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Linear Regression with Gaussian Model Uncertainty: Algorithms and Bounds

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***Abstract*—We consider the problem of estimating an unknown deterministic parameter vector in a linear regression model with random Gaussian uncertainty in the mixing matrix. We prove that the maximum likelihood (ML) estimator is a regularized least squares estimator and develop three alternative approaches for finding the regularization parameter which maximizes the likelihood. We analyze the performance using the Cramér Rao bound (CRB) on the mean squared error, and show that the degradation in performance due the uncertainty is not as severe as may be expected. Next, we address the problem again assuming that the variances of the noise and the elements in the model matrix are unknown and derive the associated CRB and ML estimator.**

We compare our methods to known results on linear regression in the error in variables (EIV) model. We discuss the similarity between these two competing approaches, and provide a thorough comparison which sheds light on their theoretical and practical differences.

***Index Terms*—Maximum likelihood estimation, Total least squares, Errors in Variables, Linear models, Random model matrix.**

I. Introduction

One of the most classical problems in statistical signal processing is that of estimating an unknown, deterministic vector parameter $\mathbf{x}$ in the linear regression model $\mathbf{y} = \mathbf{G}\mathbf{x}+\mathbf{w}$ where $\mathbf{G}$ is a linear transformation and $\mathbf{w}$ is a Gaussian noise vector. The importance of this problem stems from the fact that a wide range of problems in communications, array processing, and many other areas can be cast in this form.

Most of the literature concentrates on the simplest case, in which it is assumed that the model matrix $\mathbf{G}$ is completely specified. In this setting, the celebrated least squares (LS) estimator coincides with the maximum likelihood (ML) solution and is known to minimize the mean-squared-error (MSE) among all unbiased estimators of $\mathbf{x}$ [1], [2]. Nonetheless, it may be outperformed in terms of MSE by biased methods such as the regularized LS estimator due to Tikhonov [3], the James-Stein method [4], and the minimax MSE approach [5].

The linear regression problem for cases where $\mathbf{G}$ is not completely specified received much less attention. In this case, there are many mathematical models for describing the uncertainty in $\mathbf{G}$. Each of these models leads to different optimization criteria and accordingly to different estimation algorithms. Most of the literature can be divided into two main categories, in which the uncertainty is expressed using either deterministic or random models. A standard deterministic approach is the "robust LS" which is designed to cope with the worst-case $\mathbf{G}$ within a known deterministic set [6], [7]. Recently, the minimax MSE criterion was also considered in this problem formulation [5]. In the stochastic uncertainty models, $\mathbf{G}$ is usually known up to some Gaussian distortion. Typically, there are two approaches in this setting. First, one can use a random variables (RV) model and assume that $\mathbf{G}$ is a random Gaussian matrix with known statistics. Based on this model, different estimation methods have been considered. The ML estimator was derived in our recent letter [8]. An alternative strategy is to minimize the expected LS criterion with respect to $\mathbf{G}$ [9], [10]. The minimax MSE estimator was also generalized to this setting in [10]. The second approach is the standard Errors-in-Variables (EIV) model, where $\mathbf{G}$ is considered a deterministic unknown matrix, and an additional noisy observation on this matrix is available [11]. The ML solution for $\mathbf{x}$ in this case was addressed in [11], and coincides with the well known total LS (TLS) estimator [12] (when the additive Gaussian noise $\mathbf{w}$ is independent and identically distributed).

Evidently, there are different models and optimization criteria for estimating $\mathbf{x}$ in a linear model with uncertainty in the model matrix. The main objective of this paper is to compare the Gaussian uncertainty approaches and to shed light on their advantages and disadvantages. In particular, we consider the two classical Gaussian uncertainty formulations: the RV and EIV models. We explain the practical and theoretical differences between them, and discuss the scenarios in which each is appropriate.

The main part of this paper considers ML estimation of $\mathbf{x}$ in the RV linear regression model. We prove that the ML estimate (MLE) is a regularized (or deregularized) LS estimator, and that its regularization parameter and squared norm can be characterized as a saddle point of a concave-quasiconvex objective function. Thus, we can efficiently find the optimal parameters numerically. In fact, our previous solution in [8] can be interpreted as a minimax search for this saddle point. Using this new characterization, we present a more efficient maximin search. Furthermore, an appealing approach for finding the ML estimate in this setting is to resort to the classical expectation maximization (EM) algorithm which is known to converge to a stationary point of the ML objective (see [13], [14], [15] and references within). Due to the non-convexity of the log-likelihood function, there is no guarantee that this point will indeed be the global maximum. Fortunately, our saddle point interpretation provides a simple method for checking the global optimality of the convergence

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