Blind Compressed Sensing

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***Abstract*—The fundamental principle underlying compressed sensing is that a signal, which is sparse under some basis representation, can be recovered from a small number of linear measurements. However, prior knowledge of the sparsity basis is essential for the recovery process. This work introduces the concept of blind compressed sensing, which avoids the need to know the sparsity basis in both the sampling and the recovery process. We suggest three possible constraints on the sparsity basis that can be added to the problem in order to make its solution unique. For each constraint we prove conditions for uniqueness, and suggest a simple method to retrieve the solution. Under the uniqueness conditions, and as long as the signals are sparse enough, we demonstrate through simulations that without knowing the sparsity basis our methods can achieve results similar to those of standard compressed sensing, which relay on prior knowledge of the sparsity basis. This offers a general sampling and reconstruction system that fits all sparse signals, regardless of the sparsity basis, under the conditions and constraints presented in this work.**

I. Introduction

Sparse signal representations have gained popularity in recent years in many theoretical and applied areas [1]–[6]. Roughly speaking, the information content of a sparse signal occupies only a small portion of its ambient dimension. For example, a finite dimensional vector is sparse if it contains a small number of nonzero entries. It is sparse under a basis if its representation under a given basis transform is sparse. An analog signal is referred to as sparse if, for example, a large part of its bandwidth is not exploited [4], [7]. Other models for analog sparsity are discussed in detail in [5], [6], [8].

Compressed sensing (CS) [2], [3] focuses on the role of sparsity in reducing the number of measurements needed to represent a finite dimensional vector $x \in \mathbb{R}^m$. The vector x is measured by $b = Ax$, where A is a matrix of size $n \times m$, with $n \ll m$. In this formulation, determining x from the given measurements b is ill possed in general, since A has fewer rows than columns and is therefore non-invertible. However, if x is known to be sparse in a given basis P, then under additional mild conditions on A [9]–[11], the measurements b determine x uniquely as long as n is large enough. This concept was also recently expanded to include sub-Nyquist sampling of structured analog signals [4], [6], [12].

In principle, recovery from compressed measurements is NP-hard. Nonetheless, many suboptimal methods have been proposed to approximate its solution [1]–[3], [13]–[15]. These algorithms recover the true value of x when x is sufficiently sparse and the columns of A are incoherent [1], [9]–[11], [13]. However, all known recovery approaches use the prior knowledge of the sparsity basis P.

Dictionary learning (DL) [16]–[20] is another application of sparse representations. In DL, we are given a set of training signals, formally the columns of a matrix X. The goal is to find a dictionary P, such that the columns of X are sparsely represented as linear combinations of the columns of P. In [17], the authors study conditions under which the DL problem yields a unique solution for the given training set X.

In this work we introduce the concept of blind compressed sensing (BCS), in which the goal is to recover a high-dimensional vector x from a small number of measurements, where the only prior is that there exists some basis in which x is sparse. We refer to our setting as blind, since we do not require knowledge of the sparsity basis for the sampling or the reconstruction. This is in sharp contrast to CS, in which recovery necessitates this knowledge. Our BCS framework combines elements from both CS and DL. On the one hand, as in CS and in contrast to DL, we obtain only low dimensional measurements of the signal. On the other hand, we do not require prior knowledge of the sparsity basis which is similar to the DL problem. The goal of this work is to investigate the basic conditions under which blind recovery from compressed measurements is possible theoretically, and to propose concrete algorithms for this task.

Since the sparsity basis is unknown, the uncertainty about the signal x is larger in BCS than in CS. A straightforward solution would be to increase the number of measurements. However, we show that no rate increase can be used to determine x, unless the number of measurements is equal the dimension of x. Furthermore, we prove that even if we have multiple signals that share the same (unknown) sparsity basis, as in DL, BCS remains ill-posed. In order for the measurements to determine x uniquely we need an additional constraint on the problem. To prove the concept of BCS we begin by discussing two simple constraints on the sparsity basis, which enable blind recovery of a single vector x. We then turn to our main contribution, which is a BCS framework for structured sparsity bases. In this setting, we show that multiple vectors sharing the same sparsity pattern are needed to ensure recovery. For all of the above formulations we demonstrate via simulations that when the signals are sufficiently sparse the results of our BCS methods are similar to those obtained by standard CS algorithms which use the true, though unknown in practice, sparsity basis. When relying on the structural constraint we require in addition that the number of signals must be large enough. However, the simulations show that the number of signals needed is reasonable and much smaller than that used for DL [21]–[24].

The first constraint on the basis we consider relies on the fact that over the years there have been several bases that

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