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ON THE CLASSICAL – AND NOT SO CLASSICAL – SHANNON SAMPLING THEOREM

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Abstract. We proceed with our recently-introduced geometric approach to sampling of manifolds and investigate the relationship that exists between the classical, i.e. Shannon type, and geometric sampling concepts and formalism. Some aspects of coding and the Gaussian channel problem are considered. A geometric version of Shannon's Second Theorem is introduced. Applications to Pulse Code Modulation and Vector Quantization of Images are provided. An extension of our sampling scheme to a certain class of infinite dimensional manifolds is also considered. The relationship between real functions of bounded curvature and classical band-limited signals is investigated.

1. General background

1.1. **Introduction.** A sampling theorem for differentiable manifolds was recently presented and applied in the context image processing ([22], [23]):

Theorem 1.1. *Let $\Sigma^n \subset \mathbb{R}^{n+1}, n \geq 2$ be a connected, not necessarily compact, smooth hypersurface, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\left(\frac{1}{k(p)}\right)$, where $k(p) = \max\{|k_1|, ..., |k_n|\}$, and where $k_1, ..., k_n$ are the principal curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Moreover, the following corollary is also applicable to this problem:

Corollary 1.2. *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling scheme of Σ^n of finite density everywhere. In particular, if Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

The constructive proof of this theorem is based on the existence of the so-called fat triangulations (see [21]). The density of the vertices of the triangulation (i.e. of the sampling) is given by the inverse of the maximal principal curvature. (A concise outline of the proof of Theorem 1.1 is presented in Appendix 1.) As was shown, the resultant sampling scheme is in accord with the classical Shannon theorem, at least for the large class of

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