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Exact Random Coding Exponents for Erasure Decoding

Anelia Somekh-Baruch and Neri Merhav

***Abstract*—Random coding of channel decoding with an erasure option is studied. By analyzing the large deviations behavior of the code ensemble, we obtain *exact* single-letter formulas for the error exponents in lieu of Forney's lower bounds. The analysis technique we use is based on an enhancement and specialization of tools for assessing the moments of certain distance enumerators. We specialize our results to the setup of the binary symmetric channel case with uniform random coding distribution and derive an *explicit* expression for the error exponent which, unlike Forney's bounds, does not involve optimization over two parameters. We also establish the fact that for this setup, the difference between the exact error exponent corresponding to the probability of undetected decoding error and the exponent corresponding to the erasure event is equal to the threshold parameter. Numerical calculations indicate that for this setup, as well as for a Z-channel, Forney's bound coincides with the exact random coding exponent.**

***Index Terms*—random coding, erasure, list, error exponent, distance enumerator**

I. Introduction

In [1], Forney derived lower bounds on the random coding exponents associated with decoding rules that allow for erasure and list decoding (see also later related studies[4] -[9]). The channel model he considered was a single user discrete memoryless channel (DMC), where a codebook of block length n is randomly drawn with i.i.d. codewords having i.i.d. symbols. When erasure is concerned, the decoder may fully decode the message, or, decide to declare that an erasure has occurred. An optimum tradeoff between the probability of erasure and the probability of undetected decoding error was investigated. This tradeoff is optimally controlled by a threshold parameter T of the function e^{nT} to which one compares the ratio between the likelihood of each hypothesized message and the sum of likelihoods of all other messages. If this ratio exceeds e^{nT} for some message, a decision is made in favor of that message, otherwise, an erasure is declared. Forney's main result in [1] is a single-letter lower bound, $E_1(R,T)$, to the exponent of the probability of the event $\mathcal{E}_1$ of not making the correct decision, namely, either erasing or making the wrong decision, and a single-letter lower bound, $E_2(R,T)$, to the exponent of the probability of the event $\mathcal{E}_2$ of undetected error.

In [2, Th. 5.11], Csiszár and Körner derived universally achievable error exponents for a decoder with an erasure option for DMCs. These error exponents were obtained by analyzing a decoder which generalizes the MMI decoder for constant composition (CC) codes. Unlike Forneys decoder, no optimality claims were made for this decoder, but, in [3, Sec. 4.4.3] Telatar stated that these bounds are essentially the same as those in [1].

Inspired by a statistical–mechanical point of view on random code ensembles (offered in [10] and further elaborated on in [12]), Merhav [11] applied a different technique to derive a lower bound to the exponents of the probabilities of $\mathcal{E}_1, \mathcal{E}_2$ by assessing the moments of certain distance enumerators. This approach, which also proved fruitful in several other applications (see [13], [14], [15]), resulted in a bound that is at least as tight as Forney's bound. It is shown in [11] that under certain symmetry conditions (that often hold) on the random coding distribution and the channel, the resulting bound is also simpler in the sense that there is only one parameter to optimize rather than two. Moreover, this optimization can be carried out in closed form at least in some special cases like the binary symmetric channel (BSC). It is not clear though, whether the bounds of [11] are strictly tighter than those of Forney.

In this paper, we use the approach of distance enumerators to tackle again the problem of random of channel decoding with an erasure option. Unlike the approach of [1] and [11], our starting point is not a Gallager-type bound [16] on the probability of error, but rather the *exact* expression. This approach results in single-letter expressions for the exact exponential behavior of the probabilities of the events $\mathcal{E}_1, \mathcal{E}_2$ when random coding is used. So far, we have not been able to determine analytically whether our results coincide with Forney's bounds, i.e., we cannot say whether Forney's bounds are tight or not, but the tightness of the our expressions is guaranteed. While our analysis pertains to the ensemble of codes where each symbol of each codeword is drawn i.i.d. (mainly in order to enable a fair comparison to [1]), our technique can easily be used for other ensembles, like the ensemble where each codeword is drawn independently according to the uniform distribution within a given type class. For the case of the BSC with uniform random coding distribution we have conducted several numerical calculations, which indicate that Forney's bound coincides with the exact random coding exponent.

The outline of this paper is as follows. In Section II, we present notation conventions and in Section III, we give some necessary background in more detail. Section IV is devoted to a description of the main results. In the following sections, V-IX, we provide detailed derivations of the main results: in Section V, we derive the exact expression for the error exponent corresponding to the probability of $\mathcal{E}_1$, and in Sections VI and VII, we study two special cases of

A. Somekh-Baruch is with the School of Engineering, Bar-Ilan University, Ramat-Gan, Israel 52900. E-mail: anelia.somekhbaruch@gmail.com

N. Merhav is with the Department of Electrical Engineering, Technion – Israel Institute of Technology, Haifa, 32000, Israel. E-mail: merhav@ee.technion.ac.il.