

Minimum Variance in Biased Estimation: Bounds and Asymptotically Optimal Estimators

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Abstract

We develop a uniform Cramer-Rao lower bound (UCRLB) on the total variance of any estimator of an unknown vector of parameters, with bias gradient matrix whose norm is bounded by a constant. We consider both the Frobenius norm and the spectral norm of the bias gradient matrix, leading to two corresponding lower bounds.

We then develop optimal estimators that achieve these lower bounds. In the case in which the measurements are related to the unknown parameters through a linear Gaussian model, Tikhonov regularization is shown to achieve the UCRLB when the Frobenius norm is considered, and the shrunken estimator is shown to achieve the UCRLB when the spectral norm is considered. For more general models, the penalized maximum likelihood (PML) estimator with a suitable penalizing function is shown to *asymptotically* achieve the UCRLB. To establish the asymptotic optimality of the PML estimator, we first develop the asymptotic mean and variance of the PML estimator for any choice of penalizing function satisfying certain regularity constraints, and then derive a general condition on the penalizing function under which the resulting PML estimator asymptotically achieves the UCRLB. This then implies that from all linear and nonlinear estimators with bias gradient whose norm is bounded by a constant, the proposed PML estimator asymptotically results in the smallest possible variance.

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