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Uncertainty Relations for Analog Signals

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Abstract

In the past several years there has been a surge of research investigating various aspects of sparse representations and compressed sensing. Most of this work has focused on the finite-dimensional setting in which the goal is to decompose a finite-length vector into a given finite dictionary. Underlying many of these results is the conceptual notion of an uncertainty principle: a signal cannot be sparsely represented in two different bases. Here, we extend these ideas and results to the analog, infinite-dimensional setting by considering signals that lie in a finitely-generated shift-invariant (SI) space. This class of signals is rich enough to include many interesting special cases such as multiband signals and splines. By adapting the notion of coherence defined for finite dictionaries to infinite SI representations, we develop an uncertainty principle similar in spirit to its finite counterpart. We demonstrate tightness of our bound by considering a bandlimited low-pass comb that achieves the uncertainty principle. Building upon these results and similar work in the finite setting, we show how to find a sparse decomposition in an overcomplete dictionary by solving a convex optimization problem. The distinguishing feature of our approach is the fact that even though the problem is defined over an infinite domain with infinitely many variables and constraints, under certain conditions on the dictionary spectrum our algorithm can find the sparsest representation by solving a finite dimensional problem.

I. Introduction

Uncertainty relations date back to the work of Weyl and Heisenberg who showed that a signal cannot be localized simultaneously in both time and frequency. This basic principle was then extended by Landau, Pollack, Slepian and later Donoho and Stark to the case in which the signals are not restricted to be concentrated on a single interval [1]–[4]. The uncertainty principle has deep philosophical interpretations. For example, in the context of quantum mechanics it implies that a particle's position and momentum cannot be simultaneously measured. In harmonic analysis it imposes limits on the time-frequency resolution [5].

Recently, there has been a surge of research into discrete uncertainty relations in more general finite-dimensional bases [6]–[8]. This work has been spurred in part by the relationship between sparse representations and the emerging field of compressed sensing [9], [10]. In particular, several works have shown that discrete uncertainty relations

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