IRWIN AND JOAN JACOBS
CENTER FOR COMMUNICATION AND INFORMATION TECHNOLOGIES

Large Deviations Analysis of Variable-Rate Slepian-Wolf Coding

Nir Weinberger and Neri Merhav

CCIT Report #850
January 2014

Electronics
Computers
Communications

DEPARTMENT OF ELECTRICAL ENGINEERING
TECHNION - ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 32000, ISRAEL

CCIT Report #850 January 2014

Large Deviations Analysis of Variable-Rate Slepian-Wolf Coding

Nir Weinberger and Neri Merhav

Technion, Dept. of Electrical Engineering

Technion - Israel Institute of Technology

Technion City, Haifa 32000, Israel

{nirwein@tx, merhav@ee}.technion.ac.il

Abstract

We analyze the asymptotic performance of ensembles of random binning Slepian-Wolf codes, where each type class of the source might have a different coding rate. In particular, we first provide the exact encoder excess rate exponent as well as the decoder error exponent. Then, using the error exponent expression, we determine the optimal rate function, namely, the minimal rate for each type class needed to satisfy a given requirement on the decoder error exponent. The resulting excess rate exponent is then evaluated for the optimal rate function. Alternating minimization algorithms are provided for the calculation of both the optimal rate function and the excess rate exponent. It is thus exemplified that, compared to fixed-rate coding, larger error exponents may be achieved using variable-rate coding, at the price of a finite excess rate exponent.

Index Terms

Slepian-Wolf coding, method of types, error exponents, excess rate exponent, alternating minimization, source uncertainty.

I. Introduction

The problem of distributed encoding of correlated sources has been studied extensively since the seminal paper of Slepian and Wolf [22]. This paper addresses the case, where a memoryless source $\{(X_i, Y_i)\}$ needs to be compressed by two separate encoders, one for $\{X_i\}$ and one for $\{Y_i\}$. In a nutshell, the most significant result of [22] states that if $\{Y_i\}$ is known at the decoder side, then $\{X_i\}$ can be compressed at the rate of the *conditional* entropy of $\{X_i\}$ given $\{Y_i\}$. Since this is the minimal rate even for the case where $\{Y_i\}$ is known also to the encoder, then no rate loss is incurred by the lack of knowledge of $\{Y_i\}$ at the encoder. Early research has focused on asymptotic analysis of the decoding error probability for the ensemble of random binning codes. Gallager [14] has adapted his well known analysis techniques from random channel coding [13, Sections 5.5-5.6] to the random binning ensemble of distributed source coding. Later, it was shown in [9] and [6] that the universal *minimum entropy* decoder also achieves the same exponent. Expurgated error exponents were given in [7] assuming optimal decoding (non-universal). The expurgated exponent analysis was then generalized to coded side information in [5] (with linear codes) and [18].

More recently, a simple modification of the random binning scheme was suggested [3], [17][1]. The idea was that the encoder may inform the decoder about the type class of the source block using a short header, of negligible length. Then, a different code can be used for every type class, and in particular, different *rates* are possible. However, in both [3], [17] the average coding rate was the main concern. Since the empirical probability mass function (PMF) of the source tends to concentrate around its true PMF, then in essence, asymptotically, it is only necessary that the rate constraint will be satisfied for the true type class of the source (see [3, Thm. 1]). The average rate constraint will continue to be satisfied even if the rate for other types, distant from the true source type, is arbitrarily large. Naturally, one can increase the rates of these types up to a point in which any further increase does not improve the decoding error exponent.

This motivates us to take a somewhat different approach and address a more refined figure of merit for the rate. Specifically, we will be interested in the probability that the rate exceeds a certain threshold. Indeed, consider an online compression scheme, in which the codeword is buffered at the encoder before transmitted. If the instantaneous codeword length is larger than the buffer size, then the buffer will overflow. If the decoder is aware of this event (using a dedicated feed-forward channel, e.g.) then this is an *erasure* event. Thus, it is desirable to minimize this probability, while maintaining some given error probability. In a different case, the buffer length might be larger than the maximal codeword length, but the buffer is also used for other purposes (e.g. sending status data). If the data codewords have priority over all other uses, then it is desirable to minimize the occasions of blocking other usage of the buffer.

In this paper, we analyze the trade-off between excess rate probability and the error probability in the asymptotic regime of large block-length using error exponents. We assume the standard ensemble of

[1]In [15], the idea of sending the decoder the source type class and then using different codes for each type class was also suggested. However, [15] assumes fixed coding rate, and seeks improved error exponents for sources allowing zero-error decoding.

random binning and that source blocks from the same type class have the same coding rate. This allows us to find the exact excess rate and error exponents for any given allocation of rate. Then, for every type class, the minimal encoding rate, required to meet a prescribed value of error exponent, is found. The resulting excess rate performance of the system may then be evaluated. Since calculation of both the rate for a given type, and the excess rate exponent, lead to optimization problems without a closed-form solution, we provide alternating minimization algorithms that converge to the optimal solution.

We leave it as an open question whether allocating identical rates to all source blocks in the same type class provides the optimal trade-off between the excess rate probability and error probability.

The outline of the remaining part of the paper is as follows. In Section II, we establish notation conventions and define an ensemble of type-dependent variable-length codes. In Section III, we evaluate the exact random binning error exponent as well as the excess rate exponent for a given rate allocation for source types. Then, in Section IV, we characterize the optimal rate allocation (in a sense that will be made precise), under an average error exponent constraint. An alternating minimization algorithm is suggested for the calculation of the optimal rate, for any given source type class. In Section V, the excess rate exponent is characterized for the optimal rate function and another alternating minimization algorithm is suggested for the excess rate exponent calculation. Finally, Section VI demonstrates the results via a numerical example.

II. Problem Formulation

A. Notation Conventions

Throughout the paper, random variables will be denoted by capital letters, specific values they may take will be denoted by the corresponding lower case letters, and their alphabets will be denoted by calligraphic letters. Random vectors and their realizations will be denoted, respectively, by capital letters and the corresponding lower case letters, both in the bold face font. Their alphabets will be subscripted by their dimensions. For example, the random vector $\mathbf{X} = (X_1, \ldots, X_n)$, ($n$ positive integer) may take a specific vector value $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathcal{X}^n$, the nth order Cartesian power of $\mathcal{X}$, which is the alphabet of each component of this vector.

The source to be compressed will be denoted by the letter P, subscripted by the names of the relevant random variables/vectors and their conditionings, if applicable. We will follow the standard notation conventions, e.g., $P_X(x)$, $P_{Y|X}(y|x)$, $P_{XY}(x, y)$ and so on. The arguments will be omitted when we address the entire PMF, e.g. P_X, $P_{Y|X}$ and P_{XY}. Similarly, generic sources will be denoted by Q, $\tilde{Q}$, Q^*, and in other forms, again subscripted by the relevant random variables/vectors/conditioning. An exceptional case will be 'hat' notation. For this notation, $\hat{Q}_{\mathbf{x}}$ will denote the empirical distribution of a sequence $\mathbf{x} \in \mathcal{X}^n$, i.e., the vector of relative frequencies $\hat{Q}_{\mathbf{x}}(x)$ of each symbol $x \in \mathcal{X}$ in $\mathbf{x}$. The type class of $\mathbf{x} \in \mathcal{X}^n$, which will be denoted by $\mathcal{T}(\mathbf{x})$, is the set of all vectors $\mathbf{x}'$ with $\hat{Q}_{\mathbf{x}'} = \hat{Q}_{\mathbf{x}}$. When we wish to emphasize the dependence of the type class on the empirical distribution $\hat{Q}$, we will denote it by $\mathcal{T}(\hat{Q})$. Similarly, the empirical distribution of a pair of sequences $(\mathbf{x}, \mathbf{y})$ will be denoted by $\hat{Q}_{\mathbf{xy}}$ and the joint type class will be denoted by $\mathcal{T}(\mathbf{x}, \mathbf{y})$. The empirical conditional distribution induced by $(\mathbf{x}, \mathbf{y})$ will be denoted by $\hat{Q}_{\mathbf{x}|\mathbf{y}}(x|y)$, and the conditional type class, namely, the set $\{\mathbf{x}' : \hat{Q}_{\mathbf{x}'\mathbf{y}} = \hat{Q}_{\mathbf{xy}}\}$, will be denoted by $\mathcal{T}(\mathbf{x}|\mathbf{y})$. The set of all type classes of sequences of length n over $\mathcal{X}$ will be denoted by $\mathcal{P}_n(\mathcal{X})$, and the joint type classes over the Cartesian product alphabet $\mathcal{X} \times \mathcal{Y}$ will be denoted by $\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$. The probability simplex for $\mathcal{X}$ will be denoted by $\mathcal{S}(\mathcal{X})$, and the simplex for the alphabet $\mathcal{X} \times \mathcal{Y}$ will be denoted by $\mathcal{S}(\mathcal{X} \times \mathcal{Y})$. The entropy of the PMF Q will be denoted by $H(Q)$. The average conditional entropy of $Q_{Y|X}$ with respect to (w.r.t.) Q_X will be denoted by $H(Q_{Y|X}|Q_X) \triangleq \sum_{x \in \mathcal{X}} Q_X(x) H(Q_{Y|X}(\cdot|x))$. The information divergence between two PMFs P and Q will be denoted by $D(P||Q)$ and the average divergence between $Q_{Y|X}$ and $P_{Y|X}$ w.r.t. Q_X will be denoted by $D(Q_{Y|X}||P_{Y|X}|Q_X) \triangleq \sum_{x \in \mathcal{X}} Q_X(x) D(Q_{Y|X}(\cdot|x)||P_{Y|X}(\cdot|x))$. In all the information measures above, the PMF may also be an empirical PMF, for example, $H(\hat{Q}_{\mathbf{x}})$, $D(\hat{Q}_{\mathbf{y}|\mathbf{x}}||P_{Y|X})$ and so on.

The complement of a set $\mathcal{A}$ will be denoted by $\mathcal{A}^c$, and $\overline{\mathcal{A}}$ will be its closure. $\mathbb{P}(\mathcal{A})$ will denote the probability of the event $\mathcal{A}$, and $\mathbb{I}(A)$ will denote the indicator function of this event. The expectation operator will be denoted by $\mathbb{E}_{P_X}[\cdot]$ where, again, the subscript will be omitted if the underlying probability distribution is clear from the context.

The support of a PMF Q_X will be denoted by $\text{supp}(Q_X) \triangleq \{x : Q_X(x) \neq 0\} \subseteq \mathcal{X}$. For two positive sequences, $\{a_n\}$ and $\{b_n\}$ the notation $a_n \doteq b_n$ will mean asymptotic equivalence in the exponential scale, that is, $\lim_{n\to\infty} \frac{1}{n} \log(\frac{a_n}{b_n}) = 0$. Similarly, $a_n \dot{\leq} b_n$ will mean $\limsup_{n\to\infty} \frac{1}{n} \log(\frac{a_n}{b_n}) \leq 0$, and so on. The function $[t]_+$ will be defined as $\max\{t, 0\}$. Logarithms and exponents will be understood to be taken to the natural base, thus we will use *nats* for descriptive purposes.

B. System Model

Let $\{(X_i, Y_i)\}_{i=1}^n$ be n independent copies of a pair of random variables (X, Y). We assume that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite alphabets, are distributed according to a given PMF

$P_{XY}(x,y) = \mathbb{P}(X = x, Y = y)$. It is assumed that $\text{supp}(P_X) = \mathcal{X}$ and that $\text{supp}(P_Y) = \mathcal{Y}$, otherwise, remove the irrelevant letters from their alphabet.

A Slepian-wolf (SW) code $\mathcal{C}_n$ for sequences of length n is defined by an encoder

$$f_n : \mathcal{X}^n \to \{0,1\}^* \tag{1}$$

and a decoder

$$g_n : \{0,1\}^* \times \mathcal{Y}^n \to \mathcal{X}^n \tag{2}$$

where $\{0,1\}^*$ is the set of all finite length binary strings. The encoder maps a source block $\mathbf{x}$ into a binary string $f_n(\mathbf{x}) \in \{0,1\}^*$, and the decoder uses both $\mathbf{y}$ and $f_n(\mathbf{x})$ to obtain a decoded source block $\hat{\mathbf{x}} = g(f(\mathbf{x}), \mathbf{y})$. For $b \in \{0,1\}^*$, the inverse image of f_n is defined as

$$f_n^{-1}(b) = \{\mathbf{x} \in \mathcal{X}^n : f_n(\mathbf{x}) = b\}. \tag{3}$$

A random ensemble of SW codes is defined by a random sequence of encoders-decoders (F_n, G_n) with probability $\mathbb{P}(F_n = f_n, G_n = g_n)$. The error probability for a given code $\mathcal{C}_n$ is denoted by $e_n(\mathcal{C}_n) = \mathbb{P}(\hat{\mathbf{X}} \neq \mathbf{X})$, and the average error probability over the random ensemble of codes is defined as $\bar{e}_n = \mathbb{E}\left[e_n(\mathcal{C}_n)\right]$. The *error probability random-binning exponent* is defined as

$$\bar{E}_e \triangleq \liminf_{n\to\infty} -\frac{1}{n} \log \bar{e}_n. \tag{4}$$

The rate of $\mathbf{x} \in \mathcal{X}^n$ is defined as $r(\mathbf{x}) \triangleq \frac{|f_n(\mathbf{x})|}{n}$ where $|f_n(\mathbf{x})|$ is the length of $f_n(\mathbf{x})$. For a given target rate R, we define the *excess rate exponent function* as

$$E_r(\mathsf{R}) \triangleq \liminf_{n\to\infty} -\frac{1}{n} \cdot \log \mathbb{P}\left\{r(\mathbf{X}) \geq \mathsf{R}\right\}. \tag{5}$$

A variable-rate code is termed *type-dependent variable-length* code, if $r(\mathbf{x})$ depends on $\mathbf{x}$ only via its empirical PMF (type class). Namely, for all $\mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{X}^n$, and all n, when $\hat{Q}_{\mathbf{x}} = \hat{Q}_{\tilde{\mathbf{x}}}$ then $r(\mathbf{x}) = r(\tilde{\mathbf{x}})$. Throughout, we disregard integer codeword length constraints, as they have negligible effect on the rate, for large n.

Definition 1. Any finite function $R(Q_X) : \mathcal{S}(\mathcal{X}) \to \mathbb{R}^+$ is termed a *rate function*. A rate function is called *regular* if there exists a constant $d > 0$ and a set $\mathcal{V} \triangleq \{Q_X \in \mathcal{S}(\mathcal{X}) : D(Q_X || P_X) < d\}$, such that

$R(Q_X)$ is continuous in $\mathcal{V}$, and equals a constant R_0 for $Q_X \in \mathcal{V}^c$.

We analyze the ensemble performance of type-dependent, variable-length SW codes, that are defined as follows:

- *Codebook generation:* For a given rate function $R(Q_X)$, generate $e^{nR(Q_X)}$ bins for every $Q_X \in \mathcal{P}_n(\mathcal{X})$ and map each bin into a different binary string of length $n \cdot R(Q_X)$ nats. Next, assign to each $\mathbf{x} \in \mathcal{X}^n$ a bin by independent random selection with a uniform distribution over all the bins of type class $\mathcal{T}(\mathbf{x})$. Then, assign an index to each type class $Q_X \in \mathcal{P}_n(\mathcal{X})$. The above data is revealed to both the encoder and the decoder offline.
- *Encoding:* Upon observing $\mathbf{x}$, determine its type class $\mathcal{T}(\mathbf{x})$. Send to the decoder its type index, concatenated with its bin index (for the current type $\mathcal{T}(\mathbf{x})$).
- *Decoding:* First, recover the type class $\mathcal{T}(\mathbf{x})$ of $\mathbf{x}$. Then, we consider two options.
  - Maximum likelihood (ML) decoder: Choose $\tilde{\mathbf{x}} \in f_n^{-1}(f_n(\mathbf{x}))$ that maximizes $P_{\mathbf{X}|\mathbf{Y}}(\tilde{\mathbf{x}}|\mathbf{y})$. Since all $\tilde{\mathbf{x}} \in f_n^{-1}(f_n(\mathbf{x}))$ are in the same type class, they have the same probability $P_{\mathbf{X}}(\tilde{\mathbf{x}})$, this decoding rule is equivalent to maximizing $P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\tilde{\mathbf{x}})$.
  - Minimum conditional entropy (MCE) decoder: Choose $\tilde{\mathbf{x}} \in f_n^{-1}(f_n(\mathbf{x}))$ that minimizes $H(\hat{Q}_{\tilde{\mathbf{x}}|\mathbf{y}}|\hat{Q}_{\mathbf{y}})$. Since all $\tilde{\mathbf{x}} \in f_n^{-1}(f_n(\mathbf{x}))$ have the same empirical entropy $H(\hat{Q}_{\tilde{\mathbf{x}}})$, this decoding rule is equivalent to well-known, maximum mutual information (MMI) decoder (see, e.g., [16, Section IV.B]).

III. Exponents Analysis

It is well known that the ML decoder, which depends on the source statistics $P_{\mathbf{XY}}$, minimizes the error probability. By contrast, the MCE decoder does not use $P_{\mathbf{XY}}$ at all. In the next theorem, we evaluate the random binning error exponent of the ML decoder, and show that the MCE decoder also achieves the same exponent, and thus it is a *universal* decoder. This exponent was initially derived in [6] (for both decoders), but the proof here is simpler, and also shows that the lower bound on the ML error exponent is tight, for all rates.

