type TeD equations (with no backward components)

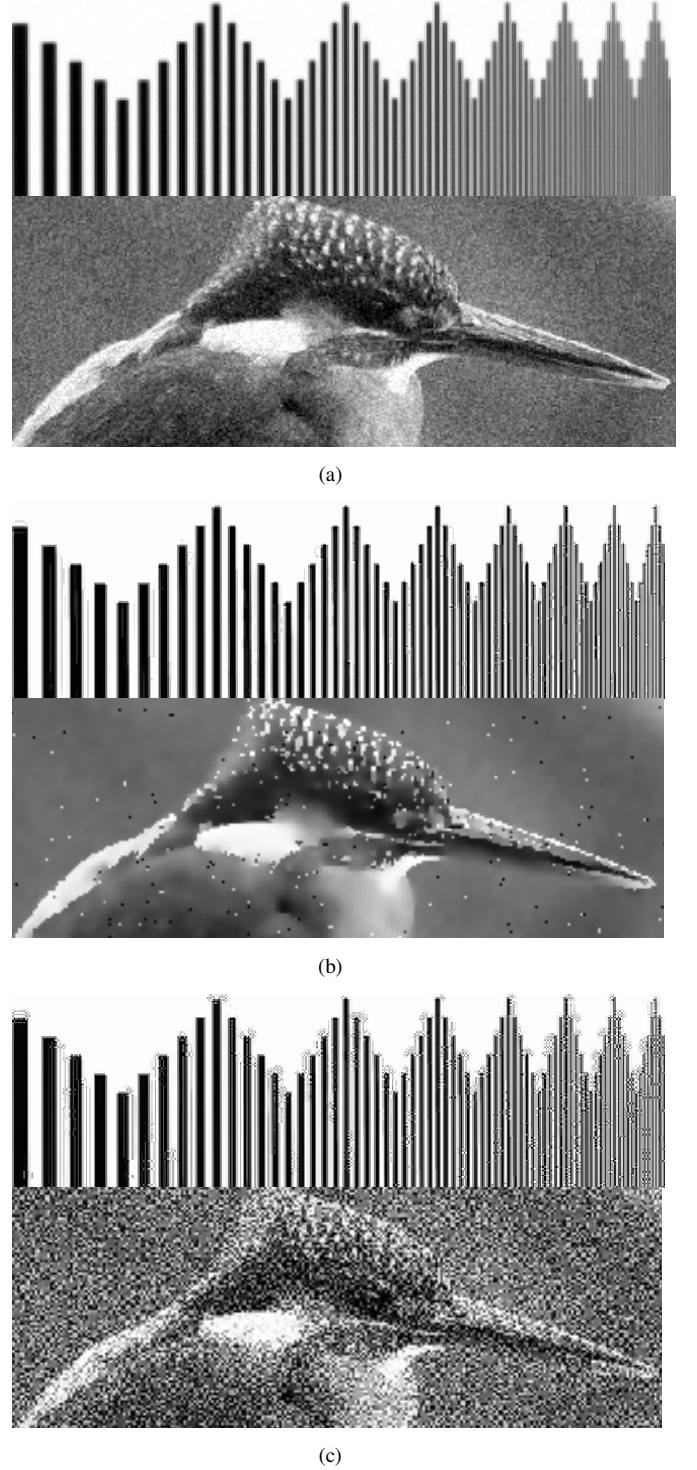


Fig. 4: Explosion of FAB-TeD vs. convergence of finite sum approximation under aggressive sharpening. (a) input image, (b) enhancement result obtained with fifth-order FAB-TeD approximation, (c) result of processing with exact FAB-TeD. The edges of the approximation result are strongly sharpened, as expected, and yet the image still resembles the input. FAB-TeD enhancement, on the other hand, causes severe deterioration of the image.

incorporating forcing terms. We then proceed to extrapolate the result to nonlinear μ .

Let us first define the terminology. To this end let $v(t, x)$ be a solution of (1), and $u(x, t, \varepsilon)$ a solution of (4). Since the solution u depends on ε (small or zero), it can be expanded into Taylor series (around $\varepsilon = 0$) as follows:

approximation	'kingfisher'	'sand'	'calibration'
1 st order	32.3	28.3	24.9
2 nd order	39.4	34	37.5
3 rd order	41.2	36.6	43.2
5 th order	41.3	36.7	44.0
7 th order	41.3	36.7	44.0

Table 1: Differences between several orders of FAB-TeD approximation and FAB-TeD assessed by a similarity measure SM (see definition in the text, (17)). The SM is summarized for the three test images illustrated in Fig.10. The approximations yield maximal SM after 5th order.

$$\begin{aligned}
u(x, t, \varepsilon) &\equiv \sum_{m=0}^{\infty} u_m(x, t) \varepsilon^m = \\
&= \sum_{m=0}^n u_m(x, t) \varepsilon^m + R_n(x, t) \equiv, \\
&\equiv \hat{u}_n(x, t, \varepsilon) + R_n(x, t),
\end{aligned} \tag{7}$$

where R_n is the residue of the series, containing higher than n powers of ε , and $\hat{u}_n$ is an approximation of u up to n^{th} power of ε (n -th order approximation). Since $u(x, t, 0) = u_0(x, t)$ solves an equation with no backward component (zero ε), it is clear that $u_0(x, t) = v(x, t)$.

We make the following assumption about R_n :

Assumption 1: There exists $\varepsilon_{\text{MAX}} > 0$, such that for every $0 \leq \varepsilon < \varepsilon_{\text{MAX}}$ the energy of R_n (as defined in (7)) decreases as n grows, i.e. the convergence radius of ε is non-zero (we will see later that experimental results support this assumption).

Alternatively, we can assume that the n -th order approximation captures the leading behavior of u , even for small n . In the case of image enhancement, the leading behavior is feature-dependent sharpening-denoising. This means that even small-order approximation should solve the enhancement problem. We will return to this point later.

