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The Cramér–Rao Bound for Sparse Estimation

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Abstract

The goal of this paper is to characterize the best achievable performance for the problem of estimating an unknown parameter having a sparse representation. Specifically, we consider the setting in which a sparsely representable deterministic parameter vector is to be estimated from measurements corrupted by Gaussian noise, and derive a lower bound on the mean-squared error (MSE) achievable in this setting. To this end, an appropriate definition of bias in the sparse setting is developed, and the constrained Cramér–Rao bound (CRB) is obtained. This bound is shown to equal the CRB of an estimator with knowledge of the support set, for almost all feasible parameter values. Consequently, in the unbiased case, our bound is identical to the MSE of the oracle estimator. Combined with the fact that the CRB is achieved at high signal-to-noise ratios by the maximum likelihood technique, our result provides a new interpretation for the common practice of using the oracle estimator as a gold standard against which practical approaches are compared.

EDICS Topics: SSP-PARE, SSP-PERF.

Index terms: Constrained estimation, Cramér–Rao bound, sparse estimation.

I. Introduction

The problem of estimating a sparse unknown parameter vector from noisy measurements has been analyzed intensively in the past few years [1]–[4], and has already given rise to numerous successful signal processing algorithms [5]–[9]. In this paper, we consider the setting in which noisy measurements of a deterministic vector $\boldsymbol{x}_0$ are available. It is assumed that $\boldsymbol{x}_0$ has a sparse representation $\boldsymbol{x}_0 = \boldsymbol{D}\boldsymbol{\alpha}_0$, where $\boldsymbol{D}$ is a given dictionary and most of the entries of $\boldsymbol{\alpha}_0$ equal zero. Thus, only a small number of "atoms," or columns of $\boldsymbol{D}$, are required to represent $\boldsymbol{x}_0$. The challenges confronting an estimation

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