Theorem 2. *Let $R(Q_X)$ be a given rate function, and let the ensemble of SW codes be as defined in Section II-B. Then for both the ML decoder and the MCE decoder, the limit in* (4) *exists and equals*

$$\bar{E}_e = \min_{Q_{XY}} D\left(Q_{XY}||P_{XY}\right) + \left[R(Q_X) - H(Q_{X|Y}|Q_Y)\right]_+ . \tag{6}$$

Proof: Suppose that $(\mathbf{x},\mathbf{y})$ was emitted from the source and its joint type is $Q_{XY}=\hat{Q}_{\mathbf{xy}}$. Let the marginals and conditional types be $Q_X=\hat{Q}_{\mathbf{x}}$, $Q_Y=\hat{Q}_{\mathbf{y}}$, and $Q_{X|Y}=\hat{Q}_{\mathbf{x}|\mathbf{y}}$.

For the ML decoder, let

$$S_o(\mathbf{x},\mathbf{y}) \triangleq \left\{\tilde{\mathbf{x}}\in\mathcal{X}^n : P_{\mathbf{X}|\mathbf{Y}}(\tilde{\mathbf{x}}|\mathbf{y}) \geq P_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})\right\}. \tag{7}$$

The conditional error probability, averaged over the random choice of binning is

$$\begin{aligned}
\bar{\Pi}_{e,o}(\mathbf{x},\mathbf{y}) &\triangleq \mathbb{P}\left\{\bigcup_{\tilde{\mathbf{x}}\in\{S_o(\mathbf{x},\mathbf{y})\cap\mathcal{T}(\mathbf{x})\}} F_n(\tilde{\mathbf{x}})=F_n(\mathbf{x})\right\} &(8)\\
&\geq \frac{1}{2}\min\left\{1, e^{-nR(Q_X)}\cdot|S_o(\mathbf{x},\mathbf{y})\cap\mathcal{T}(\mathbf{x})|\right\} &(9)\\
&\geq \frac{1}{2}\min\left\{1, e^{-nR(Q_X)}\cdot|\mathcal{T}(\mathbf{x}|\mathbf{y})|\right\} &(10)\\
&\doteq \min\left\{1, \exp\left[-n\left(R(Q_X)-H(Q_{X|Y}|Q_Y)\right)\right]\right\} &(11)\\
&= \exp\left[-n\left[R(Q_X)-H(Q_{X|Y}|Q_Y)\right]_+\right] &(12)
\end{aligned}$$

where the first inequality is due to Lemma 19 in Appendix C, and the fact that the bin indices are drawn independently in a given type class, and the second inequality is because for any pair $(\tilde{\mathbf{x}},\mathbf{y})\in\mathcal{T}(\mathbf{x}|\mathbf{y})$, we have that $\tilde{\mathbf{x}}\in\mathcal{T}(\mathbf{x})$ and that $P_{\mathbf{X}|\mathbf{Y}}(\tilde{\mathbf{x}}|\mathbf{y})=P_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})$.

For the MCE decoder, let

$$S_u(\mathbf{x},\mathbf{y}) \triangleq \left\{\tilde{\mathbf{x}}\in\mathcal{X}^n : H\left(\hat{Q}_{\tilde{\mathbf{x}}|\mathbf{y}}|Q_Y\right) \leq H\left(Q_{X|Y}|Q_Y\right)\right\}. \tag{13}$$

Similarly,

$$\begin{aligned}
\bar{\Pi}_{e,u}(\mathbf{x},\mathbf{y}) &\triangleq \mathbb{P}\left\{\bigcup_{\tilde{\mathbf{x}}\in\{S_u(\mathbf{x},\mathbf{y})\cap\mathcal{T}(\mathbf{x})\}} F_n(\tilde{\mathbf{x}})=F_n(\mathbf{x})\right\} &(14)\\
&\leq \min\left\{1, e^{-nR(Q_X)}\cdot|S_u(\mathbf{x},\mathbf{y})\cap\mathcal{T}(\mathbf{x})|\right\} &(15)\\
&\leq \min\left\{1, e^{-nR(Q_X)}\cdot|S_u(\mathbf{x},\mathbf{y})|\right\} &(16)\\
&\stackrel{\cdot}{\leq} \min\left\{1, \exp\left[-n\left(R(Q_X)-H\left(Q_{X|Y}|Q_Y\right)\right)\right]\right\} &(17)\\
&= \exp\left[-n\left[R(Q_X)-H\left(Q_{X|Y}|Q_Y\right)\right]_+\right], &(18)
\end{aligned}$$

where the first inequality is by the union bound, and the following equality is because the number of sequences in any conditional type that belongs to $S_u(\mathbf{x},\mathbf{y})$ is exponentially upper bounded by $e^{nH(Q_{X|Y}|Q_Y)}$

and the number of joint types is polynomial $|\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})|\leq(n+1)^{|\mathcal{X}||\mathcal{Y}|}$.

It can be seen that on the exponential scale, the lower bound on $\bar{\Pi}_{e,o}(\mathbf{x},\mathbf{y})$ and the upper bound on $\bar{\Pi}_{e,u}(\mathbf{x},\mathbf{y})$ are identical. Thus, when taking expectation w.r.t. the i.i.d. source P_{XY}, the resulting asymptotic bounds on the error probability are identical (lower bound for the ML decoder, and upper bound for the MCE decoder). Moreover, since the ML decoder minimizes the error probability, taking expectation w.r.t. P_{XY} we get

$$\begin{aligned}
\mathbb{E}\left\{\exp\left[-n\left[R(\hat{Q}_{\mathbf{X}})-H\left(\hat{Q}_{\mathbf{X}|\mathbf{Y}}|\hat{Q}_{\mathbf{Y}}\right)\right]_+\right]\right\} &\dot{\leq} \mathbb{E}\left\{\bar{\Pi}_{e,o}(\mathbf{X},\mathbf{Y})\right\} &(19)\\
&\leq \mathbb{E}\left\{\bar{\Pi}_{e,u}(\mathbf{X},\mathbf{Y})\right\} &(20)\\
&\dot{\leq} \mathbb{E}\left\{\exp\left[-n\left[R(\hat{Q}_{\mathbf{X}})-H\left(\hat{Q}_{\mathbf{X}|\mathbf{Y}}|\hat{Q}_{\mathbf{Y}}\right)\right]_+\right]\right\} &(21)
\end{aligned}$$

so the asymptotic average error probability of both the ML decoder and the MCE decoder is

$$\begin{aligned}
\bar{e}_n &\doteq \mathbb{E}\left\{\exp\left[-n\left[R(\hat{Q}_{\mathbf{X}})-H\left(\hat{Q}_{\mathbf{X}|\mathbf{Y}}|\hat{Q}_{\mathbf{Y}}\right)\right]_+\right]\right\} &(22)\\
&= \sum_{Q_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})}\mathbb{P}\left(\hat{Q}_{\mathbf{XY}}=Q_{XY}\right)\cdot\exp\left[-n\left[R(Q_X)-H\left(Q_{X|Y}|Q_Y\right)\right]_+\right] &(23)\\
&\doteq \sum_{Q_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})}\exp\left[-n\cdot D\left(Q_{XY}||P_{XY}\right)-n\left[R(Q_X)-H\left(Q_{X|Y}|Q_Y\right)\right]_+\right] &(24)\\
&\doteq \exp\left[-n\cdot\min_{Q_{XY}\in\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})}\left\{D\left(Q_{XY}||P_{XY}\right)+\left[R(Q_X)-H\left(Q_{X|Y}|Q_Y\right)\right]_+\right\}\right] &(25)
\end{aligned}$$

where the last inequality is again because $|\mathcal{P}_n(\mathcal{X}\times\mathcal{Y})|\leq(n+1)^{|\mathcal{X}||\mathcal{Y}|}$. Since the optimal value of the minimization problem inside the exponent is clearly finite, and the minimization argument is a continuous function, then

$$\bar{e}_n\doteq\exp\left[-n\cdot\min_{Q_{XY}}\left\{D\left(Q_{XY}||P_{XY}\right)+\left[R(Q_X)-H\left(Q_{X|Y}|Q_Y\right)\right]_+\right\}\right]. \tag{26}$$

■

Remark 3. The MCE decoder is equivalent to a decoder that estimates the unknown PMF P_{XY} for any candidate source block (generalized likelihood ratio test). Indeed, suppose that the candidate source block is $\mathbf{x}$, the side information is $\mathbf{y}$ and $(\mathbf{x},\mathbf{y})$ has the joint type Q_{XY}. The normalized log-likelihood is bounded as

$$
\begin{aligned}
\frac{1}{n}\log P_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) &= \tfrac{1}{n} \quad \log P_{\mathbf{XY}}(\mathbf{x},\mathbf{y}) - \frac{1}{n}\log P_{\mathbf{Y}}(\mathbf{y}) && (27)\\
&= \tfrac{1}{n} \quad \sum_{i=1}^{n} \log P_{XY}(x_i,y_i) - \frac{1}{n}\sum_{i=1}^{n}\log P_Y(y_i) && (28)\\
&= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} Q_{XY}(x,y)\cdot\log P_{XY}(x,y) - \sum_{y\in\mathcal{Y}} Q_Y(y)\log P_Y(y) && (29)\\
&= -H(Q_{XY}) - D(Q_{XY}||P_{XY}) + H(Q_Y) + D(Q_Y||P_Y) && (30)\\
&\leq -H(Q_{XY}) + H(Q_Y) && (31)\\
&= -H(Q_{X|Y}|Q_Y) && (32)
\end{aligned}
$$

and equality is achieved when choosing P_{XY} to be the ML estimate P_{ML}, which equals to the observed empirical PMF, i.e. $P_{ML} = Q_{XY} = \hat{Q}_{\mathbf{xy}}$. Indeed, the MCE decoder chooses the source block $\mathbf{x} \in \mathcal{X}^n$ that minimizes $H(Q_{X|Y}|Q_Y)$.

Next, we consider the excess rate exponent.

Theorem 4. *For a regular rate function* $R(Q_X)$

$$E_r(\mathsf{R}) = \min_{Q_X : R(Q_X)\geq \mathsf{R}} D(Q_X||P_X). \tag{33}$$

Proof: The proof is technical and relegated to Appendix B. ■

It should be mentioned, that the same techniques used in this section also apply to the more general case of SW coding, where the side information is also encoded. In this case, there are two encoders, f_n encoding $\mathbf{x}$ and f'_n encoding $\mathbf{y}$, while the central decoder g_n now uses both codewords $f_n(\mathbf{x})$ and $f'_n(\mathbf{y})$. Similar encodings may be defined via random binning, and a header including the type for each codeword. Now, two rate functions $R_X(Q_X)$ and $R_Y(Q_Y)$ may be defined accordingly. The resulting error exponent is given by

$$
\begin{aligned}
\bar{E}_e = \min_{Q_{XY}} \Big\{ & D\left(Q_{XY}||P_{XY}\right) + \Big[\min\{R_X(Q_X) - H(Q_{X|Y}|Q_Y), \\
& R_Y(Q_Y) - H(Q_{Y|X}|Q_X),\ R_X(Q_X) + R_Y(Q_Y) - H(Q_{XY})\}\Big]_+ \Big\}
\end{aligned} \tag{34}
$$

and the excess rate exponents clearly depend only on marginal types and rate functions. Nonetheless, due to the sum-rate term in the inner minimization, a trickle of coordination is required between the two encoders. Specifically, at least one of the encoders needs to know the current rate (or equivalently, the type class of the current source block) of the other encoder. Moreover, solving the optimization problems needed to determine the optimal rate (cf. next section) requires solving complicated equations. Thus, we do not pursue this direction here.

IV. Optimal Rate Functions

For the system described in Section II-B, a good choice of a rate function $R(Q_X)$ would achieve an error exponent requirement E_e using a minimal rate, uniformly for all Q_X. In this section, we define and characterize the optimal rate function, which achieves a specified target error exponent E_e, for a given source P_{XY}.

Definition 5. A rate function $R^*(Q_X, \mathsf{E}_e)$ is said to be optimal if it achieves an error exponent E_e, and for every other rate function $R(Q_X, \mathsf{E}_e)$ that achieves error exponent E_e, we have $R^*(Q_X, \mathsf{E}_e) \leq R(Q_X, \mathsf{E}_e)$, for all Q_X.

In the next theorem, we give an expression for the optimal rate function. Notice that $D(Q_X||P_X)$ is finite for any Q_X, since it was assumed that $\mathrm{supp}(P_X) = \mathcal{X}$.

Theorem 6. *The optimal rate function is*

$$R^*(Q_X, \mathsf{E}_e) = \begin{cases} 0, & \mathsf{E}_e \leq D(Q_X||P_X) \\ \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) & \\ - \min_{\tilde{Q}_Y} \min_{Q_{Y|X} \in \overline{\mathcal{A}}} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right)\right\}, & \textit{otherwise} \end{cases} \tag{35}$$

where

$$\mathcal{A} \triangleq \left\{Q_{Y|X} : D\left(Q_X \times Q_{Y|X}||P_{XY}\right) < \mathsf{E}_e\right\}. \tag{36}$$

Proof: Theorem 2 implies that error exponent of E_e is achieved if for all Q_{XY}

$$\mathsf{E}_e \leq D\left(Q_{XY}||P_{XY}\right) + \left[R(Q_X) - H(Q_{X|Y}|Q_Y)\right]_+. \tag{37}$$

First, notice that since $D\left(Q_{XY}||P_{XY}\right) = D(Q_X||P_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \geq D(Q_X||P_X)$, then if $D(Q_X||P_X) \geq \mathsf{E}_e$ no actual constraint is imposed on the rate, and (37) is satisfied even for $R(Q_X) = 0$. Thus, in this case $R^*(Q_X, \mathsf{E}_e) = 0$. Next, assume that $D(Q_X||P_X) < \mathsf{E}_e$. Similarly, for $Q_{Y|X} \in \mathcal{A}^c$, (37) is satisfied for $R(Q_X) = 0$. If $Q_{Y|X} \in \mathcal{A}$ then (37) can only be satisfied if $R(Q_X) > H(Q_{X|Y}|Q_Y)$. In this case, $\left[R - H(Q_{X|Y}|Q_Y)\right]_+ = R(Q_X) - H(Q_{X|Y}|Q_Y)$ and then (37) implies that to achieve E_e we must have

$$\mathsf{E}_e \leq D\left(Q_{XY}||P_{XY}\right) + R(Q_X) - H(Q_{X|Y}|Q_Y) \tag{38}$$

for all $Q_{Y|X} \in \mathcal{A}$. Thus

$$\begin{aligned} R^*(Q_X, \mathsf{E}_e) &= \mathsf{E}_e - \inf_{Q_{Y|X} \in \mathcal{A}} \left\{ D\left(Q_{XY}||P_{XY}\right) - H(Q_{X|Y}|Q_Y) \right\} \\ &= \mathsf{E}_e + H(Q_X) - \inf_{Q_{Y|X} \in \mathcal{A}} \left\{ D(Q_{XY}||Q_X \times Q_Y) + D\left(Q_{XY}||P_{XY}\right) \right\}. \end{aligned} \tag{39}$$

Introducing an auxiliary PMF $\tilde{Q}_Y$ and using Lemma 22 (Appendix C)

$$\begin{aligned} R^*(Q_X, \mathsf{E}_e) &= \mathsf{E}_e + H(Q_X) - \min_{\tilde{Q}_Y} \inf_{Q_{Y|X} \in \mathcal{A}} \left\{ D(Q_{XY}||Q_X \times \tilde{Q}_Y) + D\left(Q_{XY}||P_{XY}\right) \right\} \\ &= \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) \qquad (40) \\ &\quad - \min_{\tilde{Q}_Y} \inf_{Q_{Y|X} \in \mathcal{A}} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \right\} \\ &= \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) \qquad (41) \\ &\quad - \min_{\tilde{Q}_Y} \min_{Q_{Y|X} \in \overline{\mathcal{A}}} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \right\} \end{aligned}$$

where the last equality is because the value of the inner minimum is finite (as the divergences are non-negative). ■

The following theorem provides several properties of the optimal rate function $R^*(Q_X, \mathsf{E}_e)$.

Theorem 7. *The optimal rate function $R^*(Q_X, \mathsf{E}_e)$ has the following properties:*

1) *Let $\mathsf{E}_{e,0} \triangleq D(Q_X||P_X)$. Then, $R^*(Q_X, \mathsf{E}_e) = 0$ for $\mathsf{E}_e \leq \mathsf{E}_{e,0}$ and $R^*(Q_X, \mathsf{E}_e) > 0$ for $\mathsf{E}_e > \mathsf{E}_{e,0}$.*
2) *$R^*(Q_X, \mathsf{E}_e)$ is strictly increasing for $\mathsf{E}_e \in (\mathsf{E}_{e,0}, \infty)$.*
3) *Let $(\tilde{Q}'_Y, Q'_{Y|X}) \triangleq \arg\min_{(\tilde{Q}_Y, Q_{Y|X})} \{D(Q_{Y|X}||\tilde{Q}_Y|Q_X) + D(Q_{Y|X}||P_{Y|X}|Q_X)\}$. Then, $R^*(Q_X, \mathsf{E}_e)$ is affine with slope 1 for $\mathsf{E}_e \in (\mathsf{E}_{e,a}, \infty)$ where $\mathsf{E}_{e,a} \triangleq D(Q_X||P_X) + D(Q'_{Y|X}||P_{Y|X}|Q_X)$.*
4) *$R^*(Q_X, \mathsf{E}_e)$ is a concave function for $\mathsf{E}_e \in (\mathsf{E}_{e,0}, \infty)$.*
5) *$R^*(Q_X, \mathsf{E}_e)$ is a regular rate function.*

Proof: See Appendix B. ■

It is evident from Theorem 6 that in order to calculate $R^*(Q_X, \mathsf{E}_e)$, one needs to solve the following

Algorithm 1 Iterative alternating minimization algorithm for the solution of (42)

Input: A source P_{XY}, a type class Q_X, and a target error exponent E_e.

Output: The value of the optimization problem (42) and the optimal solution $(Q^*_{Y|X}, \tilde{Q}^*_Y)$.