Let us also assume, at first, that $k \equiv k(x, t)$ and $\mu \equiv \mu(x, t)$ are independent of u , and, therefore, do not depend on ε . Substituting (7) into (4) yields:

$$\begin{aligned}
&\left(\sum u_i \varepsilon^i \right)_n - \nabla \cdot \left((k + \varepsilon \mu) \nabla \left(\sum u_i \varepsilon^i \right) \right) + \\
&+ c \left(\sum u_i \varepsilon^i \right)_i = 0.
\end{aligned} \tag{8}$$

Since (8) is valid for all values of ε (within the convergence radius), the coefficients of every power of ε must also be zero. This yields the following system of equations:

$$\begin{aligned}
&\left((u_0)_n - \nabla \cdot (k \nabla u_0) + c(u_0)_i \right) = 0 \\
&\left((u_i)_n - \nabla \cdot (k \nabla u_i) + c(u_i)_i \right) = \nabla \cdot (\mu \nabla u_{i-1}), i \geq 1
\end{aligned} \tag{9}$$

Further, it may be possible to relax the linearity constraint on k . Allowing k , for example, to depend only on u_0 renders

the first equation of (9) to become nonlinear, while the other equations remain linear. We can, in fact, claim that it is equivalent to using fully nonlinear u -dependent k (e.g. PM-version):

$$k(u_0) \equiv \frac{1}{1 + \frac{|\nabla u_0|}{\kappa}} \approx \frac{1}{1 + \frac{|\nabla u|}{\kappa}} \equiv k(u),$$

since u and u_0 differ from each other only around edges, i.e. in areas of large gradients. We know that k is a decreasing function of the absolute gradient, which should, ideally, drop to zero around edges. Perturbations of u_0 around edges have, therefore, little or no effect on $k(u)$, as they change the gradient of u in the flat area of k (Fig. 1). This results in an effectively nonlinear k which can, however, be treated as linear (though time- and space-variant) in the inhomogeneous case.

The motivation behind the linearization of k for the inhomogeneous part of the system, while keeping it, in fact, non-linear is twofold: On the one hand, using nonlinear k in the homogeneous equation of u_0 allows us to apply the results of Nakao [7], which state that the energy of u_0 vanishes with time, providing useful bounds. On the other hand, inhomogeneous equations with linear k are much easier to analyze than their nonlinear counterparts, as will be shown in the next section.

It is interesting to note that the same process and reasoning may be applied to the FAB diffusion equation [3]:

$$u_i - \nabla \cdot \left((k(|\nabla u|)) \nabla u \right) = 0, \tag{10}$$

to yield the following system of equations:

$$\begin{aligned}
&\left((u_0)_i - \nabla \cdot (k \nabla u_0) \right) = 0 \\
&\left((u_i)_i - \nabla \cdot (k \nabla u_i) \right) = \nabla \cdot (\mu \nabla u_{i-1}), i \geq 1.
\end{aligned} \tag{11}$$

This establishes an important link between FAB-diffusion [3] and diffusion with Schrödinger's potential [4].

Another point of interest arises from the fact that the functions $\{u_i\}$ do not depend on ε . This means that a single numerical calculation of $\{u_i\}$ yields approximations of solutions of (4) for any value of ε (by substitution into (7)), which acts as an enhancement strength parameter. This is in contrast to direct solution of (4) which produces the result for a single value of ε .

Computations show that for higher indices i , the functions u_i contain higher spatial frequencies (Fig. 2). This agrees with the theory, since image enhancement should increase the bandwidth of the input. Since higher orders of approximation achieve better enhancement, higher frequencies are to be expected in higher indices of u_i . This also explains why better enhancement is achieved by increasing the weight of higher indices of u_i by using higher values of ε .

3.2 Energy behavior – linear case

We now proceed to explore the behavior of the inhomogeneous TeD equation with linear coefficients and a force term (i.e. independent of u , but time- and space-

varying). To this end we define the energy of a function u as follows:

An energy increase over time usually entails explosion of u . To avoid that, we wish to find such a scheme that would keep the resulting energy bounded for a period of time long enough to achieve image enhancement.

The convergence of energy of u_0 (the solution of the homogeneous equation (1)) to zero (faster than $(1+t)^{-1}$) can be proven under certain assumptions on the smoothness of the derivatives of u_0 , similarly to the proof presented in [7]:

Theorem 1: Let N be any positive integer and assume $k(\cdot)$ as defined in (3). Then, there exists $\delta > 0$ such that if $(u_0(t=0), (u_0)_t(t=0)) \in H^4 \times H^3$ satisfies the appropriate compatibility condition and $I_3 < \delta$ (the smallness condition), the TeD equation (1) – (2) admits a unique solution $u(t)$, with the following bounds ([7], Appendix 1):

$$\|(u_0)_x(t)\|_2^2 \leq CI_3^2(1+t)^{-1}$$

$$\|(u_0)_t(t)\|_2^2 \leq CI_0^2(1+t)^{-1},$$

where

$$\|u\|_{H^m} \equiv \sum_{k \leq m} \left\| \frac{\partial^k u}{\partial x^k} \right\|_2,$$

$$I_m \equiv \|u_0(t=0)\|_{H^{m+1}} + \|(u_0)_t(t=0)\|_{H^m},$$

and C is a positive constant. Note that C is used in various equations as a generic constant to simplify the notation, instead of denoting the various constants by C_1, C_2 , etc.

Remark 1.1: The continuous function $u_0(t=0)$ is determined by the initial value of an image under consideration, which is discrete. Therefore, by choosing an appropriate interpolation, we can assure the smoothness of any order of its spatial derivatives. In particular, we can assume that the L^2 norms of up to 4th spatial derivative of $u_0(t=0)$ are finite (and small enough), i.e.:

$$\|u_0(t=0)\|_{H^4} < \delta.$$

Since $(u_0)_t(t=0) \equiv 0$, the smallness condition is fulfilled.

Remark 1.2: Compatibility conditions for PDEs are a set of relations between the initial conditions, the PDE, and the boundary conditions which are necessary and sufficient for the solution to be sufficiently differentiable everywhere in the domain including its boundaries. We assume that the appropriate compatibility condition is satisfied since the actual boundary conditions play a minor role in the image processing task and can therefore be adjusted to fulfill any requirement. The initial conditions should be smooth enough, since they are derived from an interpolated discrete function (Appendix 1).

