An Improved Sphere-Packing Bound for Finite-Length Codes over Symmetric Memoryless Channels

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Abstract

This paper derives an improved sphere-packing (ISP) bound for finite-length error-correcting codes whose transmission takes place over symmetric memoryless channels, and the codes are decoded with an arbitrary list decoder. We first review classical results, i.e., the 1959 sphere-packing (SP59) bound of Shannon for the Gaussian channel, and the 1967 sphere-packing (SP67) bound of Shannon et al. for discrete memoryless channels. An improvement on the SP67 bound, as suggested by Valembois and Fossorier, is also discussed. These concepts are used for the derivation of a new lower bound on the error probability of list decoding (referred to as the ISP bound) which is uniformly tighter than the SP67 bound and its improved version. The ISP bound is applicable to symmetric memoryless channels, and some of its applications are exemplified. Its tightness under ML decoding is studied by comparing the ISP bound to previously reported upper and lower bounds on the ML decoding error probability, and also to computer simulations of iteratively decoded turbo-like codes. This paper also presents a technique which performs the entire calculation of the SP59 bound in the logarithmic domain, thus facilitating the exact calculation of this bound for moderate to large block lengths without the need for the asymptotic approximations provided by Shannon.

Index Terms

Block codes, error exponent, list decoding, sphere-packing bound, turbo-like codes.

I. Introduction

The theoretical study of the fundamental performance limitations of long block codes was initiated by Shannon. During the fifties and sixties, this research work attracted Shannon and his colleagues at MIT and Bell Labs (see, e.g., the collected papers of Shannon [27] and the book of Gallager [12]). An overview of these classical results and their impact was addressed by Berlekamp [2].

The 1959 sphere-packing (SP59) bound of Shannon [25] serves for the evaluation of the performance limits of block codes whose transmission takes place over an AWGN channel. This lower bound on the decoding error probability is expressed in terms of the block length and rate of the code; however, it does not take into account the modulation used, but only assumes that the signals are of equal energy. It is often used as a reference for quantifying the sub-optimality of error-correcting codes under some practical decoding algorithms.

The 1967 sphere-packing (SP67) bound, derived by Shannon, Gallager and Berlekamp [26], provides a lower bound on the decoding error probability of block codes as a function of their block length and code rate, and applies to arbitrary discrete memoryless channels. Like the random coding bound of Gallager [11], the SP67 bound decays to zero exponentially with the block length for all rates below the channel capacity. Further, the error exponent of the SP67 bound is known to be tight at the portion of the rate region between the critical rate (R_{c}) and the channel

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