1) Initialize $\tilde{Q}_Y$ randomly such that $\operatorname{supp}(\tilde{Q}_Y) = \operatorname{supp}(\sum_{x\in\mathcal{X}} Q_X P_{Y|X})$.
2) Iterate over the following steps until convergence:
   a) If $D(\overline{Q}^{1/2}_{Y|X}(y|x)||P_{Y|X}|Q_X) < \mathsf{E}_e - D(Q_X||P_X)$ then set $\alpha = 1/2$. Else, use the bisection method to find $\alpha \in [1/2, 1]$ that satisfies
   $$D\left(\overline{Q}^{\alpha}_{Y|X}||P_{Y|X}|Q_X\right) = \mathsf{E}_e - D(Q_X||P_X). \tag{44}$$
   b) Set $\tilde{Q}_Y(y) = \sum_{x\in\mathcal{X}} Q_X(x)\overline{Q}^{\alpha}_{Y|X}(y|x)$.
3) Let the converged variables be α^* and $\tilde{Q}^*_Y$. Then, set $(Q_{Y|X}, \tilde{Q}_Y) = (\overline{Q}^{\alpha^*}_{Y|X}, \tilde{Q}^*_Y)$ in (42). Return.

optimization problem:

$$\min_{\tilde{Q}_Y} \min_{Q_{Y|X}\in\overline{\mathcal{A}}} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right)\right\}. \tag{42}$$

Algorithm 1 describes an iterative minimization algorithm for solving this optimization problem, and the next lemma asserts its correctness. For a given $0 \le \alpha \le 1$ and $\tilde{Q}_Y$, we shall use the following definition for a conditional PMF $\overline{Q}^{\alpha}_{Y|X}$

$$\overline{Q}^{\alpha}_{Y|X}(y|x) \triangleq \psi_x P^{\alpha}_{Y|X}(y|x)\tilde{Q}^{1-\alpha}_Y(y), \tag{43}$$

where ψ_x is a normalization factor, such that $\sum_{y\in\mathcal{Y}} \overline{Q}^{\alpha}_{Y|X}(y|x) = 1$ for all $x \in \mathcal{X}$. In Algorithm 1 (and other algorithms in this paper), the *bisection method* (also called *binary search)* [12, Chapter 9.1] is used to find the root of a monotonic continuous function f in an interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs. In any iteration of this well-known method, the current interval is halved, and based on the sign of f in the middle of the interval, the root is known to belong to either the left or right half-interval. The bisection method always converges, but obviously, any other appropriate root finding method (that perhaps converges faster) may be used.

Lemma 8. *Algorithm 1 outputs the minimal value and the optimal solution of* (42).

Proof: See Appendix B. ■

Remark 9. The right limit of the optimal rate function $R^*(Q_X, \mathsf{E}_e)$ at its discontinuity point $\mathsf{E}_{e,0}$ can be easily evaluated. As $\mathsf{E}_e \downarrow \mathsf{E}_{e,0}$ we have $\mathcal{A} \downarrow \{P_{Y|X}\}$ and there is no need to optimize over $Q_{Y|X}$. The

Algorithm 2 Computation of optimal rate function $R^*(Q_X, \mathsf{E}_e)$

Input: A source P_{XY}, a type class Q_X, and a target error exponent E_e.

Output: The value of the optimal rate function $R^*(Q_X, \mathsf{E}_e)$.

1) Set $\mathsf{E}_{e,0} = D(Q_X||P_X)$.
2) Find $(\tilde{Q}'_Y, Q'_{Y|X}) \triangleq \arg\min_{(\tilde{Q}_Y, Q_{Y|X})}\{D(Q_{Y|X}||\tilde{Q}_Y|Q_X) + D(Q_{Y|X}||P_{Y|X}|Q_X)\}$ using the modification of Algorithm 1 suggested in Lemma 10, and find $\mathsf{E}_{e,a}$ from Theorem 7 part 3.
3) If $\mathsf{E}_e \leq \mathsf{E}_{e,0}$ then set $R^*(Q_X, \mathsf{E}_e) = 0$. Return.
   If $\mathsf{E}_{e,0} < \mathsf{E}_e \leq \mathsf{E}_{e,a}$ then find α^* and $(\overline{Q}^{\alpha^*}_{Y|X}, \tilde{Q}^*_Y)$ using Algorithm 1 and set
$$R^*(Q_X, \mathsf{E}_e) = H(Q_X) - D\left(\overline{Q}^{\alpha^*}_{Y|X}||P_{Y|X}|Q_X\right). \tag{47}$$
   Return.
   If $\mathsf{E}_{e,a} < \mathsf{E}_e$ set
$$R^*(Q_X, \mathsf{E}_e) = \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) + 2 \cdot \sum_{x\in\mathcal{X}} Q_X(x) \log\left(\sum_{y\in\mathcal{Y}} \sqrt{P_{Y|X}(y|x)\tilde{Q}'_Y(y)}\right). \tag{48}$$
   Return.

optimal $\tilde{Q}_Y$ is simply (see the proof of Lemma 8 in Appendix B)

$$\tilde{Q}^*_Y(y) = \sum_{x\in\mathcal{X}} Q_X(x)P_{Y|X}(y|x) \tag{45}$$

and from (41) the resulting rate is

$$\lim_{\mathsf{E}_e \downarrow \mathsf{E}_{e,0}} R^*(Q_X, \mathsf{E}_e) = H(Q_X) - D(P_{Y|X}||\tilde{Q}^*_Y|Q_X). \tag{46}$$

Namely, the resulting rate is the conditional entropy $H(Q_{X|Y}|Q_Y)$ of the PMF $Q_{XY} = Q_X \times P_{Y|X}$. Especially, for $Q_X = P_X$ we have that $R^*(P_X, \epsilon) \geq H(P_{X|Y}|P_Y)$, for all $\epsilon > 0$, as expected.

Notice that Theorem 7, part 3 implies that for $\mathsf{E}_e > \mathsf{E}_{e,a}$, the optimal rate function is an affine function of E_e with slope 1. Thus, for $\mathsf{E}_e > \mathsf{E}_{e,a}$ a modification of Algorithm 1 can be used to find the optimal rate function in a simpler way. The next Lemma states how to find the optimal optimal solution $(Q^*_{Y|X}, \tilde{Q}^*_Y)$ for this case, and then the resulting procedure for calculating the optimal rate function $R^*(Q_X, \mathsf{E}_e)$, for all possible cases of E_e, is summarized in Algorithm 2 . The correctness of this algorithm is asserted in Theorem 11.

Lemma 10. *For $\mathsf{E}_e > \mathsf{E}_{e,a}$, Algorithm 1 converges to the optimal solution $(\overline{Q}^{1/2}_{Y|X}, \tilde{Q}^*_Y)$ if in step 2a we always set $\alpha = 1/2$.*

Proof: See Appendix B. ■

Theorem 11. *Algorithm 2 returns* $R^*(Q_X, \mathsf{E}_e)$.

Proof: See Appendix B. ■

In Appendix A, we provide two examples for optimal rate functions. Example 17 shows that even in the simple case of a double binary source $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, where $P_{Y|X}$ is a binary symmetric channel, the calculation of the optimal rate function requires the solution of a non-trivial numerical equation. This emphasizes the importance of Algorithm 1. Nonetheless, in Example 18 it is shown that for very weakly correlated sources, an analytic approximation for the optimal rate function is indeed possible, and the resulting expression is provided.

V. Excess Rate Performance

In this section, we evaluate the excess rate exponent for the optimal rate function $R^*(Q_X, \mathsf{E}_e)$, which we denote by $E_r^*(\mathsf{R})$. Since $R^*(Q_X, \mathsf{E}_e)$ is a regular rate function (Theorem 7 part 5), then in essence, Theorem 4 can be used. However, as we have seen, $R^*(Q_X, \mathsf{E}_e)$ is not given analytically, and performing the maximization in (33) directly may be prohibitively complex, especially when $|\mathcal{X}|$ is large. Thus, in this section, we describe an indirect method to evaluate $E_r^*(\mathsf{R})$. Throughout, it is assumed that E_e is given and fixed.

For a given R, the curve $E_r^*(\mathsf{R})$ may be characterized by a condition that verifies whether $(\mathsf{R}, \mathsf{E}_r)$ is either below or above the curve. While verifying such a condition is still difficult for $R^*(Q_X, \mathsf{E}_e)$, it can nonetheless be verified for a surrogate rate function $\hat{R}(Q_X; \mathsf{R}, \mathsf{E}_r)$ which has simpler structure, but also attains $(\mathsf{R}, \mathsf{E}_r)$. Before going into details we mention few properties of $E_r^*(\mathsf{R})$.

Theorem 12. *Let* $R(Q_X)$ *be a rate function, and* $\mathsf{R}_{\max} \triangleq \max_{Q_X} R(Q_X)$. *Also, if* $R(Q_X)$ *is regular then let* $\mathsf{R}'_{\max} = \sup_{Q_X \in \mathcal{V}} R(Q_X)$. *The excess rate exponent* $E_r(\mathsf{R})$ *for the rate function* $R(Q_X)$ *has the following properties:*

1) $E_r(\mathsf{R}) = 0$ *for* $\mathsf{R} \in [0, R(P_X)]$.
2) $E_r(\mathsf{R}) = \infty$ *for* $\mathsf{R} \in (\mathsf{R}_{\max}, \infty)$.
3) $E_r(\mathsf{R})$ *is an increasing function of* R *in* $[R(P_X), \mathsf{R}_{\max}]$. *If* $R(Q_X)$ *is regular, then* $R(Q_X)$ *is strictly increasing for* $[R(P_X), \mathsf{R}'_{\max}]$.
4) $E_r(\mathsf{R})$ *is a continuous function of* R *almost everywhere in* $[R(P_X), \mathsf{R}_{\max}]$. *If* $R(Q_X)$ *is regular, then* $E_r(\mathsf{R})$ *is left-continuous for* $[R(P_X), \mathsf{R}'_{\max}]$.

Proof: See Appendix B. ■

Now, for a given R we define the aforementioned surrogate rate function as[2]

$$\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r) \triangleq \begin{cases} \mathsf{R}, & D(Q_X||P_X) < \mathsf{E}_r \\ \mathsf{R}_0, & \text{otherwise} \end{cases}. \tag{49}$$

The exact value of R_0 is immaterial, as if it is chosen properly, it affects neither the error exponent nor the excess rate exponent. A precise condition on R_0 will be stated in the proof of Proposition 14 that will appear in what follows. For now, we require that at least

$$\mathsf{R}_0 \geq \max_{Q_X:D(Q_X||P_X)\geq \mathsf{E}_r} R\left(Q_X,\mathsf{E}_e\right). \tag{50}$$

Notice that like the optimal rate function, $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ is also a regular rate function. The next lemma shows that $(\mathsf{R},\mathsf{E}_r)$ is achieved simultaneously for both rate functions, $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ and $R^*(Q_X,\mathsf{E}_e)$.

Lemma 13. *The optimal rate function $R^*(Q_X,\mathsf{E}_e)$ achieves the pair $(\mathsf{R},\mathsf{E}_r)$ with an error exponent E_e iff the rate function $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ also achieves $(\mathsf{R},\mathsf{E}_r)$ with an error exponent E_e.*

Proof: We show that $(\mathsf{R},\mathsf{E}_r)$ is achievable by both rate functions simultaneously.

($\Leftarrow$) Assume that $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ achieves $(\mathsf{R},\mathsf{E}_r)$ with an error exponent E_e. Clearly the definition of *optimal* rate functions imply that $R^*(Q_X,\mathsf{E}_e)$ also achieves $(\mathsf{R},\mathsf{E}_r)$.

($\Rightarrow$) Assume that $R^*(Q_X,\mathsf{E}_e)$ achieves $(\mathsf{R},\mathsf{E}_r)$. If Q_X satisfies $D(Q_X||P_X)\geq\mathsf{E}_r$ then the condition on R_0 implies that $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)\geq R^*(Q_X,\mathsf{E}_e)$. Else, if $R^*(Q_X,\mathsf{E}_e) > \mathsf{R}$ for some Q_X that satisfies $D(Q_X||P_X) < \mathsf{E}_r$, then $R^*(Q_X,\mathsf{E}_e)$ does not achieve $(\mathsf{R},\mathsf{E}_r)$ using (33). Thus, we must have $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)\geq R^*(Q_X,\mathsf{E}_e)$ for all Q_X and this implies that $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ also achieves error exponent E_e. It is easy to see that $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ satisfies the large deviation constraint $(\mathsf{R},\mathsf{E}_r)$ directly from its construction and (33). ■

Thus, for any given $(\mathsf{R},\mathsf{E}_r)$ we may construct the proper $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$, and check if $(\mathsf{R},\mathsf{E}_r)$ is below or above the curve $E_r^*(\mathsf{R})$, using $\hat{R}(Q_X;\mathsf{R},\mathsf{E}_r)$ instead of $R^*(Q_X,\mathsf{E}_e)$. The following proposition states a proper condition.

[2]Notice that the second argument in this rate function is a target rate and excess rate exponent, unlike the optimal rate function $R^*(Q_X,\mathsf{E}_e)$ in which the second argument is a target error exponent.

Proposition 14. *Let*

$$\begin{aligned}\Gamma\left(t, Q_X, Q_{Y|X}, \tilde{Q}_Y\right) \ \triangleq\ & D(Q_X||P_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \\ & + t\cdot\left(\mathsf{R} - H(Q_X) + D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right)\right)\end{aligned} \tag{51}$$

and

$$\upsilon(t) \triangleq \min_{Q_X : D(Q_X||P_X)\le \mathsf{E}_r} \min_{\tilde{Q}_Y} \min_{Q_{Y|X}} \Gamma\left(t, Q_X, Q_{Y|X}, \tilde{Q}_Y\right). \tag{52}$$

Then $(\mathsf{R}, \mathsf{E}_r)$ *is achievable for the exponent* E_e *iff* $\max_{0\le t\le 1}\upsilon(t) \ge \mathsf{E}_e$.

Proof: The sufficient and necessary condition on $\hat{R}(Q_X; \mathsf{R}, \mathsf{E}_r)$ to achieve E_e in (37) may be restated as

$$\mathsf{E}_e \le \min_{Q_{XY}} \max_{0\le t\le 1} \left\{ D\left(Q_{XY}||P_{XY}\right) + t\cdot\left(\hat{R}(Q_X; \mathsf{R}) - H(Q_{X|Y}|Q_Y)\right)\right\}. \tag{53}$$

For Q_X that satisfies $D(Q_X||P_X) \ge \mathsf{E}_r$, we can choose large enough R_0 that satisfies this requirement. Thus we are left with

$$\mathsf{E}_e \le \min_{Q_X : D(Q_X||P_X)<\mathsf{E}_r} \min_{Q_{Y|X}} \max_{0\le t\le 1} \left\{ D\left(Q_{XY}||P_{XY}\right) + t\cdot\left(\mathsf{R} - H(Q_{X|Y}|Q_Y)\right)\right\} \tag{54}$$

and using Lemma 22, this is equivalent to

$$\begin{aligned}\mathsf{E}_e \ \le\ & \min_{Q_X : D(Q_X||P_X)<\mathsf{E}_r} \min_{\tilde{Q}_Y} \min_{Q_{Y|X}} \max_{0\le t\le 1} \Big\{ D(Q_X||P_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \\ & + t\cdot\left[\mathsf{R} - H(Q_X) + D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right)\right]\Big\}\end{aligned} \tag{55}$$

The minimization problem in (55) is jointly convex in Q_X (over the convex set $\{Q_X \in \mathcal{S}(\mathcal{X}) : D(Q_X||P_X) < \mathsf{E}_r\}$), $\{Q_{Y|X}\}$ and $\tilde{Q}_Y$. The maximization problem is linear in t (over the convex set $[0,1]$), and thus trivially concave. Therefore, we can interchange the maximization and minimizations [21] order to obtain the resulting condition $\max_{0\le t\le 1}\upsilon(t) \ge \mathsf{E}_e$. ■

The maximization over t can be performed via a simple line search, over the finite interval $[0,1]$ (or using more sophisticated methods, e.g., Brent's method [12, Section 10.2]). However, for a given t, $\upsilon(t)$ needs to be found. Algorithm 3 provides a method to calculate $\upsilon(t)$. The technique is somewhat similar to Algorithm 1, but here an additional optimization is carried over Q_X. Its convergence proof is given in

Algorithm 3 Iterative alternating minimization algorithm for calculation of $\upsilon(t)$

Input: A source P_{XY}, a target error exponent E_e, a target rate R, a target excess rate E_r, and $0 \le t \le 1$.
Output: The value of $\upsilon(t)$.

1) Initialize $\tilde{Q}_Y$ randomly such that $\mathrm{supp}(\tilde{Q}_Y) = \mathcal{Y}$.
2) Iterate over the following steps until convergence:
   a) If $D(\overline{Q}_X^{0,t}||P_X) < \mathsf{E}_r$ then set $\lambda = 0$. Else, use the bisection method to find $\lambda > 0$ that satisfies $D(\overline{Q}_X^{\lambda,t}||P_X) = \mathsf{E}_r$.
   b) Set $\tilde{Q}_Y(y) = \sum_{x\in\mathcal{X}} \overline{Q}_X^{\lambda,t}(x)\overline{Q}_{Y|X}^{\frac{1}{1+t}}(y|x)$ for all $y \in \mathcal{Y}$.
3) Let the converged variables be λ^* and $\tilde{Q}_Y^*$. Set $\upsilon(t) = \Gamma(t, \overline{Q}_X^{\lambda^*,t}, \overline{Q}_{Y|X}^{\frac{1}{1+t}}, \tilde{Q}_Y^*)$. Return.

the lemma that follows. We shall utilize the the definition in (43) as well as

$$\begin{aligned}\delta_{1,t}(x) &\triangleq D\left(\overline{Q}_{Y|X}^{\frac{1}{1+t}}(\cdot|x)||P_{Y|X}(\cdot|x)\right)\\ \delta_{2,t}(x) &\triangleq D\left(\overline{Q}_{Y|X}^{\frac{1}{1+t}}(\cdot|x)||\tilde{Q}_Y\right)\end{aligned}$$

and

$$\overline{Q}_X^{\lambda,t} \triangleq \psi \cdot P_X^{\frac{1+\lambda}{1+\lambda+t}}(x) \cdot \exp\left(-\frac{1}{1+t+\lambda} \cdot \delta_{1,t}(x) - \frac{t}{1+t+\lambda} \cdot \delta_{2,t}(x)\right), \tag{56}$$

where ψ is a normalization factor.