We use this result to estimate the energy of the inhomogeneous equations of u_i , for $i > 0$. In order to

$$E_u(t) = \iint_{\Omega} (|u_t|^2 + |\nabla u|^2) dx dy = \|u_t\|_2^2 + \|\nabla u\|_2^2 \quad (12)$$

$$u : (\Omega \times [0, \infty)) \rightarrow \mathbb{R}.$$

proceed, we recall Duhamel's principle ([15]). This principle states that given an inhomogeneous equation with linear coefficients,

$$((u_i)_t)_t - \nabla \cdot (k \nabla u_i) + c(u_i)_t = (p_i)_x(x, t),$$

its solution, u_i , can be expressed as follows:

$$u_i(x, t) = \int_0^t w_i(x, t; \tau) d\tau,$$

where $w_i(x, t; \tau)$ is the solution of the system (13):

$$\begin{aligned} (w_i)_t - (k(x, t)(w_i)_x)_x + c w_i &= 0 \\ w_i(x, \tau; \tau) &= 0, \\ (w_i)_t(x, \tau; \tau) &= (p_i)_x(x, \tau) \equiv (\mu(u_{i-1}))_x, \\ \tau &\geq 0, \end{aligned} \quad (13)$$

i.e. $w_i(x, t; \tau)$ is a homogeneous elastic deformation process starting at time τ , with initial conditions defined by u_{i-1} . Therefore, its energy decays according to theorem 1:

$$\|(w_i(x, t; \tau))_x\|_2^2 \leq CI_{3,i}^2(\tau)(1+t-\tau)^{-1}$$

$$\|(w_i(x, t; \tau))_t\|_2^2 \leq CI_{0,i}^2(\tau)(1+t-\tau)^{-1},$$

where C is some positive constant and

$$\begin{aligned} I_{i,j}(\tau) &= \sum_{k=0}^{j+1} \left\| \frac{\partial^k w_i}{\partial x^k} \right\|_2 + \sum_{k=0}^j \left\| \frac{\partial^k (w_i)_t}{\partial x^k} \right\|_2 = \\ &= \sum_{k=0}^j \left\| \frac{\partial^k (p_i)_x}{\partial x^k} \right\|_2 = \sum_{k=0}^j \left\| \frac{\partial^{k+1} (\mu(u_{i-1}))_x(x, \tau)}{\partial x^{k+1}} \right\|_2. \end{aligned}$$

We can express the derivatives of u_i in terms of w_i :

$$(u_i)_t(x, t) = w_i(x, t; t) + \int_0^t (w_i)_t(x, t; \tau) d\tau =$$

$$\int_0^t (w_i)_t(x, t; \tau) d\tau$$

$$(u_i)_x(x, t) = \int_0^t (w_i)_x(x, t; \tau) d\tau.$$

The energy of u_i is therefore given by:

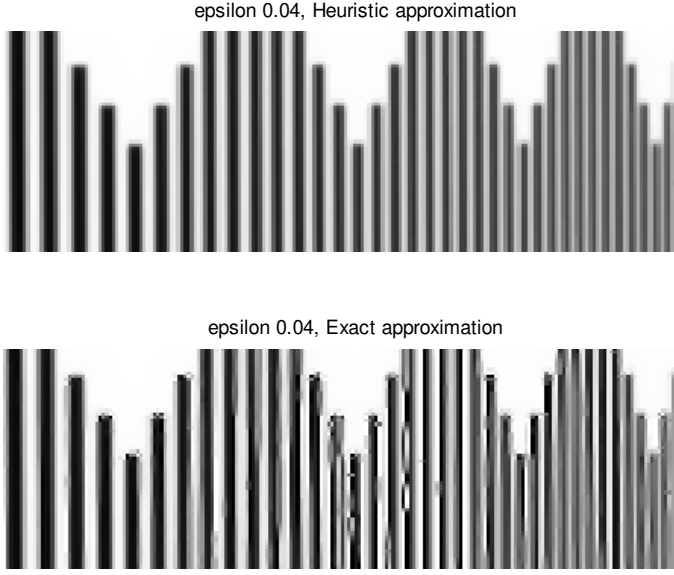


Fig. 6: Instability artifacts induced by the theoretical approximation (lower), compared with the artifact-free heuristic approximation (upper). As indicated in text, areas containing high-order derivatives (such as edges) are prone to instabilities when using the theoretical approximation.

$$E_i(t) = \|(u_i)_t\|_2^2 + \|(u_i)_x\|_2^2 = \left\| \int_0^t (w_i)_x(x, t; \tau) d\tau \right\|_2^2 + \left\| \int_0^t (w_i)_t(x, t; \tau) d\tau \right\|_2^2. \quad (14)$$

Applying Minkowski's integral inequality we can reach the following bound on the energy function:

$$E_i(t) \stackrel{(Minkowski)}{\leq} \left(\int_0^t \|(w_i)_x(x, t; \tau)\|_2 d\tau \right)^2 + \left(\int_0^t \|(w_i)_t(x, t; \tau)\|_2 d\tau \right)^2.$$

Since w_i is the solution of a homogeneous equation with initial conditions given at time τ , Theorem 1 applies:

$$E_i(t) \stackrel{(Thm.1)}{\leq} \left(\int_0^t \sqrt{C} I_{i,3}(\tau) (1+(t-\tau))^{-1/2} dt \right)^2 + \left(\int_0^t \sqrt{C} I_{i,0}(\tau) (1+(t-\tau))^{-1/2} dt \right)^2. \quad (15)$$

For a given index i , the energy E_{i+1} contains norms of (also) higher derivatives of u_i than those in E_i . This means that the energies increase more steeply for higher indices i (for images that contain sufficient high-frequency components). This claim is supported by empirical data (Fig. 3), by means of which we can also see that although the energy functions increase more steeply for higher indices, they remain bounded for a finite time. This means that the overall energy of the approximation is bounded as well, which cannot be said for the energy of u .

Conclusion 1: The energy of u obtained from the solution of FAB TeD (7) increases at least at the rate of the fastest-

increasing E_i (with i tending to infinity), and, therefore, cannot be bounded at any finite time.

Conclusion 2: The rate of explosion of the energy of the solution of FAB-TeD depends also on ε , since it is the variable Taylor expansion, and, as such, determines the influence of higher-order energy components on the overall energy. Consequently, if the energies E_i do not begin to decline after some value of i , the total energy may climb uncontrollably even for small values of ε . Experiments show that in some cases (for example in Fig. 4, for values as low as 0.2) this behavior prohibits image enhancement.

Conclusion 3: The rate of explosion of the energy of finite (up to n^{th} power of ε) approximations of FAB-TeD, $\hat{u}_n$ (7), does not depend on ε as much as that of exact FAB, since, in this case, the steepest increasing E_i (i is small) is bounded for any given time period.

Based on the above, $\hat{u}_n$ (7) can be seen as a (more) stable approximation of FAB-TeD, in the sense that its energy does not explode over finite time. For $n=0$ we get a fully stable forward (PM-type) TeD, for any t (Appendix 1). Higher orders of approximation result in schemes that are closer to FAB (Table 1), but may not exhibit energy convergence for large t (Fig. 3). The stability of these schemes for finite time $t < T$ does not depend, however, on the enhancement rate (Fig. 4), unlike that of FAB-TeD.

