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Rate–Distortion Function via Minimum Mean Square Error Estimation

Neri Merhav

***Abstract*—We derive a simple general parametric representation of the rate–distortion function of a memoryless source, where both the rate and the distortion are given by integrals whose integrands include the minimum mean square error (MMSE) of the distortion $\Delta = d(X, Y)$ based on the source symbol X, with respect to a certain joint distribution of these two random variables. At first glance, these relations may seem somewhat similar to the I–MMSE relations due to Guo, Shamai and Verdú, but they are, in fact, quite different. The new relations among rate, distortion, and MMSE are discussed from several aspects, and more importantly, it is demonstrated that they can sometimes be rather useful for obtaining non–trivial upper and lower bounds on the rate–distortion function, as well as for determining the exact asymptotic behavior for very low and for very large distortion. Analogous MMSE relations hold for channel capacity as well.**

***Index Terms*—Rate–distortion function, Legendre transform, estimation, minimum mean square error.**

I. Introduction

It has been well known for many years that the derivation of the rate–distortion function of a given source and distortion measure, does not lend itself to closed form expressions, even in the memoryless case, except for a few very simple examples [1],[2],[3],[5]. This has triggered the derivation of some upper and lower bounds, both for memoryless sources and for sources with memory.

One of the most important lower bounds on the rate–distortion function, which is applicable for difference distortion measures (i.e., distortion functions that depend on their two arguments only through the difference between them), is the Shannon lower bound in its different forms, e.g., the discrete Shannon lower bound, the continuous Shannon lower bound, and the vector Shannon lower bound. This family of bounds is especially useful for semi-norm–based distortion measures [5, Section 4.8]. The Wyner–Ziv lower bound [14] for a source with memory is a convenient bound, which is based on the rate–distortion function of the memoryless source formed from the product measure pertaining to the single–letter marginal distribution of the original source and it may be combined elegantly with the Shannon lower bound. The autoregressive lower bound asserts that the rate–distortion function of an autoregressive source is lower bounded by the rate–distortion function of its innovation process, which is again, a memoryless source.

Upper bounds are conceptually easier to derive, as they may result from the performance analysis of a concrete coding scheme, or from random coding with respect to (w.r.t.) an arbitrary random coding distribution, etc. One well known example is the Gaussian upper bound, which upper bounds the rate–distortion function of an arbitrary memoryless (zero–mean) source w.r.t. the squared error distortion measure by the rate–distortion function of the Gaussian source with the same second moment. If the original source has memory, then the same principle generalizes with the corresponding Gaussian source having the same autocorrelation function as the original source [1, Section 4.6].

In this paper, we focus on a simple general parametric representation of the rate–distortion function which seems to set the stage for the derivation of a rather wide family of both upper bounds and lower bounds on the rate–distortion function. In this parametric representation, both the rate and the distortion are given by integrals whose integrands include the minimum mean square error (MMSE) of the distortion based on the source symbol, with respect to a certain joint distribution of these two random variables. More concretely, given a memoryless source designated by a random variable (RV) X, governed by a probability function[1] $p(x)$, a reproduction variable Y, governed by a probability function $q(y)$, and a distortion measure $d(x, y)$, the rate and the distortion can be represented parametrically via a real parameter $s \in [0, \infty)$ as follows:

$$\begin{aligned} D_s &= D_0 - \int_0^s \mathrm{d}\hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X) \\ &= D_\infty + \int_s^\infty \mathrm{d}\hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X) \qquad (1) \end{aligned}$$

and

$$\begin{aligned} R_q(D_s) &= \int_0^s \mathrm{d}\hat{s} \cdot \hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X) \\ &= R_q(D_\infty) - \int_s^\infty \mathrm{d}\hat{s} \cdot \hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X), \qquad (2) \end{aligned}$$

where D_s is the distortion pertaining to parameter value s, $R_q(D_s)$ is the rate–distortion function w.r.t. reproduction distribution q, computed at D_s, $\Delta = d(X, Y)$, and $\mathrm{mmse}_s(\Delta|X)$ is the MMSE of estimating Δ based on X, where the joint probability function of (X, Δ) is induced by the following joint probability function of (X, Y):

$$p_s(x, y) = p(x) \cdot w_s(y|x) = p(x) \cdot \frac{q(y)e^{-sd(x,y)}}{Z_x(s)} \qquad (3)$$

N. Merhav is with the Department of Electrical Engineering, Technion – Israel Institute of Technology, Haifa, 32000, Israel. E-mail: merhav@ee.technion.ac.il.

[1] Here, and throughout the sequel, the term "probability function" refers to a probability mass function in the discrete case and to a probability density function in the continuous case.