Lemma 15. *Algorithm 3 outputs $\upsilon(t)$.*

Proof: See Appendix B. ■

It can easily be seen that $\max_{0\le t\le 1} \upsilon(t)$ is a non-increasing function of E_r. Thus, for any given constraint on E_e and target rate R, it is easy to search for $E_r^*(\mathsf{R}) = \min\{\mathsf{E}_r : (\mathsf{R}, \mathsf{E}_r)\,\text{is achievable for}\ \mathsf{E}_e\}$, e.g. using a simple bisection algorithm.

For the sake of comparison, we mention fixed-rate coding. In this case, to ensure an error exponent of E_e one must use $R(Q_X) = \mathsf{R}^*_{\max}(\mathsf{E}_e) \triangleq \max_{Q'_X} R^*(Q'_X, \mathsf{E}_e)$ for all Q_X. Thus, the excess rate exponent is 0 for $\mathsf{R} < \mathsf{R}^*_{\max}(\mathsf{E}_e)$ and ∞ for $\mathsf{R} > \mathsf{R}^*_{\max}(\mathsf{E}_e)$.

Remark 16. In many practical cases, there is some uncertainty regarding the source $P_{XY} = P_X \times P_{Y|X}$. Clearly, if independence between X and Y is a possible scenario, then in this worst case, the side information Y^n is useless (when no feedback link exists). In other cases, it may be known that $P_{XY} \in \mathcal{F} \subset \mathcal{S}(\mathcal{X} \times \mathcal{Y})$ for some family of PMFs $\mathcal{F}$. In this case, a possible requirement is that the rate function $R(Q_X)$ will be chosen to achieve error exponent of E_e uniformly for all sources in $\mathcal{F}$. With a slight change and abuse of notation, we define the optimal rate function for the source P_{XY} as $R^*(Q_X, \mathsf{E}_e; P_{XY})$ and the optimal rate function for the family $\mathcal{F}$ as

$$R^*\left(Q_X, \mathsf{E}_e; \mathcal{F}\right) \triangleq \max_{P_{XY} \in \mathcal{F}} R^*\left(Q_X, \mathsf{E}_e; P_{XY}\right). \tag{57}$$

This maximization is (relatively) easy to perform if, e.g., the conditional probability $P_{Y|X}$ is known exactly, and in addition, a *nominal* $\tilde{P}_X$ is known such that the actual P_X satisfies $D(\tilde{P}_X||P_X) \leq \mathsf{U}$, for some given uncertainty level $\mathsf{U} > 0$ (recall Pinsker's inequality [4, Lemma 11.6.1] and see also the discussion in [19]). Observing (41), it is noticed that $R^*(Q_X, \mathsf{E}_e)$ depends on P_X only via $D(Q_X||P_X)$, both as an additive factor and in the constraint set $\mathcal{A}$. Nonetheless, the maximum of $R^*(Q_X, \mathsf{E}_e)$ is obtained when $D\left(Q_X||P_X\right)$ is maximized. Thus, to perform that maximization (57), the following optimization problem should be solved

$$\min_{P_X : D(\tilde{P}_X||P_X) \leq \mathsf{U}} D\left(Q_X||P_X\right) \tag{58}$$

for any given Q_X. A standard Lagrange method, like the one used in previous sections, results in the solution $P^*_{X,\alpha} = \alpha \tilde{P}_X + (1-\alpha) Q_X$, where α is either chosen to satisfy the constraint $D(\tilde{P}_X||P^*_X) = \mathsf{U}$ or $\alpha = 0$ (and then the minimum is 0). The value $D(Q_X||P^*_X)$ can be substituted into the optimal rate function expression instead of $D(Q_X||P_X)$ in Theorem 6. The calculation of the resulting excess rate exponent curve $E^*_r(\mathsf{R})$ is more involved, since the minimizer P^*_X depends on Q_X. Nonetheless, if in Proposition 14, we add the dependence in P_X to (51), to obtain $\Gamma(t, Q_X, Q_{Y|X}, P_X, \tilde{Q}_Y)$, then we have that Γ is jointly convex function of $(Q_X, Q_{Y|X}, P_X, \tilde{Q}_Y)$. Thus, arguments similar to the ones used in Lemma 15, show that an alternating minimization algorithm can be used, iterating between finding the optimal $(Q_X, Q_{Y|X}, P_X)$ for a given $\tilde{Q}_Y$, and the optimal $\tilde{Q}_Y$ for a given $(Q_X, Q_{Y|X}, P_X)$ (i.e., in Algorithm 3, P_X is optimized after step 2a).

VI. A Numerical Example

In this section, we provide a simple numerical example to illustrate the results obtained in previous sections. Assume the source symbols are generated from the alphabets $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \{0, 1, 2, 3\}$, where P_X is given by $P_X(0) = 1/4 = 1 - P_X(1)$ and $P_{Y|X}$ is given by the following transition probability matrix

$$P_{Y|X} = \frac{1}{10} \cdot \begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}. \tag{59}$$

Figure 1 shows the optimal rate function $R^*(Q_X, \mathsf{E}_e)$ for Q_X given by $Q_X(0) = 0.4 = 1 - Q_X(1)$ as a function of E_e. The three parts of the function - zero rate part, a concave part, and an affine part with a unity slope, are evident. Figure 2 shows the optimal rate function $R^*(Q_X, \mathsf{E}_e)$ for all possible types (indexed by $Q_X(0)$) for various values of E_e. It can be seen that indeed this optimal function is in the form of a regular rate function. The excess rate exponent is calculated for the optimal rate function $R^*(Q_X, \mathsf{E}_e = 10^{-1})$, and plotted in Figure 3.

Following Remark 16, suppose that P_X is not precisely known, but $D(\tilde{P}_X||P_X) \leq \mathsf{U}$ for a nominal PMF $\tilde{P}_X$ given by $\tilde{P}_X(0) = 1/4 = 1 - \tilde{P}_X(1)$, and uncertainty level of $\mathsf{U} = 10^{-2}$. The resulting optimal rate function and excess rate exponents, are also shown in the aforementioned figures.

It can be verified that Figure 2 and Figure 3 are consistent. Indeed, when there is no uncertainty, it can be seen in Figure 2 that when the type is $Q_X = P_X$ the rate is $R^*(P_X, 10^{-1}) \approx 0.62$ so the excess rate exponent is $E_r^*(0.62) = 0$. Then, as $Q_X(0)$ increases, the rate also increases, up to its maximal value of $R^*(Q_X, 10^{-1}) \approx 0.67$, for $Q_X(0) \approx 0.36$. The excess rate exponent is determined by the divergence of this type from the true source P_X, and given by $E_r^*(0.67) \approx D([0.25,\, 0.75]||[0.36,\, 0.64]) \approx 2.8 \cdot 10^{-2}$. This is the maximal value of $E_r(\mathsf{R})$ shown in Figure 3, and for larger rates, clearly $E_r(\mathsf{R}) = \infty$. For the case of uncertainty with $\mathsf{U} = 10^{-2}$, it should be observed that any source P_X with $0.25 \leq P_X(0) \leq 0.315$ satisfies the uncertainty constraint, and such types result $E_r^*(\mathsf{R}) = 0$. The rate for $Q_X(0) = 0.315$ is $R^*(Q_X, 10^{-1}) \approx 0.67$, which can be seen in Figure 3 as the minimal rate for which $E_r^*(\mathsf{R}) > 0$. For larger $Q_X(0)$ the excess rate is determined by the divergence of Q_X w.r.t. the worst source in the family $\mathcal{F}$, namely $P_X^*(0) = 0.315$. For example, in the maximal rate of $R^*(Q_X, 10^{-1}) \approx 0.7$ the type is $Q_X(0) \approx 0.4$ and $E_r^*(0.7) \approx D([0.315,\, 0.685]||[0.4,\, 0.6]) \approx 1.55 \cdot 10^{-2}$.

For the sake of comparison, we also consider fixed-rate coding. First, notice that from Figure 3 for $\mathsf{E}_e = 10^{-1}$ we have $E_r^*(\mathsf{R} = 0.65) = 3.75 \cdot 10^{-3}$. Second, it can be found that if one uses fixed-rate coding, with $R(Q_X) = 0.65$ for all Q_X then the error exponent achieved is only $\mathsf{E}_e = 0.78 \cdot 10^{-1}$. Therefore, if the *finite* excess rate exponent of variable rate coding is tolerated, then this provides an improvement in the error exponent.

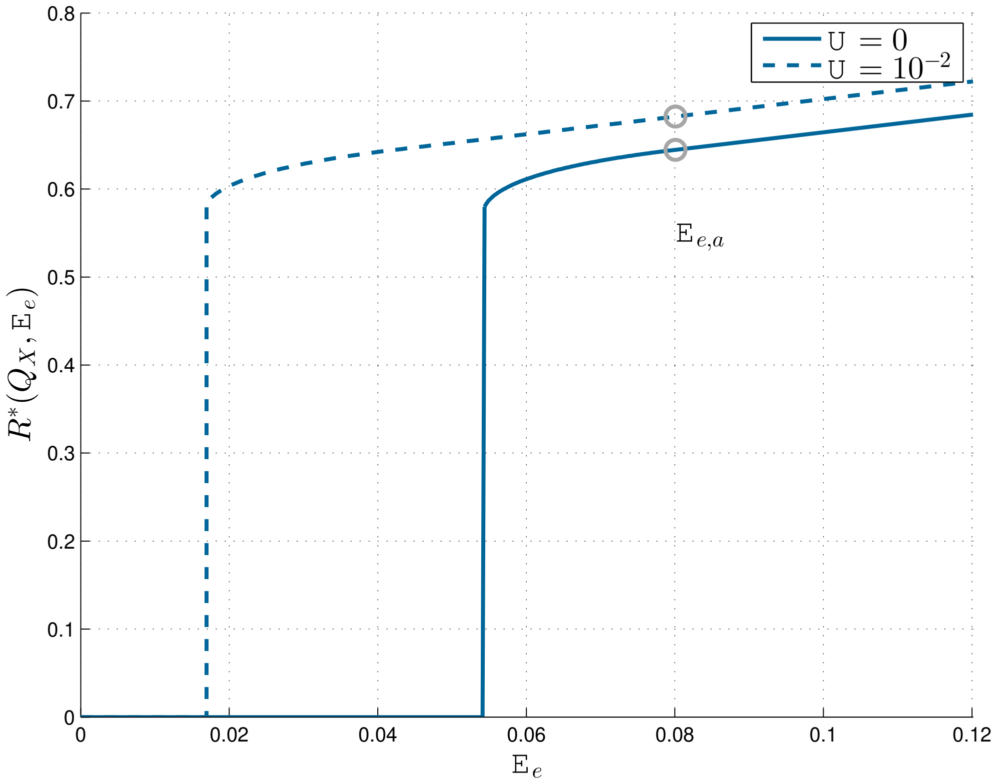

Figure 1. Optimal rate function $R^*(Q_X, \mathsf{E}_e)$ for the type $Q_X = (0.4, 0.6)$.

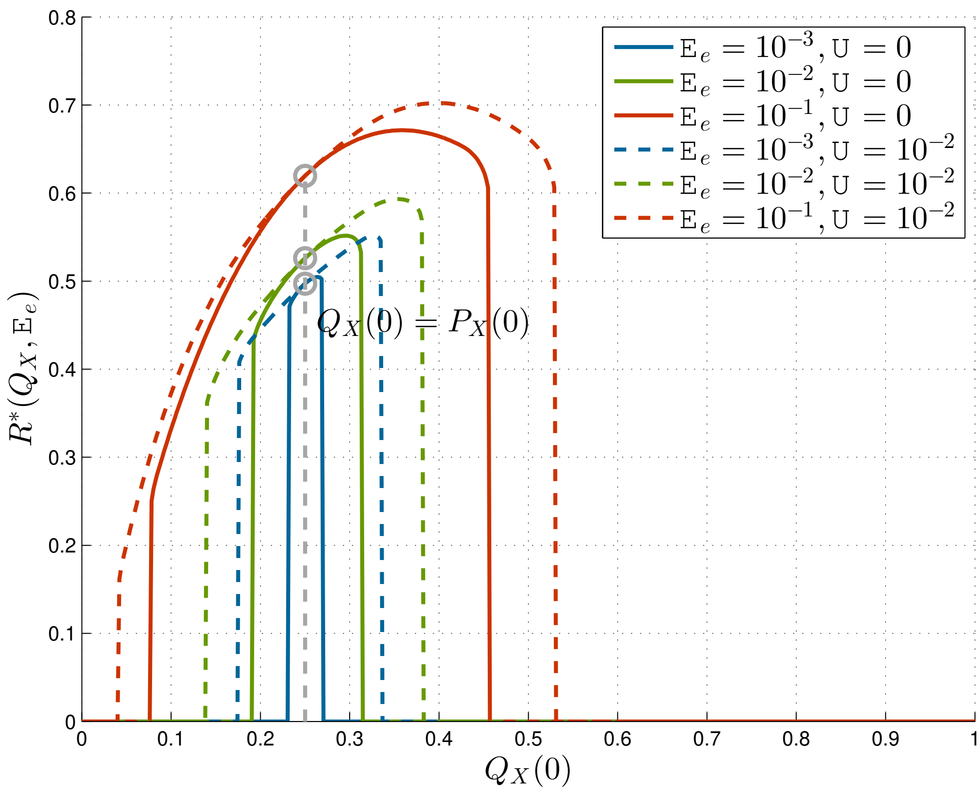

Figure 2. Optimal rate function $R^*(Q_X, \mathsf{E}_e)$.

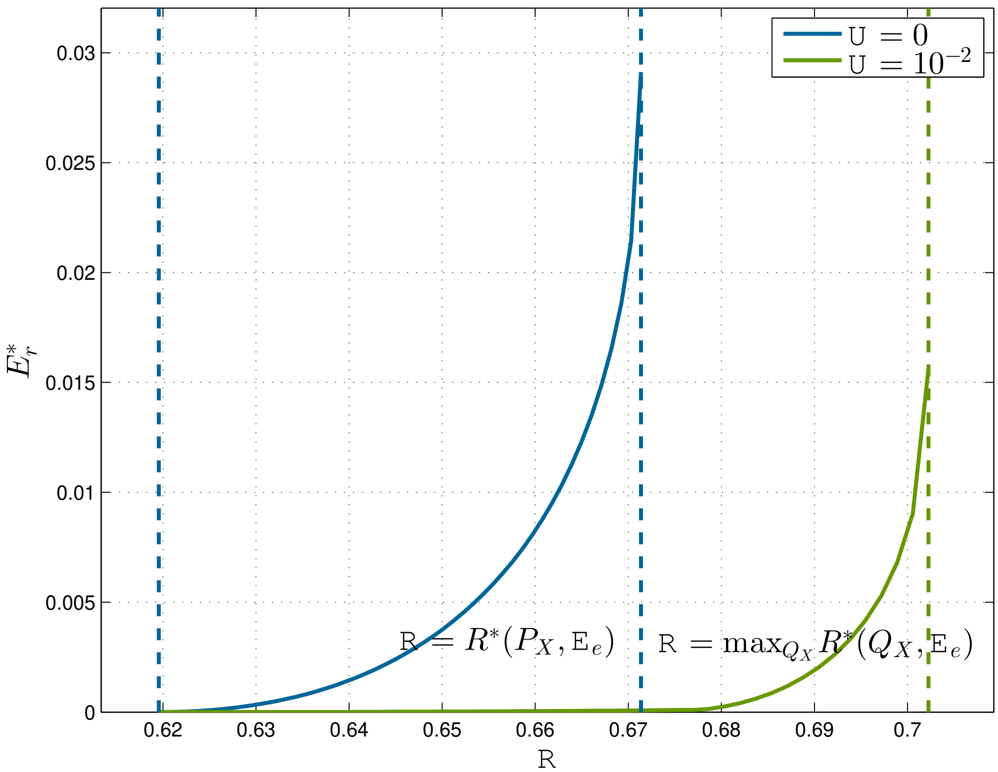

Figure 3. Excess rate exponent $\mathsf{E}_r^*(\mathsf{R})$ for $\mathsf{E}_e = 10^{-1}$.

