Zero Forcing Precoding and Generalized Inverses

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Abstract— **We consider the problem of linear zero forcing precoding design, and discuss its relation to the theory of generalized inverses in linear algebra. Special attention is given to a specific generalized inverse known as the pseudo-inverse. We begin with the standard design under the assumption of a total power constraint and prove that precoders based on the pseudo-inverse are optimal in this setting. Then, we proceed to examine individual per-antenna power constraints. In this case, the pseudo-inverse is not necessarily the optimal generalized inverse. In fact, finding the optimal inverse is non-trivial and depends on the specific performance measure. We address two common criteria, fairness and throughput, and show that the optimal matrices may be found using standard convex optimization methods. We demonstrate the improved performance offered by our approach using computer simulations.**

Index Terms— **Zero forcing precoding, Beamforming, Generalized inverses, Semidefinite relaxation, per-antenna constraints.**

I. Introduction

Transmitter design for the multiple input single output (MISO) multiuser broadcast channel is an important problem in modern wireless communication systems. The main difficulty in this channel is that coordinated receive processing is not possible and that all the signal processing must be employed at the transmitter side. From an information theory perspective, the capacity region of this channel was only recently characterized [1]. From a signal processing point of view there are still many open questions and there is ongoing search aimed at finding efficient yet simple transmitter design algorithms. In particular, linear precoding schemes which seem to provide a promising tradeoff between performance and complexity received considerable attention [2]–[4].

The most common linear precoding scheme is zero forcing (ZF) beamforming. This simple method decouples the multiuser channel into multiple independent sub-channels, and reduces the design into a power allocation problem. It performs very well in the high signal-to-noise-ratio (SNR) regime or when the number of users is sufficiently large, and is known to provide full degrees of freedom [1]. Moreover, it is easy to generalize this method to incorporate non-linear dirty paper coding (DPC) mechanisms [1]. There are dozens of papers on ZF precoding focusing on different design criteria [4]–[10]. Among these, two common criteria are maximal fairness and maximum throughput. Due to its simplicity, ZF precoding is also an appealing transmission method in multiple input multiple output (MIMO) broadcast channels [11]–[15].

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Traditionally, the transmitter is designed under the assumption of a total power constraint [1]–[10]. In practice, there is increasing interest in addressing more complicated scenarios, such as individual per-antenna power constraints. These are more realistic since each transmit antenna has its own power amplifier. Moreover, state-of-the-art communication systems will utilize multiple transmitters, which are geographically separated, but cooperatively send data to the receiving units. In such systems, it is clear that each transmitter has its own power restrictions. Recently, our work on linear precoding [2] was generalized to incorporate per-antenna power constraints in [16]. The problem with these methods is their prohibitive computational complexity. Therefore, ZF precoding methods were also generalized to address per-antenna power constraints [17]–[19].

Interestingly, ZF precoding design is highly related to the concept of generalized inverses in linear algebra [20]. This is easy to understand as the ZF precoder basically inverts the multiuser channel. Previous works using total power constraints [4]–[10] as well as individual per-antenna power constraints [17]–[19] began with the assumption that the precoder has the form of a specific generalized inverse known as the pseudo-inverse. We prove that the pseudo-inverse based precoder is optimal for maximizing any performance measure under a total power constraint. However, when per-antenna power constraints are involved, it is no longer optimal and other generalized inverses may outperform it. Finding the optimal inverse is non-trivial and depends on the specific performance criterion. We consider the two classical criteria, fairness and throughput, and transform the design problems into convex optimization programs which can be solved efficiently using off-the-shelves numerical packages.

The ZF precoding design for maximizing throughput turns out to be a non-convex optimization problem. One of the methods for handling such problems is to lift it into a higher dimension and then relax the non-convex constraints. Consequently, there is an increasing interest in analyzing the tightness of such relaxations [21], [22]. We apply this method and use Lagrange duality to prove that the relaxation is always tight in our setting.

The paper is organized as follows. In Section II we introduce the ZF precoding design problem. A brief review on generalized inverses is provided in Section III. Next, precoding under total power constraint is addressed in Section IV, whereas precoding under individual per-antenna power constraints is considered in Section V. A few numerical results are demonstrated in Section VI.

The following notation is used. Boldface upper case letters denote matrices, boldface lower case letters denote column vectors, and standard lower case letters denote scalars. The superscripts $(\cdot)^T$, $(\cdot)^{-1}$, $(\cdot)^{-}$ and $(\cdot)^{\dagger}$ denote the transpose,