On Universal Properties of Capacity-Approaching LDPC Ensembles

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CCIT Report # 638 September 2007

Abstract

This paper provides some universal information-theoretic bounds related to capacity-approaching ensembles of low-density parity-check (LDPC) codes. These bounds refer to the behavior of the degree distributions of such ensembles, and also to the graphical complexity and the fundamental system of cycles associated with the Tanner graphs of LDPC ensembles. The transmission of these ensembles is assumed to take place over an arbitrary memoryless binary-input output-symmetric (MBIOS) channel. The universality of the bounds derived in this paper stems from the fact that they do not depend on the full characterization of the LDPC ensembles but rather depend on the achievable gap between the channel capacity and the design rate of the ensemble, and also on the required bit error (or erasure) probability at the end of the decoding process. Some of these bounds hold under maximum-likelihood decoding (and hence, they also hold under any sub-optimal decoding algorithm) whereas the others hold particularly under the sum-product iterative decoding algorithm. The tightness of some of these bounds is exemplified numerically for capacity-approaching LDPC ensembles under sum-product decoding; the bounds are reasonably tight for general MBIOS channels, and are tightened for the binary erasure channel.

Index Terms

Cycles, low-density parity-check (LDPC) codes, maximum-likelihood decoding, memoryless binary-input output-symmetric (MBIOS) channels, stability condition, sum-product decoding algorithm, Tanner graphs.

I. Introduction

Low-density parity-check (LDPC) codes were introduced by Gallager [7] in the early 1960s. These linear block codes are characterized by sparse parity-check matrices which facilitate their efficient decoding with sub-optimal iterative message-passing algorithms. In spite of the seminal work of Gallager [7], LDPC codes were ignored for a long time. Following the breakthrough in coding theory made by the introduction of turbo codes [2] and the rediscovery of LDPC codes [13] in the mid 1990s, it was realized that these codes and lots of other variants of capacity-approaching error-correcting codes can all be understood as codes defined on graphs. Graphs not only describe the codes, but more importantly, they structure the operation of efficient iterative decoding algorithms which are used to decode these codes. Various iterative algorithms, used to decode codes defined on graphs, enable to closely approach the channel capacity while maintaining reasonable decoding complexity. This breakthrough attracted coding theorists, and lots of research activity has been conducted during the last decade on these modern coding techniques and their practical decoding algorithms; the reader is referred to the special issue of the *IEEE Transactions on Information Theory* on codes on graphs and iterative algorithms [26].

This paper derives some universal information-theoretic bounds related to capacity-approaching LDPC ensembles whose transmission takes place over a memoryless binary-input output-symmetric (MBIOS) channel. The universality of the bounds derived in this paper stems from the fact that they do not depend on the full characterization of the LDPC ensembles but rather depend on the achievable gap between the channel capacity and the design rate of the ensemble, and also on the required bit error (or erasure) probability at the end of the decoding process. Most of these bounds apply to the asymptotic case where we let the block length tend to infinity, and the bounds are compared with some capacity-approaching LDPC ensembles. The design of such ensembles under iterative decoding lies on a solid background due to the *density evolution* technique which was developed by Richardson and Urbanke (see [10], [18], [19]). This technique is commonly used for a numerical search of the degree distributions of capacity-approaching LDPC ensembles where the target is to minimize the gap to capacity for infinite block length while

The paper was submitted in September 2007 as a regular paper to the *IEEE Trans. on Information Theory*. This research work was supported by the Israel Science Foundation (grant no. 1070/07).

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