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Combinatorial Ricci Curvature and Laplacians for Image Processing

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Abstract. A new Combinatorial Ricci curvature and Laplacian operators for grayscale images are introduced and tested on $2D$ synthetic, natural and medical images. Analogue formulae for voxels are also obtained. These notions are based upon more general concepts developed by R. Forman. Further applications, in particular a fitting Ricci flow, are discussed.

1 Introduction

Curvature analysis is of paramount importance in Image Processing, Computer Graphics, Computer Vision and related fields, for example in applications such as reconstruction, segmentation and recognition (e.g. [4], [10], [17], [18]). The conventional approach practiced in most studies implements curvature estimation of a polygonal (polyhedral) mesh, approximating the ideally smooth ($\mathcal{C}^2$) image under study. The curvature measures of the mesh converge in this case to the classical, differential, curvature measure of the investigated surface. In the case of surfaces, the most important curvature is the *intrinsic* Gaussian (or total) curvature.

A great deal of interest was generated recently by Perelman's important contribution to the Ricci flow [13] and by its application in the proof of Thurston's Geometrization Conjecture, and, implicitly of the Poincaré Conjecture ([12]), resulting in discrete versions of the Ricci flow and related flows ([3], [6], [8], [11]).

Ricci curvature measures the deviation of the manifold from being locally Euclidean in various tangential directions. More precisely, it appears in the second term of the formula for the $(n-1)$-volume $\Omega(\varepsilon)$ generated within a solid angle (i.e. it controls the growth of measured angles) – see Fig. 1. Moreover,

$$\mathbf{v}\cdot Ricci(\mathbf{v}) = \frac{n-1}{vol\left(\mathbb{S}^{n-2}\right)}\int_{\mathbf{w}\in T_p(M^n),\ \mathbf{w}\perp\mathbf{v}} K(<\mathbf{v},\mathbf{w}>)\,,$$

where $<\mathbf{v},\mathbf{w}>$ denote the plane spanned by $\mathbf{v}$ and $\mathbf{w}$, i.e. Ricci curvature represents an average of sectional curvatures. The analogy with mean curvature is further emphasized by the fact that Ricci curvature behaves as the Laplacian of the metric g ([2]). It is also important to note that in dimension $n = 2$, that