Duhamel's principle can also be applied to the system of inhomogeneous equations resulting from FAB-diffusion. This should provide further insights into FAB-diffusion stability, based on stability properties of the homogeneous diffusion equation ([12], [6]), and on the work of Welk, Gilboa and Weickert [13].

3.3 Nonlinear case

In the previous sections we have explored the behavior of a system of inhomogeneous equations which approximated the FAB-TeD equation with linear negative elasticity component term. We now apply similar reasoning in generalizing the results to nonlinear μ .

We consider two approaches to dealing with nonlinear μ . The heuristic approach, is simply to substitute the nonlinear function into the system (9) that results from calculations that assume linear μ . The theoretic approach is to expand μ into a Taylor series,

$$\mu(x, t, \varepsilon) = \sum_{m=0}^{\infty} \mu_m(x, t) \varepsilon^m, \quad (16)$$

and follow the calculations of section 3.1. We begin with the latter.

Substituting (16) and (7) into (4) and comparing to zero, in a way similar to that used in section 3.1, we can derive the following system of inhomogeneous equations:

$$\begin{aligned} ((u_0)_t - \nabla \cdot (k \nabla u_0) + c(u_0))_t &= 0 \\ ((u_i)_t - \nabla \cdot (k \nabla u_i) + c(u_i))_t &= \sum_{\substack{j+l=i-1 \\ j, l \geq 0}} \nabla \cdot (\mu_j \nabla u_l); \quad i > 0. \end{aligned}$$

Using the same arguments as in section 3.2 we can show, for a smooth bounded μ , that the energy of each of the equations converges to zero. The Taylor coefficients μ_j

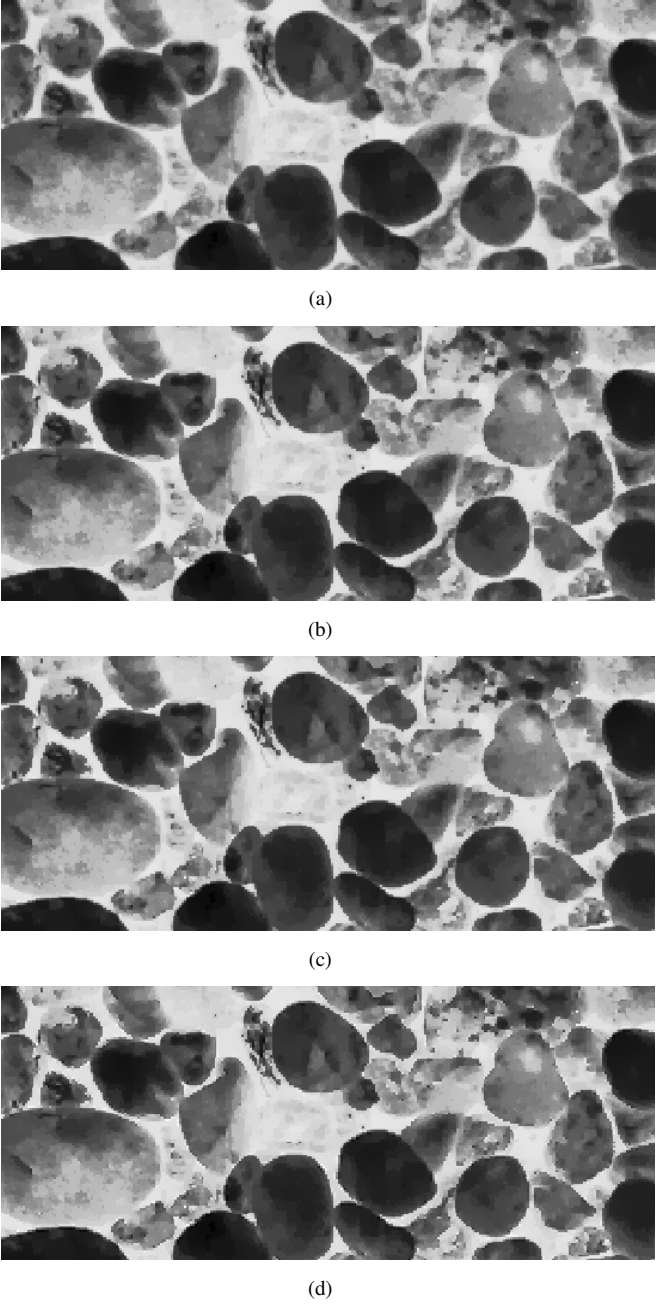


Fig. 8: Processing results obtained with various orders of approximation of FAB-TeD. (a) – 1st order (forward TeD); (b) – 2nd order; (c) – 5th order; (d) – FAB-TeD. While all the results appear sharper than regular (forward) TeD (a), the difference between FAB-TeD and its higher-order approximations is barely visible, although, as indicated by Fig. 9, it exists.

can be expressed in terms of $\{u_i\}_{i < j}$ for any given μ by differentiating it by ε . Let us take, for example, the coefficient defined in (5) for one-dimensional signal, and calculate the first three terms μ_0 and μ_1 :

$$\mu = -\left(1 + \left(\left|u_x\right| - \kappa_f\right)/w\right)^8\right)^{-1} = -\left(1 + \left(\left|\sum_{m=0}^{\infty} (u_m)_x(x, t) \varepsilon^m\right| - \kappa_f\right)/w\right)^8\right)^{-1}$$

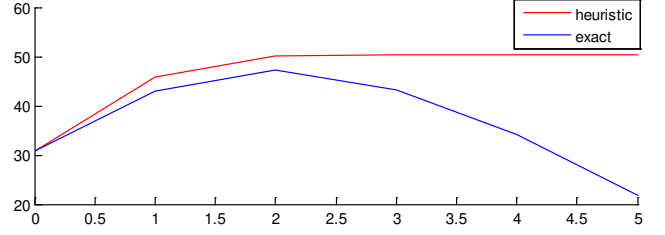


Fig. 7: Similarity measure (SM) of the Heuristic and the Theoretical (exact) approximations (see text) of FAB-TeD applied to the 'calibration' image. The SM results are presented as a function of the approximation order. As expected, higher orders of the theoretic approximation perform poorly when compared with the heuristic version. Experiments on other images exhibit similar behavior.

$$\mu_0 = \mu|_{\varepsilon=0} = -\left(1 + \left(\left|(u_0)_x\right| - \kappa_f\right)/w\right)^8\right)^{-1}$$

$$\mu_1 = \frac{\partial \mu}{\partial \varepsilon}\bigg|_{\varepsilon=0} =$$

$$8w^8 \text{sign}((u_0)_x)(u_1)_x \left(\left|(u_0)_x\right| - \kappa_f\right)^7 \left(w^8 + \left(\left|(u_0)_x\right| - \kappa_f\right)^8\right)^{-2}.$$

