CCIT Report #547
August 2005

Comparing Between Estimation Approaches: Admissible and Dominating Linear Estimators

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Abstract

We treat the problem of evaluating the performance of linear estimators for estimating a deterministic parameter vector $\mathbf{x}$ in a linear regression model, with the mean-squared error (MSE) as the performance measure. Since the MSE depends on the unknown vector $\mathbf{x}$, direct comparison between estimators is a difficult problem. Here we consider a framework for examining the MSE of different linear estimation approaches based on the concepts of admissible and dominating estimators. We develop a general procedure for determining whether or not a linear estimator is MSE admissible, and for constructing an estimator strictly dominating a given inadmissible method, so that its MSE is smaller for all $\mathbf{x}$. In particular we show that both problems can be addressed in a unified manner for arbitrary constraint sets on $\mathbf{x}$ by considering a certain convex optimization problem. We then demonstrate the details of our method for the case in which $\mathbf{x}$ is constrained to an ellipsoidal set, and for unrestricted choices of $\mathbf{x}$.

As a by product of our results, we derive a closed form solution for the minimax MSE estimator on an ellipsoid, which is valid for arbitrary model parameters, as long as the signal-to-noise-ratio exceeds a certain threshold.

Key Words—Linear estimation, regression, admissible estimators, dominating estimators, mean-squared error (MSE) estimation, minimax MSE estimation.

I. Introduction

An important estimation problem that has been treated extensively in the literature is the problem of estimating a deterministic parameter vector $\mathbf{x}$ in the regression model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$, where $\mathbf{y}$ are the observations, $\mathbf{H}$ is a given model matrix, and $\mathbf{w}$ is a random noise vector with positive definite covariance matrix. In an estimation context, the goal is to construct an estimator $\hat{\mathbf{x}}$ of $\mathbf{x}$ from the observations $\mathbf{y}$ that is close to $\mathbf{x}$ in some sense. In this paper, we focus on *linear* estimation, in which the estimator $\hat{\mathbf{x}}$ is linear in the observations $\mathbf{y}$.

A popular measure of estimator performance is the mean-squared error (MSE), which is the average squared-norm of the estimation error $\hat{\mathbf{x}} - \mathbf{x}$. Due to the fact that the parameter vector $\mathbf{x}$ is assumed to be fixed, the averaging is only over the noise, and not over $\mathbf{x}$, typically resulting in a parameter-dependent MSE. Since the MSE generally depends on $\mathbf{x}$, it cannot be minimized directly. Thus, alternative criteria for constructing estimators must be sought.

Beginning with the celebrated least-squares estimator proposed by Gauss, a myriad of linear and nonlinear estimators have been developed for the regression model with the common goal of leading to"good" MSE performance. The first estimators considered for this problem where restricted to be linear and unbiased [1], [2], [3]. In the past 30 years attempts have been made to develop linear estimators that may be biased but

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