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Uniformly Improving the Cramér-Rao Bound and Maximum-Likelihood Estimation

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Abstract

An important aspect of estimation theory is characterizing the best achievable performance in a given estimation problem, as well as determining estimators that achieve the optimal performance. The traditional Cramér-Rao type bounds provide benchmarks on the variance of any estimator of a deterministic parameter vector under suitable regularity conditions, while requiring a-priori specification of a desired bias gradient. In applications, it is often not clear how to choose the required bias. A direct measure of the estimation error that takes both the variance and the bias into account is the mean-squared error (MSE), which is the sum of the variance and the squared-norm of the bias. Here, we develop bounds on the MSE in estimating a deterministic parameter vector $\mathbf{x}_0$ over all bias vectors that are linear in $\mathbf{x}_0$, which includes the traditional unbiased estimation as a special case. In some settings, it is possible to minimize the MSE over all linear bias vectors. More generally, direct minimization is not possible since the optimal solution depends on the unknown $\mathbf{x}_0$. Nonetheless, we show that in many cases we can find bias vectors that result in an MSE bound that is smaller than the CRLB for all values of $\mathbf{x}_0$. Furthermore, we explicitly construct estimators that achieve these bounds in cases where an efficient estimator exists, by performing a simple linear transformation on the standard maximum likelihood (ML) estimator. This leads to estimators that result in a smaller MSE than the ML estimator for all possible values of $\mathbf{x}_0$.

I. Introduction

One of the prime goals of statistical estimation theory is the development of bounds on the best achievable performance in estimating parameters of interest in a given model, as well as determining estimators that achieve these bounds. Such bounds provide benchmarks against which we can compare the performance of any proposed estimator, and insight into the fundamental limitations of the problem.

Here, we consider the class of estimation problems in which we seek to estimate an unknown deterministic parameter vector $\mathbf{x}_0$ from measurements $\mathbf{y}$, where the relationship between $\mathbf{y}$ and $\mathbf{x}_0$ is described by the probability density function (pdf) $p(\mathbf{y};\mathbf{x}_0)$ of $\mathbf{y}$ characterized by $\mathbf{x}_0$.

A classic performance bound is the Cramér-Rao lower bound (CRLB) [1], [2], [3], which characterizes the smallest achievable total variance of any *unbiased* estimator of $\mathbf{x}_0$. Although other variance bounds exist in the literature, this bound is relatively easy to determine, and can often be achieved. Specifically, in the

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