The resulting system of equations does not depend on ε , which means that after performing a single calculation of $\{\mu_i\}$ and $\{u_i\}$, one may immediately find the result, u , for any given value of ε . A disadvantage of this system is that high powers of $\left(\left|(u_0)_x\right| - \kappa_f\right)$ in the higher-order derivatives of μ (7 in μ_1 , 14 in μ_2) render the system to become numerically unstable. This instability introduces artifacts (Fig. 6) and reduces the quality of the processed image (Fig. 7). Calculation of $\{\mu_i\}$ also increases significantly the computational load (using unoptimized, straightforward implementations of the algorithms, the theoretical version can take up to 6 times more processor time than the heuristic version which, in turn, takes up to 3 times more time than FAB-TeD).

The heuristic version doesn't suffer from computational inaccuracies and is more efficient than its theoretical counterpart. However, in the heuristic case, we can no longer claim that a single simulation provides the results for all values of ε , since here the functions $\{u_i\}$ depend on μ which, in turn, depends on u and therefore on ε .

In practice, the heuristic version achieves a good approximation of FAB-TeD, and a calculation of the signal components $\{u_i\}$ for some value of ε yields reasonably good results for other values as well. Thus, we have obtained an approximation of FAB-TeD, the energy of which converges with time. This has an immediate application in image processing, since it allows edge enhancement on the one hand and ensures a well-behaved system on the other.

It remains, however, to show how good the approximation is. In the beginning of this exposition, we have assumed that the convergence radius of the Taylor series in (7) is non-zero, i.e. that we can use positive ε and get a residue R_n that diminishes as n increases. While the analysis of the nonlinear system is challenging, the

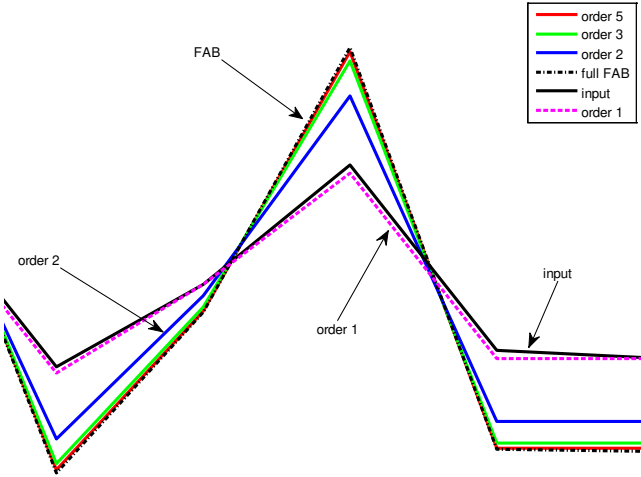


Fig. 9: Results of different orders of approximation of FAB-TeD of a 1D image cross-cut (a horizontal sample of the 'calibration' image). The orders presented are 1st (forward TeD), 2nd, 3rd and 5th. They are compared with the input (solid black line) and full FAB-TeD (dash-dot black line). The approximation clearly improves with an increase of the order, with 5th order being virtually identical to the FAB-TeD result. Significant sharpening is already achieved for 2nd order approximation.

experimental results show that this assumption is justified. Fig. 8 presents the results of approximations with several values of n , all of which achieve results similar to FAB-TeD. This is even more pronounced in Figs. 9 and 11, where the enhancement schemes were applied to a 1D function.

4 Experiments

We have conducted two types of experiments. One was performed in the stable regime of FAB-TeD, and tested the performance of the heuristic and the theoretical approximations of FAB-TeD (see section 3.3). The other was performed in the unstable regime (i.e. when the FAB solution energy fails to converge in finite time). The experiment demonstrated the stability of the proposed approximation.

In the stable regime we've tested Assumption 1 by comparing the results of FAB-TeD obtained with several orders (values of n in (7)) of the heuristic approximation. The quality of the results was assessed by means of visual inspection and by implementing a quantitative similarity measure (SM) defined as follows:

$$SM(a, b) = -20 \log_{10} \left(\frac{1}{m} \sum_{j=1}^m (a_j - b_j)^2 \right), \quad (17)$$

where a and b represent two compared images of identical size, and m is the number of pixels in the images.

We've used the test images of kingfisher, sand and calibration (Fig. 10). The parameters used for all the images were $\varepsilon = 0.03$, $k_b = 0.3$, $w = 0.12$ (for image values between 0 and 1). In these and other images, the results of heuristic approximation closely resemble the

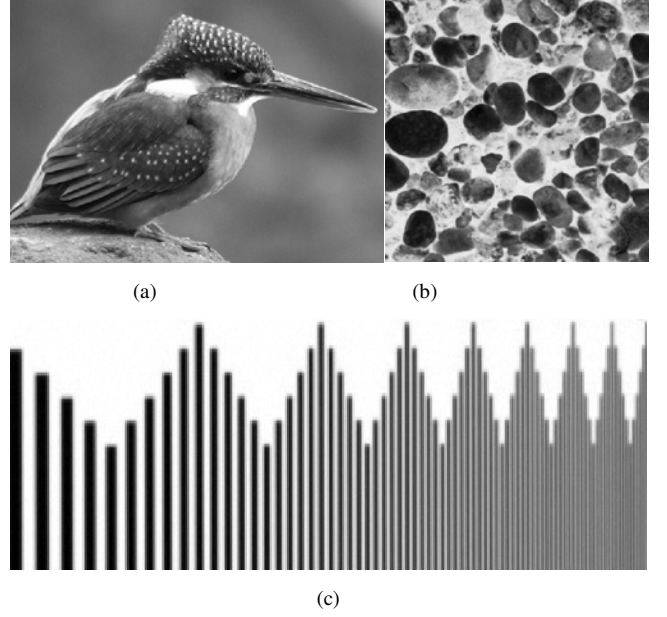


Fig. 10: Test images used in this work: (a) – kingfisher; (b) – sand; (c) – calibration.

FAB-TeD output, both visually (Figs. 8, 11) and according to the similarity measure (Table 1, Fig. 7).

We also performed similar tests to compare the heuristic and the theoretical approximations. The SM results of different orders of both approximations are depicted in Fig. 7. The theoretical approach achieves worse results in practice. Although it is theoretically more precise, its higher orders of approximation contain increasingly higher powers of the first order spatial derivative of the signal. This greatly increases the sensitivity to noise, introducing artifacts in the resulting images (Fig. 6).