Appendix A
Examples for Optimal Rate Functions

Example 17 (Double Binary Source)**.** From the proof of Lemma 8 (Appendix B) we may deduce that the optimal $\tilde{Q}_Y^*$ satisfies

$$\tilde{Q}_Y^*(y) = \sum_{x\in\mathcal{X}} Q_X(x) Q_{Y|X}^*(y|x) \tag{A.1}$$

or

$$\tilde{Q}_Y^*(y) = \sum_{x\in\mathcal{X}} Q_X(x)\psi_x P_{Y|X}^{\alpha}(y|x)\tilde{Q}_Y^{*1-\alpha}(y) \tag{A.2}$$

for some α. Writing ψ_x explicitly

$$\tilde{Q}_Y^{\alpha}(y) = \sum_{x\in\mathcal{X}} \frac{Q_X(x) P_{Y|X}^{\alpha}(y|x)}{\sum_{y'\in\mathcal{Y}} P_{Y|X}^{\alpha}(y'|x)\tilde{Q}_Y^{1-\alpha}(y')} \tag{A.3}$$

and using the common denominator of the fractions in the right term we get

$$\tilde{Q}_Y^{\alpha}(y) = \frac{\sum_{x\in\mathcal{X}} Q_X(x) P_{Y|X}^{\alpha}(y|x)\prod_{x'\in\mathcal{X}, x'\neq x}\left(\sum_{y'\in\mathcal{Y}} P_{Y|X}^{\alpha}(y'|x')\tilde{Q}_Y^{1-\alpha}(y')\right)}{\prod_{x\in\mathcal{X}}\left(\sum_{y'\in\mathcal{Y}} P_{Y|X}^{\alpha}(y'|x)\tilde{Q}_Y^{1-\alpha}(y')\right)}. \tag{A.4}$$

It can be noticed that the denominator does not depend on y and thus

$$\tilde{Q}_Y^\alpha(y) = \psi \sum_{x\in\mathcal{X}} Q_X(x) P_{Y|X}^\alpha(y|x) \prod_{x'\in\mathcal{X}, x'\neq x} \left(\sum_{y'\in\mathcal{Y}} P_{Y|X}^\alpha(y'|x') \tilde{Q}_Y^{1-\alpha}(y') \right) \tag{A.5}$$

for some normalization constant $\psi \triangleq \left[\prod_{x\in\mathcal{X}} \left(\sum_{y'\in\mathcal{Y}} P_{Y|X}^\alpha(y'|x)\tilde{Q}_Y^{1-\alpha}(y')\right)\right]^{-1}$. We attempt to solve this equation in the simple case of $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, and $P_{Y|X}$ represents a binary symmetric channel (BSC) with crossover probability ϵ. In this case, (A.5) is given by

$$\tilde{Q}_Y^\alpha(y) = \psi \sum_{x\in\mathcal{X}} Q_X(x) P_{Y|X}^\alpha(y|x) \left(\sum_{y'\in\mathcal{Y}} P_{Y|X}^\alpha(y'|x\oplus 1) \tilde{Q}_Y^{1-\alpha}(y') \right) \tag{A.6}$$

or

$$\begin{aligned} \tilde{Q}_Y^\alpha(0) &= \psi \left\{ \tilde{Q}_Y^{1-\alpha}(0) \cdot [\epsilon(1-\epsilon)]^\alpha + \tilde{Q}_Y^{1-\alpha}(1) \cdot \left[Q_X(0)(1-\epsilon)^{2\alpha} + Q_X(1)\epsilon^{2\alpha}\right] \right\} &\text{(A.7)} \\ \tilde{Q}_Y^\alpha(1) &= \psi \left\{ \tilde{Q}_Y^{1-\alpha}(0) \cdot \left[Q_X(0)\epsilon^{2\alpha} + Q_X(1)(1-\epsilon)^{2\alpha}\right] + \tilde{Q}_Y^{1-\alpha}(1) \cdot [\epsilon(1-\epsilon)]^\alpha \right\}. &\text{(A.8)} \end{aligned}$$

Letting $\rho = \frac{\epsilon}{1-\epsilon}$ and $\gamma = \frac{\tilde{Q}_Y(0)}{\tilde{Q}_Y(1)}$, we get

$$\gamma = \frac{\gamma^{1-\alpha} + [Q_X(0)\rho^{-\alpha} + Q_X(1)\rho^\alpha]}{\gamma^{\alpha-1} + [Q_X(0)\rho^\alpha + Q_X(1)\rho^{-\alpha}]}. \tag{A.9}$$

In the special cases of $\alpha = 1$ and $\alpha = 1/2$ an explicit solution can be found, and it is equal to

$$\gamma = \frac{Q_X(0)\rho^{-1} + Q_X(1)\rho}{Q_X(0)\rho + Q_X(1)\rho^{-1}} \tag{A.10}$$

both for $\alpha = 1$ and $\alpha = 1/2$. However, generally, (A.9) may only be solved numerically. It also evident that even in this simple case, the optimal $\tilde{Q}_Y$ is not necessarily uniform (unless for some simple cases such as $Q_X(0) = Q_X(1) = 1/2$ or $\epsilon = 1/2$).

Example 18 (Very Weakly Correlated Sources)**.** Consider the case of very weakly correlated sources [3], namely

$$P_{Y|X}(y|x) = P_Y(y) \cdot (1 + \epsilon_{xy}) \tag{A.11}$$

where for all $x \in \mathcal{X}$ we have $\sum_{y\in\mathcal{Y}} \epsilon_{xy} = 0$ and $|\epsilon_{xy}| \ll 1$ for all $(x, y) \in (\mathcal{X}, \mathcal{Y})$. Consider again the

[3]In channel coding, this is referred to as *"very noisy channel"*.

minimization problem in (40)

$$\min_{\tilde{Q}_Y} \min_{Q_{Y|X} \in \mathcal{A}} \left\{ D(Q_{Y|X}||\tilde{Q}_Y|Q_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \right\}. \tag{A.12}$$

and notice that if X and Y are independent, then the optimal solution is $\tilde{Q}_Y^* = Q_{Y|X} = P_Y$ for all $x \in \mathcal{X}$ and both divergences vanish. A continuity argument then implies that for the low dependence case, the two divergences at the optimal solution are close to 0. Therefore, we can use the following Euclidean approximation [10, Theorem 4.1]: For two PMFs P_X, Q_X such that $\text{supp}(P_x) = \mathcal{X}$ and $Q_X \approx P_X$ we have that

$$D(Q_X||P_X) \approx \frac{1}{2}\chi^2(Q_X, P_X) \triangleq \frac{1}{2}\sum_{x\in\mathcal{X}} \frac{(Q_X(x) - P_X(x))^2}{P_X(x)}. \tag{A.13}$$

Moreover, for another PMF $\tilde{P}_X$, if $\tilde{P}_X \approx P_X$ then

$$D(Q_X||P_X) \approx \frac{1}{2}\sum_{x\in\mathcal{X}} \frac{(Q_X(x) - P_X(x))^2}{\tilde{P}_X(x)} \tag{A.14}$$

which also shows that $D(P_X||Q_X) \approx D(Q_X||P_X)$. Now, the objective function of the minimization problem can be approximated as

$$\begin{aligned} & D(Q_{Y|X}||\tilde{Q}_Y|Q_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) && \text{(A.15)} \\ \approx\ & \tfrac{1}{2}\mathbb{E}_{Q_X}\left\{\textstyle\sum_{y\in\mathcal{Y}} \frac{(Q_{Y|X}(y|X) - \tilde{Q}_Y(y))^2}{\tilde{Q}_Y(y)} + \sum_{y\in\mathcal{Y}} \frac{\left(Q_{Y|X}(y|X) - P_{Y|X}(y|x)\right)^2}{P_{Y|X}(y|X)}\right\} && \text{(A.16)} \\ \approx\ & \tfrac{1}{2}\mathbb{E}_{Q_X}\left\{\textstyle\sum_{y\in\mathcal{Y}} \frac{(Q_{Y|X}(y|X) - \tilde{Q}_Y(y))^2 + \left(Q_{Y|X}(y|X) - P_{Y|X}(y|X)\right)^2}{P_Y(y)}\right\} && \text{(A.17)} \end{aligned}$$

and similarly the constraint $Q_{Y|X} \in \mathcal{A}$ is approximated by

$$\frac{1}{2} \cdot \mathbb{E}_{Q_X}\left\{\sum_{y\in\mathcal{Y}} \frac{\left(Q_{Y|X}(y|X) - P_{Y|X}(y|X)\right)^2}{P_Y(y)}\right\} \leq \mathsf{E}_e - D(Q_X||P_X). \tag{A.18}$$

The Lagrangian for a given $\tilde{Q}_Y$ (ignoring positivity constraints for the moment) is

$$\begin{aligned} L\left(Q_{Y|X}, \lambda, \mu_x\right) &= \tfrac{1}{2} \cdot \sum_{x\in\mathcal{X}} Q_X(x) \sum_{y\in\mathcal{Y}} \frac{(Q_{Y|X} - \tilde{Q}_Y(y))^2 + (1+\lambda)\left(Q_{Y|X}(y|x) - P_{Y|X}(y|x)\right)^2}{P_Y(y)} && \text{(A.19)} \\ &\quad + \sum_{x\in\mathcal{X}} \mu_x \sum_{y\in\mathcal{Y}} Q_{Y|X}(y|x) \end{aligned}$$

with $\lambda > 0$ and $\mu_x \in \mathbb{R}$ for $x \in \mathcal{X}$. Differentiating w.r.t. some $Q_{Y|X}(y'|x')$ for $x' \in X, y' \in \mathcal{Y}$ we have

$$\frac{\partial L}{\partial Q_{Y|X}(y'|x')} = \frac{1}{2} \cdot Q_X(x') \cdot \frac{2\left(Q_{Y|X}(y'|x') - \tilde{Q}_Y(y')\right) + (1+\lambda)2\left(Q_{Y|X}(y'|x') - P_{Y|X}(y'|x')\right)}{P_Y(y')} + \mu_{x'} \tag{A.20}$$

and equating the derivative to zero in this case is equivalent to

$$\frac{\left(Q_{Y|X}(y'|x') - \tilde{Q}_Y(y')\right) + (1+\lambda)\left(Q_{Y|X}(y'|x') - P_{Y|X}(y'|x')\right)}{P_Y(y')} + \mu'_{x'} = 0. \tag{A.21}$$

Thus, for some $\lambda > 0$

$$Q^*_{Y|X}(y|x) = \frac{1+\lambda}{2+\lambda} P_{Y|X}(y|x) + \frac{1}{2+\lambda}\tilde{Q}_Y(y) - \frac{\mu'_x}{2+\lambda} P_Y(y). \tag{A.22}$$

It can be easily seen that $\mu'_x = 0$ for all $x \in \mathcal{X}$ so

$$Q^*_{Y|X} = \alpha P_{Y|X} + (1-\alpha)\tilde{Q}_Y \tag{A.23}$$

for some $\alpha = \frac{1+\lambda}{2+\lambda}$, where α is either chosen to satisfy the constraint or $\alpha = 1/2$. It is evident that indeed the solution satisfies the positivity constraints. Now, for any given α the resulting value of the optimization problem is

$$\left[\frac{\alpha^2 + (\alpha-1)^2}{2}\right] \sum_{x\in\mathcal{X}} Q_X(x) \sum_{y\in\mathcal{Y}} \frac{\left(P_{Y|X}(y|x) - \tilde{Q}_Y(y)\right)^2}{P_Y(y)} \tag{A.24}$$

and by differentiating w.r.t. some $\tilde{Q}_Y(y')$ for $y' \in \mathcal{Y}$ we have

$$\left[\frac{\alpha^2 + (\alpha-1)^2}{2}\right] \cdot \sum_{x\in\mathcal{X}} \frac{Q_X(x)}{P_Y(y')} \left[-2\left(P_{Y|X}(y'|x) - \tilde{Q}_Y(y')\right)\right] \tag{A.25}$$

and equating to zero we obtain that the optimal solution is

$$\tilde{Q}^*_Y(y) = \sum_{x\in\mathcal{X}} Q_X(x) P_{Y|X}(y|x). \tag{A.26}$$

Notice that the optimal solution $\tilde{Q}^*_Y$ does not depend on α. Thus, for a given $\mathsf{E}_e \geq \mathsf{E}_{e,0}$ the optimal value of α is given by $\alpha^* \approx \max(\tilde{\alpha}, 1/2)$ where $\tilde{\alpha}$ achieves equality in (A.18),

$$\tilde{\alpha} = 1 - \sqrt{\frac{\mathsf{E}_e - D(Q_X||P_X)}{\frac{1}{2}\sum_{x\in\mathcal{X}} Q_X(x) \sum_{y\in\mathcal{Y}} \frac{\left(P_{Y|X}(y|x) - \tilde{Q}^*_Y(y)\right)^2}{P_Y(y)}}} \approx 1 - \sqrt{\frac{\mathsf{E}_e - D(Q_X||P_X)}{D(P_{Y|X}||\tilde{Q}^*_Y|Q_X)}} \tag{A.27}$$

using again (A.14). Then, in the case of very weakly correlated sources, the optimal rate function can be approximated by

$$R^*(Q_X, \mathsf{E}_e) \approx \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) - \left[\alpha^{*2} + (\alpha^* - 1)^2\right] D(P_{Y|X}||\tilde{Q}_Y^*|Q_X) \tag{A.28}$$

where α^* is given analytically as a function of E_e. In addition, similar approximations for the unconstrained minimization problem (B.43) show that

$$\mathsf{E}_{e,a} \approx D(Q_X||P_X) + \frac{1}{4} D(P_{Y|X}||\tilde{Q}_Y^*|Q_X). \tag{A.29}$$

Thus, for $\mathsf{E}_{e,0} \leq \mathsf{E}_e \leq \mathsf{E}_{e,a}$ we have $\tilde{\alpha} \leq 1/2$ and by substituting $\tilde{\alpha}$ in (A.28) we obtain

$$\begin{aligned} R^*(Q_X, \mathsf{E}_e) \quad &\approx \quad H(Q_X) - \mathsf{E}_e + D(Q_X||P_X) \\ &\quad -D(P_{Y|X}||\tilde{Q}_Y^*|Q_X) + 2\sqrt{D(P_{Y|X}||\tilde{Q}_Y^*|Q_X)\left(\mathsf{E}_e - D(Q_X||P_X)\right)}. \end{aligned}$$

Notice that $D(P_{Y|X}||\tilde{Q}_Y^*|Q_X)$ is the mutual information of the joint PMF $Q_X \times P_{Y|X}$ and thus is a measure of the independence between X and Y. As $D(P_{Y|X}||\tilde{Q}_Y^*|Q_X) \to 0$ then X and Y become "more" independent and $\mathsf{E}_{e,a} \to \mathsf{E}_{e,0}$. Thus, in the case of a small $D(P_{Y|X}||\tilde{Q}_Y^*|Q_X)$, the optimal rate function $R^*(Q_X, \mathsf{E}_e)$ is affine for almost the entire range $\mathsf{E}_e \geq \mathsf{E}_{e,0}$. Indeed, in this case, the main error event is associated with "bad binning", i.e. at least two source blocks of the same type are mapped to the same bin by the random binning procedure.

APPENDIX B
PROOFS

Proof of Theorem 4: For a target rate R

$$\begin{aligned} \mathbb{P}\left\{R(\hat{Q}_{\mathbf{X}}) \geq \mathsf{R}\right\} \quad &= \quad \sum_{Q_X \in \mathcal{P}_n(\mathcal{X})} \mathbb{P}(\hat{Q}_{\mathbf{X}} = Q_X) \cdot \mathbb{I}\left(R(Q_X) \geq \mathsf{R}\right) &\text{(B.1)} \\ &\doteq \quad \exp\left(-n \cdot \min_{Q_X \in \mathcal{P}_n(\mathcal{X}): R(Q_X) \geq \mathsf{R}} D(Q_X||P_X)\right). &\text{(B.2)} \end{aligned}$$

Because $\mathrm{supp}(P_X) = \mathcal{X}$ then $D(Q_X||P_X)$ is continuous and finite for all Q_X and as $n \to \infty$

$$\lim_{n\to\infty} -\frac{1}{n} \cdot \log \mathbb{P}\left\{R(\hat{Q}_{X^n}) \geq \mathsf{R}\right\} = \min_{Q_X: R(Q_X) \geq \mathsf{R}} D(Q_X||P_X). \tag{B.3}$$

Now, for a source block $\mathbf{x}$ of type class $\mathcal{T}(\mathbf{x})$, the total rate is the sum of the description rate of its type class index and $R(\hat{Q}_{\mathbf{x}})$. Using the fact that $|\mathcal{P}_n(\mathcal{X})| \leq (n+1)^{|\mathcal{X}|}$ for any given $\epsilon > 0$

$$\begin{aligned} E_r(\mathsf{R}) &= \lim_{n\to\infty} -\frac{1}{n}\cdot\log\mathbb{P}\left\{R(\hat{Q}_{\mathbf{X}}) + |\mathcal{X}|\cdot\frac{\log(n+1)}{n} \geq \mathsf{R}\right\} \\ &\geq \lim_{n\to\infty} -\frac{1}{n}\cdot\log\mathbb{P}\left\{R(\hat{Q}_{\mathbf{X}}) \geq \mathsf{R} - \epsilon\right\} \end{aligned} \tag{B.4}$$

and similarly

$$E_r(\mathsf{R}) \leq \lim_{n\to\infty} -\frac{1}{n}\cdot\log\mathbb{P}\left\{R(\hat{Q}_{\mathbf{X}}) \geq \mathsf{R}\right\}. \tag{B.5}$$

Thus, to conclude the proof we show that $\min_{Q_X:R(Q_X)\geq\mathsf{R}} D(Q_X||P_X)$ is left continuous as a function of R, since this will imply that when taking $\epsilon\to 0$, the upper and lower bounds on $E_r(\mathsf{R})$ will coincide.

To show this, let $\delta > 0$ be given. Recall that $R(Q_X)$ is regular, thus there exists $\mathcal{V}\subset\mathcal{S}(\mathcal{X})$ such that $R(Q_X)$ is continuous in $\mathcal{V}$, and equals a constant $R(Q_X) = \mathsf{R}_0$, for $Q_X\in\mathcal{V}^c$. Also, since $R(Q_X)\geq 0$ and $\mathcal{S}(\mathcal{X})$ is a compact set, then exists a minimizer $Q_X^*\in\mathcal{S}(\mathcal{X})$ for the right hand side of (B.3). For any $\epsilon > 0$ we clearly have

$$\min_{Q_X:R(Q_X)\geq\mathsf{R}-\epsilon} D(Q_X||P_X) \leq \min_{Q_X:R(Q_X)\geq\mathsf{R}} D(Q_X||P_X) = D(Q_X^*||P_X). \tag{B.6}$$

To obtain an inequality in the reversed direction, we split the proof into two cases, depending whether $Q_X^*\in\mathcal{V}$ or not.

