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On the Role of Exponential Functions in Image Interpolation

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Abstract

A reproducing-kernel Hilbert space approach to image interpolation is introduced. In particular, the reproducing kernels of Sobolev spaces are shown to be exponential functions. These functions, in turn, give rise to alternative interpolation kernels that outperform presently available designs. Both theoretical and experimental results are presented.

I. INTRODUCTION

Interpolation is needed in image processing tasks such as rotation, translation, resizing and derivative evaluation. The underlying idea in current interpolation methods corresponds to regularity constraints imposed on the continuous-domain image where the pixel values provide its sampled version. For example, sinc-based interpolation kernels assume bandlimitedness (apodized sinc, discrete sinc [1]) while other methods assume piecewise polynomial models (nearest neighbor, linear, Schaum, Keys, Dodgson, B-spline, Meijering and OMOMS [2]). Every model converges to the original function as the sampling interval shortens and the corresponding approximation error can be characterized by [3]:

$$\|\mathbf{x}-\hat{\mathbf{x}}\|_{L_2} \propto C \cdot \Delta^L \cdot \left\|\mathbf{x}^{(L)}\right\|_{L_2}. \tag{1}$$

Here, $\mathbf{x}$ is the original continuous domain signal, $\mathbf{x}^{(L)}$ is its Lth derivative, $\hat{\mathbf{x}}$ is the interpolated signal and Δ is the sampling interval. In such a formulation, the parameters L and C are the approximation order and the proportional constant, respectively; they provide a means for comparing the various reconstruction (interpolation) methods. Both theoretical and experimental studies have shown that B-spline interpolation kernels perform better in this regard [2], [4], [5].

In spite of the power-law convergence of the approximation error to zero, current interpolation kernels do not necessarily provide the best possible continuous domain model for the whole set of finite-energy functions and a less restrictive regularization constraint other than the piecewise polynomial or bandlimited functions may be considered in this regard. It is suggested here to use the Sobolev space framework for this purpose instead.

Sobolev spaces consist of smooth functions and they serve as the underlying continuous-domain model in several image processing algorithms [6]–[9]. Nevertheless, it seems that the reproducing-kernel Hilbert space (RKHS) property of these spaces has not been investigated within the context of image interpolation; Sobolev functions are dense in L_2 and the suggested approach may further reduce the approximation error of (1). It will be shown that the reproducing kernels of certain Sobolev spaces correspond to exponentials that give rise to interpolating functions. These functions will be then shown to have attractive properties in terms of approximation error characterization while experimental results will be further shown to support these findings.

II. REPRODUCING KERNELS OF SOBOLEV SPACES

Let H_2^p be the Sobolev space of order p. This space consists of all one-dimensional finite-energy functions defined on the real line for which their first p derivatives are of finite energy as well [10]. The corresponding inner product is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle_{H_2^p} = \sum_{n=0}^{p} \lambda_n \cdot \left\langle \mathbf{x}^{(n)}, \mathbf{y}^{(n)} \right\rangle_{L_2}, \tag{2}$$