In the unstable regime of FAB-TeD, we've tested the hypothesis of section 3.2, namely, that for short time finite-order approximation of FAB-TeD has bounded energy, unlike FAB-TeD itself. We used a relatively large value of ε (0.2) to ensure energy dispersion of the FAB-TeD solution. All the parameters used in simulating the FAB-TeD and its approximation were identical (including timestep and number of iterations). Indeed, instability artifacts are clearly visible on the FAB-TeD processing result in Fig. 4. Despite that, the results of FAB-TeD approximations remain stable, while achieving strong enhancement of the image. Even stronger sharpening ($\varepsilon = 2, 3$), resulting in complete deterioration of FAB-TeD-processed images, causes only some ringing effects in the approximation.

5 Conclusions

The proposed inhomogeneous methods permit the development of powerful and stable image enhancement schemes. The theoretical approach outlined concisely in this paper provides further insight into the fundamental issue of stability regimes of FAB-TeD image enhancement and processing schemes. Whereas we focused on its application to the FAB-TeD, it can also be applied to existing diffusion-based methods.

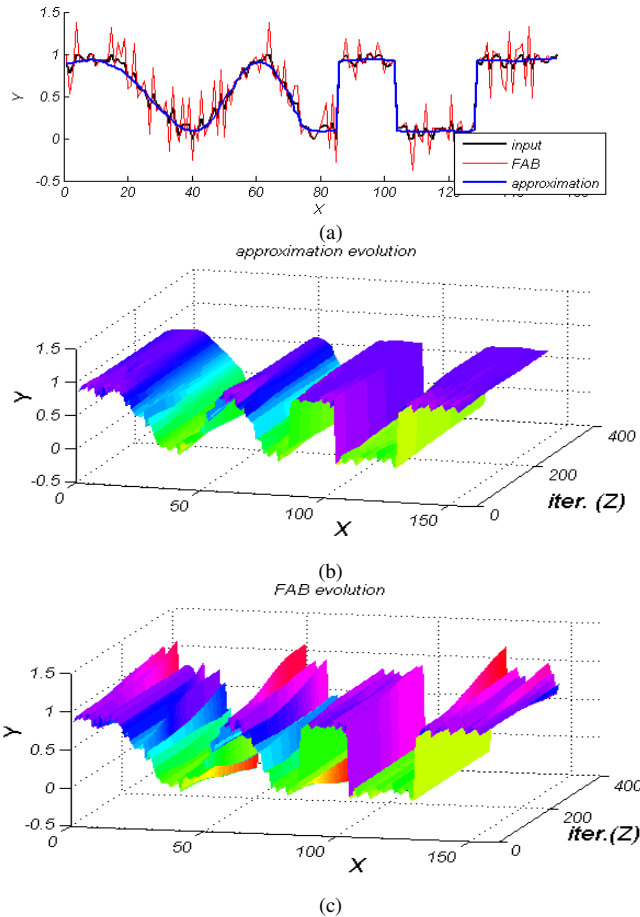


Fig. 11: Comparison of the evolution of a 1D image cross-cut under the FAB and the approximated FAB enhancement, in the unstable regime of FAB ($\varepsilon = 0.3$). The initial (input) and the resulting signals are shown in (a). Approximation evolution is depicted as a function of a number of iterations (Z-axis) in (b). The same is shown for the FAB in (c). Note that the approximated operator quickly eliminates noise, while preserving sharp edges, whereas the unstable FAB enhances most of the existing noise peaks.

In practice, the proposed scheme follows closely the results of other PDE-based methods that lack stability (in their stable regime). The proposed scheme has also the advantage of simultaneously yielding a wide range of magnitudes of enhancement in a single simulation at a small computational cost.

There still remains to be done a more rigorous analysis of energy convergence of finite-order approximations, as well as that of full FAB-TeD. Such analysis should facilitate the identification of stable parameter sets. This should yield a stable FAB-TeD enhancement, thus eliminating the need to use approximations.

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Appendix 1: Existence and Convergence of Denoising-TeD Solution

Notations

$\Omega \subseteq \mathbb{R}^N$ - domain of the image

$\overline{\Omega}$ - completion of Ω (contains the limits of all its Cauchy sequences)

$C_0^\infty(\Omega), C_0^\infty(\overline{\Omega})$ - sets of infinitely many times

differentiable functions over Ω and $\overline{\Omega}$.

$W^{m,p}(\Omega), W_0^{m,p}(\Omega)$ - completions of $C_0^\infty(\overline{\Omega})$,

$C_0^\infty(\Omega)$ with respect to the norm $\sum_{|\alpha| \leq m} \|D^\alpha u\|_p$, with

multi index

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N), |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$

and differentiation operator

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u$$

(all real-valued functions on Ω whose first m weak derivatives are functions in L^p).

What this means in terms of image processing:

Completion of a metric space A : the metric space A' or $\overline{A}$ which contains A and all the points to which Cauchy series in A converge.

The norm $\sum_{|\alpha| \leq m} \|D^\alpha u\|_p$:

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u$$

$$|\alpha| = \alpha_1 + \dots + \alpha_N$$

$$\sum_{|\alpha| \leq m} \|D^\alpha u\|_p = \sum_{|\alpha| \leq m} \left\| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u \right\|_p =$$

$$\sum_{|\alpha| \leq m} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u \right|^p d\omega \right)^{\frac{1}{p}} =$$

$$\left(\int_{\Omega} |u|^p d\omega \right)^{\frac{1}{p}} + \sum_{i=1}^N \left(\int_{\Omega} \left| \frac{\partial}{\partial x_i} u \right|^p d\omega \right)^{\frac{1}{p}} +$$

$$+ \sum_{j=1}^N \sum_{i=1}^N \left(\int_{\Omega} \left| \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u \right|^p d\omega \right)^{\frac{1}{p}} + \dots$$

in our case $p=2$, so the norm is an L_2 norm of u (if m is zero), a sum of L_2 norms of u and all its partial derivatives up to m .

