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Singular Control with State Constraints on Unbounded Domain

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Abstract

We study a class of stochastic control problems where a cost of the form

$$\mathbb{E}\int_{[0,\infty)} e^{-\beta s}[\ell(X_s)ds + h(Y_s^\circ)d|Y|_s] \tag{0.1}$$

is to be minimized over control processes Y whose increments take values in a cone $\mathbb{Y}$ of $\mathbb{R}^p$, keeping the state process $X = x + B + GY$ in a cone $\mathbb{X}$ of $\mathbb{R}^k$, $k \le p$. Here, $x \in \mathbb{X}$, B is a Brownian motion with drift b and covariance Σ, G is a fixed matrix, and Y° is the Radon-Nikodym derivative $dY/d|Y|$. Let $\mathcal{L} = -(1/2)\text{trace}(\Sigma D^2) - b\cdot D$ where D denotes the gradient. Solutions to the corresponding dynamic programming PDE

$$[(\mathcal{L}+\beta)f - \ell] \vee \sup_{y\in\mathbb{Y}:|Gy|=1}[-Gy\cdot Df - h(y)] = 0, \tag{0.2}$$

on $\mathbb{X}^o$ are considered with a polynomial growth condition and are required to be supersolution up to the boundary (corresponding to a "state constraint" boundary condition on $\partial\mathbb{X}$). Under suitable conditions on the problem data, including continuity and nonnegativity of ℓ and h, and polynomial growth of ℓ, our main result is the unique viscosity-sense solvability of the PDE by the control problem's value function in appropriate classes of functions. In some cases where uniqueness generally fails to hold in the class of functions that grow at most polynomially (e.g., when $h = 0$), our methods provide uniqueness within the class of functions that, in addition, have compact level sets. The results are new even in the following special cases: (1) The one-dimensional case $k = p = 1$, $\mathbb{X} = \mathbb{Y} = \mathbb{R}_+$; (2) The first order case $\Sigma = 0$; (3) The case where ℓ and h are linear. The proofs combine probabilistic arguments and viscosity solution methods. Our framework covers a wide range of diffusion control problems that arise from queueing networks in heavy traffic.

Keywords. Singular Control, State Constraints, Viscosity Solutions, Hamilton-Jacobi-Bellman Equations, Skorohod Problem, Brownian Control Problems, Stochastic Networks.

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