Case 1: $Q_X^*\in\mathcal{V}$. Recall that $R(Q_X)$ is continuous and finite inside the interior of $\mathcal{V}$, and $D(Q_X||P_X)$ is continuous function of Q_X. Now, we may define for any $Q_X\in\mathcal{V}$ such that $R(Q_X)\geq\mathsf{R}$, the closed neighborhood

$$\mathcal{D}(Q_X,\mathsf{R},\epsilon) \triangleq \left\{\tilde{Q}_X : D(\tilde{Q}_X||P_X)\geq D(Q_X||P_X)-\delta\right\}\cap\left\{\tilde{Q}_X : R(\tilde{Q}_X)\geq\mathsf{R}-\epsilon\right\}\cap\overline{\mathcal{V}}. \tag{B.7}$$

Then, the continuity of both $R(Q_X)$ and $D(Q_X||P_X)$ implies that there exists a closed ball (e.g. in $\mathcal{L}_1$ norm) $\mathcal{B}(Q_X,\rho)$ with center in Q_X and radius $\rho > 0$ such that $\mathcal{B}(Q_X,\rho)\subset\mathcal{D}(Q_X,\mathsf{R},\epsilon)$. Also, we may define the set

$$\mathcal{V}'(\mathsf{R}) \triangleq \left\{Q_X\in\partial\mathcal{V} : \lim_{\tilde{Q}_X\to Q_X} R(\tilde{Q}_X) = \mathsf{R}\right\} \tag{B.8}$$

where $\partial\mathcal{V} = \overline{\mathcal{V}}\backslash\mathcal{V}$ is the boundary of $\mathcal{V}$, and for any $Q_X\in\mathcal{V}'(\mathsf{R})$

$$\mathcal{D}'(Q_X,\mathsf{R},\epsilon) \triangleq Q_X\cup\mathcal{D}(Q_X,\mathsf{R},\epsilon). \tag{B.9}$$

Here too, continuity arguments imply that there exists a closed ball $\mathcal{B}(Q_X,\rho)$ with center in Q_X and radius $\rho>0$ such that $\mathcal{B}(Q_X,\rho)\cap\overline{\mathcal{V}}\subset\mathcal{D}'(Q_X,\mathsf{R},\epsilon)$. Now, consider the set

$$\mathcal{U}\triangleq\mathcal{V}\backslash\left\{\left\{\bigcup_{\{Q_X\in\mathcal{V}'(\mathsf{R})\}}\mathcal{D}'(Q_X,\mathsf{R},\epsilon)\right\}\cup\left\{\bigcup_{\{Q_X\in\mathcal{V}:R(Q_X)\geq\mathsf{R}\}}\mathcal{D}(Q_X,\mathsf{R},\epsilon)\right\}\right\} \tag{B.10}$$

and let $\mathsf{R}'\triangleq\max_{Q_X\in\overline{\mathcal{U}}}R(Q_X)$. Then we must have $\mathsf{R}'<\mathsf{R}$. To see this, assume by contradiction that $\mathsf{R}'=\mathsf{R}$ and let $\overline{Q}_X$ achieve the maximum, namely, $R(\overline{Q}_X)=\mathsf{R}$. Now, either $\overline{Q}_X\in\mathcal{V}$ or $\overline{Q}_X\in\partial\mathcal{V}$, but both cases lead to contradiction. Indeed, the definition of $\mathcal{U}$, and the fact that for some $\rho>0$ we have $\mathcal{B}(\overline{Q}_X,\rho)\cap\overline{\mathcal{V}}\subset\mathcal{D}(\overline{Q}_X,\mathsf{R},\epsilon)$ imply that $\overline{Q}_X\notin\mathcal{V}$. Similar arguments show that $\overline{Q}_X\notin\partial\mathcal{V}$. Now, consider two sub-cases:

1) $\mathsf{R}>\mathsf{R}_0$. If we choose $\epsilon'=\min\{\epsilon,\mathsf{R}-\mathsf{R}',\mathsf{R}-\mathsf{R}_0\}$ we have

$$\min_{Q_X:R(Q_X)\geq\mathsf{R}-\epsilon'}D(Q_X||P_X)\geq\min_{\tilde{Q}_X\in\bigcup_{Q_X:R(Q_X)\geq\mathsf{R}}\mathcal{B}(Q_X,\mathsf{R},\epsilon)}D(\tilde{Q}_X||P_X)\geq D(Q_X^*||P_X)-\delta \tag{B.11}$$

since the right minimization is over a smaller set.

2) $\mathsf{R}\leq\mathsf{R}_0$. Since Q_X^* is the minimizer for the right hand side of (B.3) then

$$\min_{\tilde{Q}_X\in\mathcal{V}^c}D(\tilde{Q}_X||P_X)\geq D(Q_X^*||P_X) \tag{B.12}$$

and if we choose $\epsilon'=\min\{\epsilon,\mathsf{R}-\mathsf{R}'\}$ we also have

$$\min_{Q_X:R(Q_X)\geq\mathsf{R}-\epsilon'}D(Q_X||P_X)\geq\min_{\tilde{Q}_x\in\left\{\bigcup_{Q_X:R(Q_X)\geq\mathsf{R}}\mathcal{B}(Q_X,\mathsf{R},\epsilon)\right\}}D(\tilde{Q}_X||P_X)\geq D(Q_X^*||P_X)-\delta \tag{B.13}$$

and thus

$$\min_{Q_X:R(Q_X)\geq\mathsf{R}-\epsilon'}D(Q_X||P_X)\geq\min_{\tilde{Q}_X\in\mathcal{V}^c\cup\left\{\bigcup_{Q_X:R(Q_X)\geq\mathsf{R}}\mathcal{B}(Q_X,\mathsf{R},\epsilon)\right\}}D(\tilde{Q}_X||P_X)\geq D(Q_X^*||P_X)-\delta. \tag{B.14}$$

Case 2: $Q_X^*\in\mathcal{V}^c$. In this case we clearly have $\mathsf{R}_0\geq\mathsf{R}$. Now, let $\overline{\mathsf{R}}\triangleq\sup_{Q_X\in\mathcal{V}}R(Q_X)$ and let $\overline{Q}_X\in\overline{\mathcal{V}}$ any PMF achieving the supremum. Then we must have that either $\overline{Q}_X\in\partial\mathcal{V}$ or $\overline{\mathsf{R}}<\mathsf{R}$. To see this, assume by contradiction that both $\overline{Q}_X\in\mathcal{V}$ and $\overline{\mathsf{R}}\geq\mathsf{R}$. Recall that $\mathcal{V}\triangleq\{Q_X:D(Q_X||P_X)<d\}$. This implies that Q_X^* could be any point on the boundary of $\mathcal{V}$. Now, let

$$Q_X^\alpha\triangleq(1-\alpha)P_X+\alpha\overline{Q}_X. \tag{B.15}$$

and since $D(Q_X^\alpha||P_X)$ is an increasing function of α (Lemma 20 in Appendix C), then for some $\tilde{\alpha} > 1$, we have $D(Q_X^\alpha||P_X) = d$ and $Q_X^\alpha \in \partial\mathcal{V}$. But this implies that while both $R(Q_X^{\tilde{\alpha}}) = \mathsf{R}_0 \geq \mathsf{R}$ and $R(\overline{Q}_X) = \overline{\mathsf{R}} \geq \mathsf{R}$ we have

$$D(\overline{Q}_X||P_X) \leq D(Q_X^*||P_X), \tag{B.16}$$

and this contradicts the optimality of Q_X^*. Thus, we must have that either $\overline{Q}_X \in \partial\mathcal{V}$ (for any $\overline{Q}_X$ achieving $\overline{\mathsf{R}}$) or $\overline{\mathsf{R}} < \mathsf{R}$. Now, we have two sub-cases:

1) If $\overline{\mathsf{R}} < \mathsf{R}$, we can choose $\epsilon = \mathsf{R} - \overline{\mathsf{R}} > 0$ and obtain

$$\min_{Q_X : R(Q_X) \geq \mathsf{R}-\epsilon} D(Q_X||P_X) \geq D(Q_X^*||P_X). \tag{B.17}$$

2) Otherwise, suppose $\overline{Q}_X \in \partial\mathcal{V}$ and $\overline{\mathsf{R}} \geq \mathsf{R}$. If $\overline{\mathsf{R}} > \mathsf{R}$ then similar arguments as before show that this contradicts the optimality of Q_X^*. If $\overline{\mathsf{R}} = \mathsf{R}$ then similar arguments to the ones used in the first case show that

$$\min_{Q_X : R(Q_X) \geq \mathsf{R}-\epsilon'} D(Q_X||P_X) \geq D(Q_X^*||P_X) - \delta \tag{B.18}$$

for any $\delta > 0$ and small enough $\epsilon' > 0$.

To conclude, in any of the two cases, for any given $\delta > 0$ we can find $\epsilon > 0$ such that

$$\min_{Q_X : R(Q_X) \geq \mathsf{R}-\epsilon} D(Q_X||P_X) \geq D(Q_X^*||P_X) - \delta. \tag{B.19}$$

This means that $\min_{Q_X : R(Q_X) \geq \mathsf{R}} D(Q_X||P_X)$ is left continuous as a function of R, and the desired result is obtained.

■

Proof of Theorem 7:

1) The fact that $R^*(Q_X, \mathsf{E}_e) = 0$ for $\mathsf{E}_e \leq \mathsf{E}_{e,0}$ evident from (35). For $\mathsf{E}_e > \mathsf{E}_{e,0}$, notice that if $\mathsf{E}_e > D(Q_X||P_X)$ then to satisfy (37) for $Q_{Y|X} = P_{Y|X}$ we must have $R^*(Q_X, \mathsf{E}_e) > H(Q_{X|Y}|Q_Y)$, where here $Q_{X|Y}$ is induced from $Q_X \times P_{Y|X}$. If $H(Q_{X|Y}|Q_Y) > 0$ then we are done. Else, slightly alter $Q_{Y|X}$ from $P_{Y|X}$ such that $D(Q_X||P_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) < \mathsf{E}_e$ but $H(Q_{X|Y}|Q_Y) > 0$.
2) Observing (39), we see that $R^*(Q_X, \mathsf{E}_e)$ equals a linear function of E_e (which is a strictly increasing function of E_e), minus $\inf_{Q_{Y|X} \in \mathcal{A}} \left\{D\left(Q_{XY}||P_{XY}\right) - H(Q_{X|Y}|Q_Y)\right\}$, which is a non-decreasing function of E_e (since as E_e gets larger, the infimum is over a larger set).

3) Using (41) we get

$$
\begin{aligned}
R^*(Q_X, \mathsf{E}_e) &= \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) && \text{(B.20)}\\
&\quad - \min_{\tilde{Q}_Y} \min_{Q_{Y|X} \in \overline{\mathcal{A}}} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_x\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right)\right\}\\
&\geq \mathsf{E}_e + H(Q_X) && \text{(B.21)}\\
&\quad - D(Q_X||P_X) - \left\{ D\left(Q'_{Y|X}||\tilde{Q}'_Y|Q_X\right) + D\left(Q'_{Y|X}||P_{Y|X}|Q_X\right)\right\}
\end{aligned}
$$

and for $\mathsf{E}_e > \mathsf{E}_{e,a}$ equality is achieved since $Q'_{Y|X} \in \mathcal{A}$.

4) From (41), the optimal rate function

$$
\begin{aligned}
R^*(Q_X, \mathsf{E}_e) &= \mathsf{E}_e + H(Q_X) - D(Q_X||P_X) && \text{(B.22)}\\
&\quad - \min_{\tilde{Q}_Y} \min_{D(Q_X||P_X) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right) \leq \mathsf{E}_e} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right)\right\}.
\end{aligned}
$$

The objective and constraint functions are jointly convex in $\tilde{Q}_Y \times Q_{Y|X}$ and for $\mathsf{E}_e \in (\mathsf{E}_{e,0}, \infty)$ the constraint set is feasible. Thus, the conditions of Lemma 21 (Appendix C) are satisfied and $R^*(Q_X, \mathsf{E}_e)$ is concave for $\mathsf{E}_e \in (D(Q_X||P_X), \infty)$.

5) Let $\mathcal{V} = \{Q_X : D(Q_X||P_X) < \mathsf{E}_e\}$. Then clearly $R^*(Q_X, \mathsf{E}_e) = 0$ for $Q_X \in \mathcal{V}^c$. Also, for any $Q_X \in \mathcal{V}$ there exists a closed neighborhood $\mathcal{B}$ of Q_X such that $\mathcal{B} \in \mathcal{V}$. Inside $\mathcal{B}$, the optimal rate $R^*(Q_X, \mathsf{E}_e)$ is given by the second term in (35). In this second term, both the objective and constraint are continuous functions of Q_X, and thus $R^*(Q_X, \mathsf{E}_e)$ is continuous for $Q_X \in \mathcal{V}$.

■

Proof of Lemma 8: Notice that (42) is an optimization problem over $(Q_{Y|X}, \tilde{Q}_Y)$ and consider utilizing an alternating minimization algorithm, where for a given $\tilde{Q}_Y$, the minimizer $Q_{Y|X}$ is found, and vice versa. We divide the rest of the proof into two main parts. In the first part, we prove that the alternating minimization algorithm indeed converges to the optimal solution, and in the second part, we solve the two individual optimization problems (resulting from keeping one of the optimization variables fixed).

Part 1: In [10, Section 5.2], [11] sufficient conditions were derived for the convergence of an alternating minimization algorithm. Specifically, these conditions are met for a minimization problem of the form

$$\inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} D(P||Q) \tag{B.23}$$

where P and Q are two positive measures (which may not necessarily sum to 1) over a finite alphabet $\mathcal{Z}$, and $\mathcal{P}, \mathcal{Q}$ are two convex sets. To prove that alternating minimization algorithm converges for the optimization problem (42), we now show that it can be written in the form of (B.23). The objective function of (42) is given by

$$\begin{aligned} D(Q_{Y|X}||\tilde{Q}_Y|Q_X) + D(Q_{Y|X}||P_{Y|X}|Q_X) &= \sum_{x,y} Q_X(x) Q_{Y|X}(y|x) \log \frac{Q_{Y|X}^2(y|x)}{\tilde{Q}_Y(y) P_{Y|X}(y|x)} && \text{(B.24)} \\ &= 2 \sum_{x,y} Q_{XY}(x,y) \log \frac{Q_{XY}(x,y)}{\sqrt{\tilde{Q}_Y(y) P_{Y|X}(y|x)} Q_X(x)} && \text{(B.25)} \end{aligned}$$

Thus, if we let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ and consider the measures Q_{XY} and $\tilde{Q}_{XY} \triangleq \sqrt{\tilde{Q}_Y P_{Y|X}} Q_X$ then the objective function is of the form of (B.23). Now, using the definition of the set $\mathcal{A}$, the feasible set for Q_{XY} is

$$\left\{ Q_{XY} : \sum_{y \in \mathcal{Y}} Q_{XY}(x,y) = Q_X(x), D\left(Q_{XY}||P_{XY}\right) < \mathsf{E}_e \right\} \tag{B.26}$$

which is a convex set. Now using Corollary 23 of Lemma 22 (Appendix C), we have that the feasible region of $\tilde{Q}_Y$ can be extended from the simplex $\mathcal{S}(\mathcal{Y})$ to the set

$$\tilde{\mathcal{S}}(\mathcal{Y}) \triangleq \left\{ \tilde{Q}_Y : \sum_{y \in \mathcal{Y}} \tilde{Q}_Y(y) \le 1, \tilde{Q}_Y(y) \ge 0 \text{ for all } y \in \mathcal{Y} \right\} \tag{B.27}$$

which is also a convex set. Now, define the feasible set for the variables $\tilde{Q}_{XY}$ as

$$\breve{\mathcal{S}} = \left\{ \tilde{Q}_{XY} : \exists \tilde{Q}_Y \in \tilde{\mathcal{S}}(\mathcal{Y}) \text{ so that } \tilde{Q}_{XY}(x,y) = \sqrt{\tilde{Q}_Y(y) P_{Y|X}(y|x)} Q_X(x) \text{ for all } (x,y) \in \mathcal{X} \times \mathcal{Y} \right\}. \tag{B.28}$$

We show that $\breve{S}$ is also a convex set. Let $\tilde{Q}^i_{XY}(x,y) = \sqrt{\tilde{Q}^i_Y(y) P_{Y|X}(y|x)} Q_X(x)$ for $\tilde{Q}^i_Y \in \tilde{\mathcal{S}}(\mathcal{Y})$, $i = 1, 2$, and $0 \le \alpha \le 1$. Then,

$$\begin{aligned} \tilde{Q}^\alpha_{XY} &\triangleq \alpha \tilde{Q}^1_{XY} + (1-\alpha) \tilde{Q}^2_{XY} && \text{(B.29)} \\ &= \sqrt{P_{Y|X}} Q_X \cdot \left(\alpha \sqrt{\tilde{Q}^1_Y} + (1-\alpha) \sqrt{\tilde{Q}^2_Y} \right). && \text{(B.30)} \end{aligned}$$

Thus, to show that $\tilde{Q}^{\alpha}_{XY} \in \check{\mathcal{S}}$ all is needed to prove is that $\overline{Q}^{\alpha}_{Y} \triangleq \left(\alpha\sqrt{\tilde{Q}^1_Y} + (1-\alpha)\sqrt{\tilde{Q}^2_Y}\right)^2 \in \tilde{\mathcal{S}}(\mathcal{Y})$. As positivity of $\overline{Q}^{\alpha}_{Y}$ is clear, we only verify that $\sum_{y\in\mathcal{Y}} \overline{Q}^{\alpha}_{Y}(y) \le 1$. Indeed, we have

$$\begin{aligned}
\sum_{y\in\mathcal{Y}} \overline{Q}^{\alpha}_{Y}(y) &= \sum_{y\in\mathcal{Y}} \left(\alpha\sqrt{\tilde{Q}^1_Y(y)} + (1-\alpha)\sqrt{\tilde{Q}^2_Y(y)}\right)^2 && \text{(B.31)}\\
&\overset{(a)}{\le} \left(\alpha\sqrt{\sum_{y\in\mathcal{Y}}\tilde{Q}^1_Y(y)} + (1-\alpha)\sqrt{\sum_{y\in\mathcal{Y}}\tilde{Q}^2_Y(y)}\right)^2 && \text{(B.32)}\\
&\overset{(b)}{\le} (\alpha + (1-\alpha))^2 && \text{(B.33)}\\
&= 1 && \text{(B.34)}
\end{aligned}$$

where (a) follows from a variant of Minkowski inequality (Lemma 24 in Appendix C), and (b) is from the fact that both $\sqrt{t}$ and t^2 are increasing functions of $t \in \mathbb{R}^+$, and $\tilde{Q}_Y \in \tilde{\mathcal{S}}(\mathcal{Y})$. Thus the optimization problem (42) is of the form (B.23) and an alternating minimization algorithm converges to the optimal, unique, solution, which we denote by $(Q^*_{Y|X}, \tilde{Q}^*_Y)$.