$\dot{W}^{m,p}(\Omega)$ - completion of $C_0^\infty(\overline{\Omega})$ w.r.t. the norm

$\sum_{|\alpha| \leq m} \|D^\alpha u\|_p$. Here the norm is a sum of L_p norms of u 's m^{th} derivatives.

$$W^{m,2}(\Omega) \triangleq H^m(\Omega), \quad W_0^{m,2}(\Omega) \triangleq H_0^m(\Omega),$$

$$\dot{W}^{m,2}(\Omega) \triangleq \dot{H}^m(\Omega)$$

$$X_M \equiv \bigcap_{i=0}^{2M-1} C^i([0, \infty); H^{2M-i}(\Omega)) \cap C^{2M}([0, \infty); L^2(\Omega))$$

$$X_m^T \equiv$$

$$\bigcap_{i=0}^m C^i([0, T]; H^{m+1-i}(\Omega) \cap H_0^1(\Omega)) \cap C^{m+1}([0, T]; L^2(\Omega))$$

$$X_m \equiv X_m^\infty$$

$C([0, \infty); L^2(\Omega))$ - set of $L^2(\Omega)$ valued continuous functions on $[0, \infty)$

$$B_L = \{x \in \mathbb{R}^N \mid |x| \leq L\}$$

$$E_{loc}^\varepsilon(t) \equiv \frac{1}{2} \int_{\Omega \cap B_{L+\varepsilon}} (|u_t|^2 + |\nabla u|^2) dx$$

$$\phi(r) = \begin{cases} \varepsilon_0 & \text{if } 0 \leq r \leq L \\ \frac{\varepsilon_0 L}{r} & \text{if } r \geq L \end{cases} \quad \text{Lipshitz continuous function}$$

on $\mathbb{R}_+$, for ε_0, L positive constants appearing in Hyp. 2' below.

Existence and Convergence of Denoising-TeD Solution

This part deals with a system of the form:

$$u_{tt} - \nabla \cdot (\sigma(|\nabla u|^2) \nabla u) + c(x)u_t = 0 \quad (2)$$

in $\Omega \times [0, \infty)$,

with the following initial and boundary conditions:

$$u(x, 0) = \tilde{u}(x), \quad (3)$$

$$u_t(x, 0) = \tilde{u}_t(x) \text{ and } u|_{\partial\Omega} = 0.$$

It is based mainly on the work of Nakao ([7]) that proves convergence of the system and provides bounds on the energy of the solution.

We'll start by defining the assumptions made in [7] on the coefficients of (2) and on the conditions (3). We'll also test these assumptions on the denoising-TeD equation.

Hypothesis 1:

$k(\cdot)$ is a differentiable function on $\mathbb{R}^+ = [0, \infty]$ and satisfies the conditions:

$$k(v^2) \geq k_0 > 0 \text{ and } k(v^2) - 2|k'(v^2)|v^2 \geq k_0 > 0, \text{ if } |v| \leq L$$

where $L > 0$ is an arbitrarily fixed constant and

$k_0 \equiv k_0(L)$ is a positive constant.

Implications for TeD denoising:

$$k(v) = \frac{1}{1 + \frac{v}{k_f}}, \quad k'(v) = \frac{-k_f}{(k_f + v)^2}$$

$$k(v^2) = \frac{1}{1 + \frac{v^2}{k_f}} \geq \frac{1}{1 + \frac{L^2}{k_f}} > 0 \quad \forall |v| \leq L$$

$$\begin{aligned} k(v^2) - 2|k'(v^2)|v^2 &= \frac{1}{1 + \frac{v^2}{k_f}} - 2v^2 \left| \frac{-k_f}{(k_f + v^2)^2} \right| = \\ &= \frac{k_f + v^2 - 2k_f v^2}{(k_f + v^2)^2} \stackrel{k_f \ll 0.5}{\geq} \frac{k_f}{(k_f + v^2)^2} \stackrel{|v| \leq L}{\geq} \frac{k_f}{(k_f + L^2)^2} > 0 \end{aligned}$$

Remark: We can assume that k is small (smaller than 0.5), otherwise the change in $|\nabla u|$ (i.e proximity to edges) will have little effect on the smoothing process. It is also possible to use $k(v) = \frac{1}{\sqrt{1 + \frac{v}{k_f}}}$, which achieves

similar behavior (and is closer to the TV model), and is an example of a valid $k(\bullet)$, as given in [7].

Conclusion: In the case of TeD, the function $k(\bullet)$ fulfils the requirements of Hyp. 1.

Hypothesis 2

$0 \leq c(\bullet) \in L^\infty(\Omega)$ (c is nonnegative on Ω , belonging to $L^\infty(\Omega)$), and there exists a relatively open set ω in $\bar{\Omega}$

and $x_0 \in \mathbb{R}^N$, such that

$\overline{\Gamma(x_0)} \subset \omega$ and $c(x) \geq \varepsilon_0 > 0$ on ω for some ε_0 ,

where $\Gamma(x_0)$ is defined as a part of the boundary $\partial\Omega$:

$$\Gamma(x_0) = \{x \in \partial\Omega \mid \nu(x) \cdot (x - x_0) > 0\}, \quad x_0 \in \mathbb{R}^N,$$

where $\nu(x)$ is the outward normal at $x \in \partial\Omega$. Among rest, this means that if $\mathbb{R}^N / \Omega$ is star-shaped (i.e. $\Gamma(x_0) = \emptyset$), then $c(x) \equiv 0$ is allowed.

Implications to TeD denoising:

In the case of TeD denoising $c(x) \equiv \text{const.} > 0$ on all Ω and its boundary, therefore Hyp. 2 is fulfilled with no restrictions on the shape of the domain Ω .

Hypothesis 2':

(1) There exists $x_0 \in \mathbb{R}^N$ and a relatively open set

$\omega \subset \bar{\Omega}$ such that

$\overline{\Gamma(x_0)} \subset \omega$ and $c(x) \geq \varepsilon_0 > 0$ for $x \in \omega$

with some ε_0 .

(2) There exists $L > 0$ such that

$c(x) \geq \varepsilon_0 > 0$ for $|x| \geq L$

Again, as with the previous hypothesis, constant damping, as used in TeD, fulfils these conditions.

Compatibility condition of order m-1 [7]:

The initial conditions $(\tilde{u}, \tilde{u}_1) \in H_m \times H_{m-1}$ satisfy the compatibility condition of m-1th order if $u_i \in H_1^0$ and

$$\frac{\partial u_{i-1}}{\partial \nu} = -\alpha u_i \quad \text{on } \Gamma_1 \quad \text{for } 0 \leq i \leq m-1,$$

where

$$u_0 \equiv \tilde{u}, \quad u_1 \equiv \tilde{u}_1$$

$$u_m \equiv \Delta u_{m-2} - c(x)u_{m-1}$$

$$H_1^0(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Gamma_0} = 0\}$$

Γ_0 is the part of the boundary of Ω on which u is zero.