Part 2: First, suppose that $\tilde{Q}_Y$ is given. In order to find the minimizer $Q_{Y|X}$ the Karush-Kuhn-Tucker (KKT) conditions for convex problems [2, Section 5.5.3] can be utilized. Ignoring positivity constraints for the moment, and defining the Lagrangian

$$\begin{aligned}
L\left(Q_{Y|X},\lambda,\mu_x\right) &= \sum_{x\in\mathcal{X}} Q_X(x) D(Q_{Y|X}(\cdot|x)||\tilde{Q}_Y) + \sum_{x\in\mathcal{X}} Q_X(x) D\left(Q_{Y|X}(\cdot|x)||P_{Y|X}(\cdot|x)\right) && \text{(B.35)}\\
&\quad +\lambda\cdot\sum_{x\in\mathcal{X}} Q_X(x) D\left(Q_{Y|X}(\cdot|x)||P_{Y|X}(\cdot|x)\right) + \sum_{x\in\mathcal{X}}\mu_x\sum_{y\in\mathcal{Y}} Q_{Y|X}(y|x)\\
&= \sum_{x\in\mathcal{X}} Q_X(x)\sum_{y\in\mathcal{Y}} Q_{Y|X}(y|x)\log\frac{Q^{2+\lambda}_{Y|X}(y|x)}{\tilde{Q}_Y P^{1+\lambda}_{Y|X}(y|x)} + \sum_{x\in\mathcal{X}}\mu_x\sum_{y\in\mathcal{Y}} Q_{Y|X}(y|x) && \text{(B.36)}
\end{aligned}$$

where $\lambda \ge 0$ and $\mu_x \in \mathbb{R}$ for $x \in \mathcal{X}$. Differentiating w.r.t. some $Q_{Y|X}(y'|x')$ for $x' \in \mathcal{X}, y' \in \mathcal{Y}$

$$\frac{\partial L}{\partial Q_{Y|X}(y'|x')} = Q_X(x')\left((2+\lambda)\cdot\left(\log Q_{Y|X}(y'|x') + 1\right) + \log\frac{1}{\tilde{Q}_Y(y')P^{1+\lambda}_{Y|X}(y'|x')}\right) + \mu_{x'} \qquad \text{(B.37)}$$

and equating to zero we get

$$Q_X(x')\cdot\log\frac{Q^{2+\lambda}_{Y|X}(y'|x')}{P^{1+\lambda}_{Y|X}(y'|x')\tilde{Q}_Y(y')} + \mu'_{x'} = 0 \qquad \text{(B.38)}$$

where $\mu'_{x'} = \mu_{x'} + \lambda + 2$. Thus, the argument of the logarithm must not depend on x, and this implies that for any $x \in \mathcal{X}$ such that $Q_X(x) \neq 0$ we must have

$$Q^*_{Y|X}(y|x) = \psi_x P^{\alpha}_{Y|X}(y|x)\tilde{Q}^{1-\alpha}_Y(y) = \overline{Q}^{\alpha}_{Y|X}(y|x) \tag{B.39}$$

for $\alpha = \frac{1+\lambda}{2+\lambda}$, where ψ_x is a normalization constant, and the definition (43) was used. The value of $Q^*_{Y|X}$ for $x \in \mathcal{X}$ with $Q_X(x) = 0$ is immaterial as it does not affect the optimal value of the objective function. Also, it is evident that the solution $Q^*_{Y|X}$ is indeed positive.

To find the optimal $Q^*_{Y|X}$ we need to choose α in order to satisfy the constraint $\overline{Q}^{\alpha}_{Y|X} \in \overline{\mathcal{A}}$. For this, the *complementary slackness condition* [2, Section 5.5.2] implies that α should be chosen either to satisfy

$$D(\overline{Q}^{\alpha}_{Y|X}||P_{Y|X}|Q_X) = \mathsf{E}_e - D(Q_X||P_X) \tag{B.40}$$

and ${}^1/_2 \leq \alpha \leq 1$, or $\alpha = {}^1/_2$ and then

$$D(\overline{Q}^{1/2}_{Y|X}||P_{Y|X}|Q_X) < \mathsf{E}_e - D(Q_X||P_X). \tag{B.41}$$

To find α that satisfies the complementary slackness condition it is noticed that $D(\overline{Q}^{\alpha}_{Y|X}||P_{Y|X}|Q_X)$ is a monotonically decreasing function of α. Indeed, it is easy to see that if $\tilde{Q}_Y$ is initialized such that $\operatorname{supp}(\tilde{Q}_Y) = \operatorname{supp}(\sum_{x\in\mathcal{X}} Q_X(x)P_{Y|X}(y|x))$ then this remains true for all iterations. Then, it follows from Lemma 25 (Appendix C) that for any given $x \in \mathcal{X}$ such that $Q_X(x) \neq 0$ we have that $D(\overline{Q}^{\alpha}_{Y|X}(\cdot|x)||P_{Y|X}(\cdot|x))$ is a decreasing function of α, and thus their average $D(\overline{Q}^{\alpha}_{Y|X}||P_{Y|X}|Q_X)$ is also a decreasing function of α. Thus, if $D(\overline{Q}^{1/2}_{Y|X}||P_{Y|X}|Q_X) < \mathsf{E}_e - D(Q_X||P_X)$ then $\alpha = {}^1/_2$. Otherwise, we have $D(\overline{Q}^{1/2}_{Y|X}||P_{Y|X}|Q_X) > \mathsf{E}_e - D(Q_X||P_X)$ and $D(\overline{Q}^{1}_{Y|X}||P_{Y|X}|Q_X) = 0 < \mathsf{E}_e - D(Q_X||P_X)$. Thus, in the later case, a simple bisection search finds the required α.

Second, assume that $Q_{Y|X}$ is given. The minimizer $\tilde{Q}_Y$ can be found using Lemma 22 (Appendix C) to be

$$\tilde{Q}_Y(y) = \sum_{x\in\mathcal{X}} Q_X(x)Q_{Y|X}(y|x). \tag{B.42}$$

It is easily seen that Algorithm 1 indeed implements the procedure described in this proof. ■

Proof of Lemma 10: Using Theorem 7 part 3, for $\mathsf{E}_e > \mathsf{E}_{e,a}$, it is sufficient to solve the unconstrained minimization problem

$$\min_{\tilde{Q}_Y} \min_{Q_{Y|X}} \left\{ D\left(Q_{Y|X}||\tilde{Q}_Y|Q_X\right) + D\left(Q_{Y|X}||P_{Y|X}|Q_X\right)\right\}. \tag{B.43}$$

This is equivalent to setting $\mathsf{E}_e = \infty$. Thus, in step 2a of Algorithm 1, we always have

$$D(\overline{Q}_{Y|X}^{1/2}||P_{Y|X}|Q_X) < \mathsf{E}_e - D(Q_X||P_X) \tag{B.44}$$

and thus we need to set $\alpha = 1/2$. ■

Proof of Theorem 11: First, consider the case $\mathsf{E}_{e,0} < \mathsf{E}_e \leq \mathsf{E}_{e,a}$, and assume that Algorithm 1 has converged to $(Q^*_{Y|X}, \tilde{Q}^*_Y)$. Then, we must have $D(Q^*_{Y|X}||P_{Y|X}|Q_X) = \mathsf{E}_e - D(Q_X||P_X)$. Indeed, if $D(Q^*_{Y|X}||P_{Y|X}|Q_X) < \mathsf{E}_e - D(Q_X||P_X)$, then $(Q^*_{Y|X}, \tilde{Q}^*_Y)$ is a solution of the unconstrained minimization (B.43). However, since the solution of the unconstrained minimization (B.43) is unique (since its objective function is strictly convex) then we must have $(Q^*_{Y|X}, \tilde{Q}^*_Y) = (Q'_{Y|X}, \tilde{Q}'_Y)$, and thus $D(Q^*_{Y|X}||P_{Y|X}|Q_X) < \mathsf{E}_e - D(Q_X||P_X)$ implies $\mathsf{E}_e > \mathsf{E}_{e,a}$, which contradicts the assumption. Now, we can substitute $D(Q^*_{Y|X}||P_{Y|X}|Q_X) = \mathsf{E}_e - D(Q_X||P_X)$ in (35) to obtain (47).

Second, consider the case $\mathsf{E}_{e,a} < \mathsf{E}_e$. In this case, the optimal solution is clearly $(Q'_{Y|X}, \tilde{Q}'_Y)$, regardless of E_e. Using Lemma 10 we have $Q'_{Y|X} = \overline{Q}_{Y|X}^{1/2}$, and substituting this value in (35) provides (48). ■

Proof of Theorem 12:

1) This stems directly from (B.5) and (B.3) (which are valid even if $R(Q_X)$ is not regular).
2) This stems directly from (B.4) and (B.3) (which are valid even if $R(Q_X)$ is not regular), with the choice $0 < \epsilon < \mathsf{R} - \mathsf{R}_{\max}$.
3) The first statement stems directly from the definition (5). When $R(Q_X)$ is regular, we may use (33). Now, let Q^*_X be any minimizer of (33), for a given $\mathsf{R} < \mathsf{R}'_{\max}$. We begin with showing that $R(Q^*_X) = \mathsf{R}$. Assume by contradiction that $R(Q^*_X) > \mathsf{R}$. Since $\mathsf{R} < \mathsf{R}'_{\max}$ then arguments used in the proof of Theorem (4) show that $Q^*_X \in \mathcal{V}$. Now, consider

$$Q_{\alpha,X} = (1-\alpha)P_X + \alpha Q^*_X. \tag{B.45}$$

Since $R(Q_X)$ is continuous in $\mathcal{V}$ then intermediate value theorem implies that $\alpha < 1$ must exist such that $R(Q_{\alpha,X}) = \mathsf{R}$. Using Lemma (20) we have that $D(Q_{\alpha,X}||P_X) < D(Q^*_X||P_X)$ which contradicts the fact that Q_X^* is a minimizer of (33). Now, let $\mathcal{Q}(\mathsf{R})$ be the collection of all minimizers of (33), such that for all $Q_X \notin \mathcal{Q}(\mathsf{R})$ we have either $D(Q_X||P_X) > D(Q_X^*||P_X)$ or $R(Q_X) < \mathsf{R}$. Thus, for any $\mathsf{R}_1 > \mathsf{R}$ we have

$$\min_{Q_X : R(Q_X) \geq \mathsf{R}_1} D(Q_X||P_X) > D(Q_X^*||P_X). \tag{B.46}$$

4) The first statement follows from the fact that monotonic functions are continuous almost everywhere. The proof of the second is a part of the proof of Theorem 4.

■

Proof of Lemma 15: Algorithm 3 is an alternating minimization algorithm, that keeps all variables but one fixed, and optimize over the non-fixed variable. Now, for a given $0 \leq t \leq 1$, the objective function in (52) is given by

$$(1+t) \cdot \sum_{x,y} Q_{XY}(x,y) \log \frac{Q_{XY}(x,y)}{P_{XY}^{1/1+t}(x,y) \tilde{Q}_Y^{t/1+t}(y)} + t\mathsf{R}. \tag{B.47}$$

The exact same technique which was used in the proof of lemma 8 shows that this optimization problem is of the form (B.23). Thus, an alternating minimization algorithm converges to the optimal solution.

We now turn to the minimization of individual variables, assuming that all other variables are fixed, for a given t. First, consider the minimization over Q_{XY}, which itself can be separated to an unconstrained minimization over $Q_{Y|X}$ and constrained minimization over Q_X. The minimizer $Q_{Y|X}^*$ can again be found using similar Lagrange methods as in the proof of Lemma 8. The result is

$$Q_{Y|X}^*(y|x) = \psi_x P_{Y|X}^{\frac{1}{1+t}}(y|x) \tilde{Q}_Y^{\frac{t}{1+t}}(y) = \overline{Q}_{Y|X}^{\frac{1}{1+t}} \tag{B.48}$$

for all $x \in \mathcal{X}$ such that $Q_X(x) \neq 0$ (and arbitrary otherwise, since the value of $Q_{Y|X}^*(\cdot|x)$ for $x \in \mathcal{X}$ such that $Q_X(x) = 0$ does not affect the value of the optimization problem). For this optimal choice, using the definitions of $\delta_{1,t}(x)$ and $\delta_{2,t}(x)$ we obtain

$$\min_{\tilde{Q}_Y} \min_{Q_X : D(Q_X||P_X) \leq \mathsf{E}_r} \left\{ D(Q_X||P_X) + \sum_{x \in \mathcal{X}} Q_X(x)\delta_{1,t}(x) + t \cdot \left(\mathsf{R} - H(Q_X) + \sum_{x \in \mathcal{X}} Q_X(x)\delta_{2,t}(x) \right) \right\}. \tag{B.49}$$

Next, we optimize over Q_X using the KKT conditions. The Lagrangian with $\lambda \geq 0$ and μ is given by

$$\begin{aligned} L\left(Q_X, \lambda, \mu\right) &\triangleq D(Q_X||P_X) + \sum_{x\in\mathcal{X}} Q_X(x)\delta_{1,t}(x) + t\cdot\left(\mathsf{R} - H(Q_X) + \sum_{x\in\mathcal{X}} Q_X(x)\delta_{2,t}(x)\right) \quad \text{(B.50)} \\ &\quad + \lambda\cdot D(Q_X||P_X) + \mu\cdot\sum_{x\in\mathcal{X}} Q_x \\ &= t\cdot\mathsf{R} + \sum_{x\in\mathcal{X}} Q_X(x)\left[\log\left(\frac{Q_X^{1+t+\lambda}(x)}{P_X^{1+\lambda}(x)}\cdot\exp(\delta_{1,t}(x)+t\cdot\delta_{2,t}(x))\right)+\mu\right]. \quad \text{(B.51)} \end{aligned}$$

Differentiating w.r.t. some $Q_X(x')$ for $x' \in \mathcal{X}$ we get

$$\frac{\partial L}{\partial Q_X(x')} = \log\left(\frac{Q_X^{1+t+\lambda}(x')}{P_X^{1+\lambda}(x')}\cdot\exp(\delta_{1,t}(x')+t\cdot\delta_{2,t}(x'))\right) + (1+t+\lambda)+\mu \quad \text{(B.52)}$$

and equating to zero results

$$Q_X^*(x) = \psi\cdot P_X^{\frac{1+\lambda}{1+\lambda+t}}(x)\cdot\exp\left(-\frac{1}{1+t+\lambda}\cdot\delta_{1,t}(x) - \frac{t}{1+t+\lambda}\cdot\delta_{2,t}(x)\right) = \overline{Q}_X^{\lambda,t}(x) \quad \text{(B.53)}$$

where ψ is a normalization constant, and the definition (56). Using the complementary slackness condition [2, Section 5.5.2], λ is chosen to either satisfy $D(\overline{Q}_X^{\lambda,t}||P_X) = \mathsf{E}_r$ or $\lambda = 0$. To show that a bisection method can be used to find the desired λ, we use Lemma 25 (Appendix (C)) to show that $D(\overline{Q}_X^{\lambda,t}||P_X)$ is a monotonic decreasing function of λ. To see that the conditions of Lemma 25 are met, notice that initializing $\tilde{Q}_Y$ with support $\mathcal{Y}$ implies that in the first iteration $\mathrm{supp}(Q^*_{Y|X}) = \mathrm{supp}(P_{Y|X})$ which assures that $\delta_{1,t}(x)$ and $\delta_{2,t}(x)$ are finite. As $\mathrm{supp}(\overline{Q}_X^{\lambda,t}) = \mathrm{supp}(P_X) = \mathcal{X}$ for all $\lambda > 0$ and $t \in [0,1]$ then $\mathrm{supp}(\tilde{Q}_Y) = \mathcal{Y}$ for all iterations (cf. (B.58)). Thus, for any $t \neq 0$ we may express $\overline{Q}_X^{\lambda,t}$ as

$$\overline{Q}_X^{\lambda,t}(x) = \psi\cdot P_X^{\frac{1+\lambda}{1+\lambda+t}}(x)\cdot\breve{P}_X^{\frac{t}{1+\lambda+t}}(x) \quad \text{(B.54)}$$

where

$$\breve{P}_X(x) \triangleq \breve{\psi}\cdot\exp\left(-\frac{\delta_{1,t}(x)}{t} - \delta_{2,t}(x)\right) \quad \text{(B.55)}$$

and $\breve{\psi}$ is a normalization factor. Setting $\alpha = \frac{1+\lambda}{1+\lambda+t}$ we get that $D(\overline{Q}_X^{\lambda,t}||P_X)$ is a decreasing function of α. Since α is a monotonic increasing function of λ this implies that $D(\overline{Q}_X^{\lambda,t}||P_X)$ is also a decreasing function of λ. For $t = 0$ we may write again

$$\overline{Q}_X^{\lambda,0}(x) = \psi\cdot P_X^{\frac{\lambda}{1+\lambda}}(x)\cdot\breve{P}_X^{\frac{1}{1+\lambda}}(x) \quad \text{(B.56)}$$

where now

$$\breve{P}_X(x) \triangleq \breve{\psi} \cdot \exp\left(-\delta_{1,t}(x) + \log P_X\right). \tag{B.57}$$

Similar arguments show that $D(\overline{Q}_X^{\lambda,0}||P_X)$ is a decreasing function of λ.

The optimal $\tilde{Q}_Y^*$ for a given t and $Q_X, Q_{Y|X}$ is simply

$$\tilde{Q}_Y^*(y) = \sum_{x\in\mathcal{X}} Q_X(x)Q_{Y|X}(y|x) \tag{B.58}$$

using Lemma 22. ■

Appendix C
Useful Lemmas

In this appendix, we provide several useful lemmas.

Lemma 19 (Tightness of the union bound)**.** *Let $\mathcal{A}_1, \mathcal{A}_2 \ldots, \mathcal{A}_M$ be pairwise independent events from a probability space. Then*

$$\mathbb{P}\left\{\bigcup_{i=1}^{M} \mathcal{A}_i\right\} \geq \frac{1}{2} \cdot \min\left\{1, \sum_{i=1}^{M} \mathbb{P}\left(\mathcal{A}_i\right)\right\} \tag{C.1}$$

Proof: See [20, Lemma A.2, pp. 109]. ■

Lemma 20. *Let P, Q be two PMFs over some alphabet $\mathcal{X}$ such that* $\text{supp}(P) = \text{supp}(Q) = \mathcal{X}$*, $P \neq Q$, and*

$$Q_\alpha \triangleq (1-\alpha)P + \alpha Q. \tag{C.2}$$

Also, let $\alpha_{max} = \max\{\alpha : Q_\alpha \in \mathcal{S}(\mathcal{X})\}$. Then, $D(Q_\alpha||P)$ is a strictly increasing function of α for $\alpha \in (0, \alpha_{max})$.

Proof: Let $0 < \alpha_1 < \alpha_2 \leq \alpha_{max}$. Then,

$$\begin{aligned}
Q_{\alpha_1} &= (1-\alpha_1)P + \alpha_1 Q &\text{(C.3)}\\
&= \frac{\alpha_1}{\alpha_2}\left(\frac{\alpha_2}{\alpha_1} - \alpha_2 + 1 - \frac{\alpha_1}{\alpha_1}\right) P + \frac{\alpha_1}{\alpha_2}\alpha_2 Q &\text{(C.4)}\\
&= \frac{\alpha_1}{\alpha_2}\left(1 - \alpha_2 + \frac{\alpha_2 - \alpha_1}{\alpha_1}\right) P + \frac{\alpha_1}{\alpha_2}\alpha_2 Q &\text{(C.5)}\\
&= \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2)P + \alpha_2 Q\right) + \frac{(\alpha_2 - \alpha_1)}{\alpha_2} P &\text{(C.6)}\\
&= \frac{\alpha_1}{\alpha_2} Q_{\alpha_2} + \left(1 - \frac{\alpha_1}{\alpha_2}\right) P &\text{(C.7)}
\end{aligned}$$

thus $Q_{\alpha_1}$is a convex combination of Q_{α_2} and P with coefficient $\beta \triangleq \frac{\alpha_1}{\alpha_2}$, and $0 < \beta < 1$. Now, since divergence is strictly convex function then

$$\begin{aligned} D(Q_{\alpha_1}||P) &= D(\beta Q_{\alpha_2} + (1-\beta)P||P) && \text{(C.8)} \\ &< \beta D(Q_{\alpha_2}||P) + (1-\beta)D(P||P) && \text{(C.9)} \\ &= \beta D(Q_{\alpha_2}||P) && \text{(C.10)} \\ &< D(Q_{\alpha_2}||P) && \text{(C.11)} \end{aligned}$$

and thus $D(Q_\alpha||P)$ is strictly increasing in α. ■

Lemma 21. *Let $f_i(z) : \mathbb{R}^N \to \mathbb{R}$ be convex functions for $i = 1, 2$. Consider the optimization problem*

$$W(E) = \min_{f_1(z) \le E} f_2(z). \tag{C.12}$$

assuming that the constraint is feasible for some interval $E \in \mathcal{J}$. Then $W(E)$ is a convex function of E in $\mathcal{J}$ and $E - W(E)$ is a concave function E in *$\mathcal{J}$.*

Proof: This is a standard result. For example, in [1, Theorem 3], this theorem is proved for the case that f_1 and f_2 are information divergences. The proof may be used verbatim for any convex functions. ■

Lemma 22. *Let $P_X \times P_{Y|X}$ be a given joint distribution over $\mathcal{X} \times \mathcal{Y}$. Then the distribution Q_Y that minimizes $D(P_X \times P_{Y|X}||P_X \times Q_Y)$ is the marginal distribution Q^*_Y corresponding to $P_{Y|X}$ namely, $Q^*_Y(y) = \sum_x P_X(x)P_{Y|X}(y|x)$.*

Proof: See [4, Lemma 10.8.1]. ■

Corollary 23. *Let $P_X \times P_{Y|X}$ be a given joint distribution over $\mathcal{X} \times \mathcal{Y}$. Then the vector $Q_Y \in \mathbb{R}^{|\mathcal{Y}|}$ that minimizes $D(P_X \times P_{Y|X}||P_X \times Q_Y)^4$ under the constraint $\sum_{y\in\mathcal{Y}} Q_Y(y) \le 1$ and $Q_Y(y) \ge 0$ for all $y \in \mathcal{Y}$, is $Q^*_Y(y) = \sum_x P_X(x)P_{Y|X}(y|x)$.*

Proof: Suppose that the minimizer vector Q^*_Y satisfies $\sum_{y\in\mathcal{Y}} Q^*_Y(y) < 1$. Then for some $y' \in \mathcal{Y}$, we

[4]Notice that the divergence is well defined even if $\{Q_Y\}$ do not sum exactly to 1.

can increase $Q_Y^*(y')$ by $1-\sum_{y\in\mathcal{Y}} Q_Y^*(y) > 0$ and obtain $\bar{Q}_Y$ which satisfies $\sum_{y\in\mathcal{Y}} \overline{Q}_Y(y) = 1$. But,

$$\begin{aligned} D(P_X \times P_{Y|X}||P_X \times Q_Y^*) &= \sum_{x,y} P_{XY}(x,y) \log \frac{P_{XY}(x,y)}{P_X(x)Q_Y^*(y)} && \text{(C.13)} \\ &= \sum_{x,y\neq y'} P_{XY}(x,y) \log \frac{P_{XY}(x,y)}{P_X(x)Q_Y^*(y)} && \text{(C.14)} \\ &\quad + \sum_{x} P_{XY}(x,y') \log \frac{P_{XY}(x,y')}{P_X(x)Q_Y^*(y')} \\ &> \sum_{x,y\neq y'} P_{XY}(x,y) \log \frac{P_{XY}(x,y)}{P_X(x)\overline{Q}_Y(y)} && \text{(C.15)} \\ &\quad + \sum_{x} P_{XY}(x,y') \log \frac{P_{XY}(x,y')}{P_X(x)\overline{Q}_Y(y')} \end{aligned}$$

and this contradicts the fact that Q_Y^* is a minimizer. Thus, we must have $\sum_{y\in\mathcal{Y}} Q_Y^*(y) = 1$. In this case, Lemma 22 shows that the optimal solution is $Q_Y^*(y) = \sum_x P_X(x)P_{Y|X}(y|x)$. ■

Lemma 24 (Variant of Minkowski inequality)**.** *Let* $0 \leq \lambda \leq 1$*, and let* Q_X *be a PMF over a finite alphabet* $\mathcal{X}$*, and let* $\{a_x(i)\}$ *be a set of non-negative numbers for* $1 \leq i \leq I$ *and* $x \in \mathcal{X}$ *. Then,*

$$\sum_{i=1}^{I} \left(\sum_{x\in\mathcal{X}} Q_X(x) a_x(i)^\lambda \right)^{1/\lambda} \leq \left(\sum_{x\in\mathcal{X}} Q_X(x) \left(\sum_{i=1}^{I} a_x(i) \right)^\lambda \right)^{1/\lambda} \tag{C.16}$$

Proof: This variant of Minkowski inequality is stated and proved in [23, Section 3A.1, inequality (k)]. ■

Lemma 25. *Let* P_1, P_2 *be two PMFs over some alphabet* $\mathcal{X}$*, such that* $\text{supp}(P_2) \subseteq \text{supp}(P_1)$*. Define for any* $x \in \mathcal{X}$

$$Q_\alpha(x) \triangleq \psi_\alpha P_1^\alpha(x) P_2^{1-\alpha}(x) \tag{C.17}$$

where $\alpha \in [0,1]$ *and* ψ_α *is a normalization factor such that* $Q_\alpha \in \mathcal{S}(\mathcal{X})$*. Then,* $D\left(Q_\alpha||P_1\right)$ *is a continuous function of* α *whose limit as* $\alpha \to 0$ *is* $D(Q'||P)$ *where*

$$P_2'(x) = \begin{cases} \psi' \cdot P_2(x) & P_1(x) > 0 \\ 0 & P_1(x) = 0 \end{cases} \tag{C.18}$$

for some normalization factor ψ'*. Moreover,* $D\left(Q_\alpha||P_1\right)$ *is monotonic strictly decreasing function of* α *unless* $P_2' = P_1$*.*

Proof: This is [8, Ex. 2.14, pp. 30-31] but for completeness, we provide a proof here based on [1]. First, notice that $P_1(x) = 0 \Rightarrow Q_\alpha(x) = 0$ and thus all $x \in \mathcal{X}$ such that $P_1(x) = 0$ are immaterial to the divergence, assuming the regular convention, that any summand of the form $0 \cdot \frac{0}{0}$ is 0. Thus it may be assumed without loss of generality that $\text{supp}(P_1) = \mathcal{X}$ and $P_2' = P_2$.

Continuity: Since $\text{supp}(P_1) = \mathcal{X}$ then $D\left(Q_\alpha || P_1\right)$ is a continuous function of Q_α in $\mathcal{S}(\mathcal{X})$. As Q_α is a continuous function of α we get that $D\left(Q_\alpha || P_1\right)$ is a continuous function of α.

Limit for $\alpha \to 0$: Since $\text{supp}(P_1) = \mathcal{X}$ we get that $\text{supp}(Q_\alpha) = \text{supp}(P_2)$. It is easily seen that as $\alpha \to 0$ we have $Q_\alpha(x) \to P_2(x)$.

Monotonicity: Consider the following optimization problem

$$W(E) = \min_{D(Q||P_2) \leq E} D(Q||P_1). \tag{C.19}$$

Standard Lagrange techniques, as used in this paper, show that the optimal solution is

$$Q(x) = \psi P_1^{\frac{1}{1+\lambda}}(x) P_2^{\frac{\lambda}{1+\lambda}}(x) \tag{C.20}$$

where $\lambda \geq 0$ is either chosen such that the constraint is satisfied with equality, or $\lambda = 0$. When $\lambda > 0$ defining $\alpha \triangleq \frac{1}{1+\lambda}$ we get $W(E) = D(Q_\alpha || P_1)$. Thus, if we show that $W(E)$ is a monotonic increasing function of λ, then the proof is finished because α is an increasing function of λ. To this end, notice that:

1) $W(E)$ is a strictly decreasing function of E.
2) Using Lemma 21, $W(E)$ is a strictly convex function of E which implies that $\frac{dW(E)}{dE}$ is a strictly increasing function of E.
3) We have that

$$\frac{dW(E)}{dE} = -\lambda. \tag{C.21}$$

To see this relation, suppose that λ is chosen to satisfy the constraint E. Then, we get

$$\begin{aligned}
W(E) &= D\left(Q_\alpha || P_1\right) && \text{(C.22)}\\
&= \sum_{x\in\mathcal{X}} Q_\alpha(x)\cdot\log\frac{Q_\alpha(x)}{P_1(x)} && \text{(C.23)}\\
&= \sum_{x\in\mathcal{X}} Q_\alpha(x)\cdot\log\frac{P_2^\lambda(x)}{Q_\alpha^\lambda(x)} + \sum_{x\in\mathcal{X}} Q_\alpha(x)\cdot\log\frac{Q_\alpha^{1+\lambda}(x)}{P_1(x)\cdot P_2^\lambda(x)} && \text{(C.24)}\\
&= -\lambda E - (\lambda+1)\log(\psi) && \text{(C.25)}\\
&= -\lambda E - (\lambda+1)\log\left(\sum_{x\in\mathcal{X}} P_1^{\frac{1}{1+\lambda}}(x) P_2^{\frac{\lambda}{1+\lambda}}(x)\right). && \text{(C.26)}
\end{aligned}$$

When differentiating we obtain

$$\frac{dW(E)}{dE} = -\lambda - E\frac{d\lambda}{dE} - \frac{d\lambda}{dE}\cdot\frac{d}{d\lambda}\left[(\lambda+1)\log\left(\sum_{x\in\mathcal{X}} P_1^{\frac{1}{1+\lambda}}(x)P_2^{\frac{\lambda}{1+\lambda}}(x)\right)\right], \tag{C.27}$$

and because $\frac{d}{d\lambda}\left[(\lambda+1)\log\left(\sum_{x\in\mathcal{X}} P_1^{\frac{1}{1+\lambda}}(x)P_2^{\frac{\lambda}{1+\lambda}}(x)\right)\right] = -E$ we obtain the desired result.

These properties imply that as E increases $W(E)$ decreases and $\frac{dW(E)}{dE} = -\lambda$ increases. This results that $W(E)$ is a monotonic increasing function of λ, and concludes the proof.

Strict monotonicity can be verified by noticing that all monotonicity relations are strict.

■

References

[1] R. Blahut. Hypothesis testing and information theory. *Information Theory, IEEE Transactions on*, 20(4):405–417, 1974.

[2] S. P. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge university press, 2004.

[3] J. Chen, D. He, A. Jagmohan, and L. A. Lastras-Montaño. On universal variable-rate Slepian-Wolf coding. In *Proc. of IEEE International Conference on Communications*, pages 1426–1430. IEEE, 2008.

[4] T. M. Cover and J. A. Thomas. *Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing)*. Wiley-Interscience, 2006.

[5] I. Csiszár. Linear codes for sources and source networks: Error exponents, universal coding. *Information Theory, IEEE Transactions on*, 28(4):585–592, 1982.

[6] I. Csiszar and J. Körner. Towards a general theory of source networks. *Information Theory, IEEE Transactions on*, 26(2):155–165, 1980.

[7] I. Csiszár and J. Körner. Graph decomposition: A new key to coding theorems. *Information Theory, IEEE Transactions on*, 27(1):5–12, 1981.

[8] I. Csiszar and J. Körner. *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Cambridge University Press, 2011.

[9] I. Csiszár, J. Körner, and K. Marton. A new look at the error exponent of discrete memoryless channels. In *Proc. of International Symposium on Information Theory*, page 107 (abstract), 1977.

[10] I. Csiszár and P. C. Shields. *Information Theory and Statistics: A Tutorial*. Foundations and Trends in Communications and Information Theory. Now Publishers Inc, 2004.

[11] I. Csiszár and G. Tusnády. Information Geometry and Alternating Minimization Procedures. *Statistics and Decisions*, Supplement Issue 1, 1984.

[12] W.H. Press et.al. *Numerical Recipes in C*. Cambridge University Press, New York, 1992.

[13] R. G. Gallager. *Information Theory and Reliable Communication*. Wiley, 1968.

[14] R. G. Gallager. Source coding with side information and universal coding. LIDS- P-937, M.I.T., 1976. Available online: http://web.mit.edu/gallager/www/papers/paper5.pdf.

[15] B. G. Kelly and A. B. Wagner. Improved source coding exponents via Witsenhausen's rate. *Information Theory, IEEE Transactions on*, 57(9):5615–5633, 2011.

[16] A. Lapidoth and P. Narayan. Reliable communications under channel uncertainty. *IEEE Trans. Inform. Theory*, 44(6):2148–2177, October 1998.

[17] N. Merhav. Erasure/list exponents for Slepian-Wolf decoding. *Submitted to IEEE Transactions on Information Theory*, June 2013. Available online: http://arxiv.org/pdf/1305.5626.pdf.

[18] Y. Oohama and Han T. S. Universal coding for the Slepian-Wolf data compression system and the strong converse theorem. *Information Theory, IEEE Transactions on*, 40(6):1908–1919, 1994.

[19] F. Rezaei and C. D. Charalambous. Robust coding for uncertain sources: a minimax approach. In *Proc. of International Symposium on Information Theory*, pages 1539–1543. IEEE, 2005.

[20] N. Shulman. *Communication Over an Unknown Channel via Common Broadcasting*. PhD thesis, Tel Aviv University, 2003. http://www.eng.tau.ac.il/~shulman/papers/Nadav_PhD.pdf.

[21] M. Sion. On general minimax theorems. *Pacific Journal of Mathematics*, 8(1):171–176, 1958.

[22] D. Slepian and J. Wolf. Noiseless coding of correlated information sources. *Information Theory, IEEE Transactions on*, 19(4):471–480, 1973.

[23] A.J. Viterbi and J.K. Omura. *Principles of Digital Communication and Coding*. Dover Publications, 2009.