Γ_1 is the part of the boundary of Ω on which u is non-

zero and $\frac{\partial u}{\partial \nu}$ is some function of u_i (non-zero), so that

$$\Gamma_0 \cap \Gamma_1 = \emptyset.$$

$\nu(x)$ is the outward normal to a point x on the boundary.

The case of interest in the context of TeD denoising is $m=3$. Therefore:

$$u_0 = u(t=0)$$

$$u_1 = u_t(t=0) \equiv 0$$

$$u_2 = \Delta u_0 - cu_1 = (u_0)_{xx} + (u_0)_{yy}$$

$$u_3 = \Delta u_1 - cu_2 = -c(u_0)_{xx} - c(u_0)_{yy}.$$

A general image does not fulfill the compatibility condition of order 3. However, one may take any image and pad it with zeroes (frame it in black). Thus, we obtain another, legitimate image, for which the boundary condition $u|_{\partial\Omega} = 0$ holds and also $\Gamma_0 = \partial\Omega$ (therefore $\Gamma_1 = \emptyset$).

This does not affect the denoising task, and leaves us only with smoothness conditions on the input image - continuous spatial derivative up to 3rd degree. We assume that the input is smooth enough to fulfill this requirement.

Theorem 2:

Let $N \geq 1$ be any integer and assume that $k(\cdot) \in C^{m+1}(\mathbb{R}^+)$ and $c(\cdot) \in C^{m+1}(\bar{\Omega})$ with an integer $m > \lceil N/2 \rceil + 1$. Then, under Hyp. 2' and Hyp. 1, there

exists $\delta > 0$ such that if $(\tilde{u}, \tilde{u}_1) \in H^{m+1} \times H^m$ satisfy the compatibility condition of the m^{th} order and smallness condition $I_m \equiv \|\tilde{u}\|_{H^{m+1}} + \|\tilde{u}_1\|_{H^m} < \delta$, the problem (2)-(3) admits a unique solution $u(t)$ in the class X_m :

$$\begin{aligned} X_m^T &\equiv \\ \bigcap_{i=0}^m C^i \left([0, T]; H^{m+1-i}(\Omega) \cap \right) &\cap C^{m+1}([0, T]; L^2(\Omega)) \\ X_m &\equiv X_m^\infty \end{aligned}$$

Further, the following estimates hold:

$$\|D_t^{l+1} u(t)\|_{H^{m-l}}^2 + \|D_t^l \nabla u(t)\|_{H^{m-l}}^2 \leq CI_m^2 (1+t)^{-l-1}$$

for $0 \leq l \leq m$

and

$$\|\nabla u(t)\|_{H^m}^2 \leq CI_m^2 (1+t)^{-1} \text{ for } 0 \leq l \leq m$$

Implications to TeD denoising:

In our case N is 2, i.e. $m > 2$

$k(\cdot) \in C^3(\mathbb{R}^+)$: - fulfilled, in both cases as

$$\frac{1}{1 + \frac{\nu}{\kappa}} \in C^\infty(\mathbb{R}^+)$$

$c(\cdot) \in C^{m+1}(\bar{\Omega})$: is also always true for constant c .

$(\tilde{u}, \tilde{u}_1) \in H^{m+1} \times H^m$: Since in our case $\tilde{u}$ and $\tilde{u}_1$ are $m+1$ differentiable, with derivatives in L_2 , they belong to H^{m+1} and H^m respectively.

$$I_m \equiv \|\tilde{u}\|_{H^{m+1}} + \|\tilde{u}_1\|_{H^m} = \sum_{|\alpha| \leq m+1} \|D^\alpha \tilde{u}\|_2 + \sum_{|\alpha| \leq m} \|D^\alpha \tilde{u}_1\|_2 < \delta$$

$$I_2 \equiv \|\tilde{u}\|_{H^3} + \|\tilde{u}_1\|_{H^2} = \sum_{|\alpha| \leq 3} \|D^\alpha \tilde{u}\|_2 + \sum_{|\alpha| \leq 2} \|D^\alpha \tilde{u}_1\|_2 < \delta$$

$\tilde{u}_1$ is zero, therefore $\|\tilde{u}_1\|_{H^m} \equiv 0$

$\tilde{u}$ is bounded, at least $m+1$ times differentiable (with all derivatives bounded) and has finite support, therefore

$$I_m \equiv \|\tilde{u}\|_{H^{m+1}} < \infty.$$

$$\|D_t^{k+1} u(t)\|_{H^{m-k}}^2 + \|D_t^k \nabla u(t)\|_{H^{m-k}}^2 \leq CI_m^2 (1+t)^{-k-1} \text{ if}$$

for $0 \leq k \leq m$

we take $k=0$ we get:

$$\|u_t(t)\|_{H^{m-k}}^2 + \|\nabla u(t)\|_{H^{m-k}}^2 \leq CI_m^2 (1+t)^{-1} \text{ for } 0 \leq k \leq m$$

$$\begin{aligned} &\left(\sum_{|\alpha| \leq m} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u_t \right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right)^2 + \\ &+ \left(\sum_{|\alpha| \leq m} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \nabla u \right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right)^2 \leq CI_m^2 (1+t)^{-1} \end{aligned}$$

for $0 \leq k \leq m$.

Since all members of the sum are positive, we can say that:

$$\begin{aligned} CI_m^2 (1+t)^{-1} &\geq \left(\sum_{|\alpha| \leq m} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u_t \right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right)^2 + \\ &+ \left(\sum_{|\alpha| \leq m} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \nabla u \right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right)^2 \geq \\ &\geq \left(\sum_{|\alpha| \leq 0} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u_t \right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right)^2 + \\ &+ \left(\sum_{|\alpha| \leq 0} \left(\int_{\Omega} \left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \nabla u \right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right)^2 = \end{aligned}$$

$$\left(\int_{\Omega} |u_t|^2 d\mathbf{x} \right) + \left(\int_{\Omega} |\nabla u|^2 d\mathbf{x} \right) = \|u_t\|^2 + \|\nabla u\|^2$$

i.e. the total variation decreases with t .

Additionally, the solution u belongs to X^m , which means that

$$\sup_{0 \leq t < T} \sum_{k=0}^{m+1} \|D_t^k u(t)\|_{H^{m+1-k}(\Omega)} < \infty$$

i.e. all partial derivatives (spatial and temporal) of u up to order $m+1$ belong to L_2 .

Therefore, all the conditions to Theorem 2 are fulfilled by the denoising-TeD problem.