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Tightened Exponential Bounds for Discrete Time, Conditionally Symmetric Martingales with Bounded Jumps

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Abstract

This letter derives some new exponential bounds for discrete time, real valued, conditionally symmetric martingales with bounded jumps. The new bounds are extended to conditionally symmetric sub/ supermartingales, and are compared to some existing bounds.

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I. Introduction and Main Results

Classes of exponential bounds for discrete-time real-valued martingales were extensively studied in the literature (see, e.g., [1], [2], [4], [5], [6]–[9], [12], and [15]–[18]). This letter further assumes conditional symmetry of these martingales, as it is defined in the following:

Definition 1: Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$, where $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$, be a discrete-time and real-valued martingale, and let $\xi_k \triangleq X_k - X_{k-1}$ for every $k \in \mathbb{N}$ designate the jumps of the martingale. Then $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ is called a *conditionally symmetric martingale* if, conditioned on $\mathcal{F}_{k-1}$, the random variable ξ_k is symmetrically distributed around zero.

Our goal in this letter is to demonstrate how the assumption of the conditional symmetry improves existing exponential inequalities for discrete-time real-valued martingales with bounded increments. Earlier results, serving as motivation, appear in [7, Section 4] and [15, Section 6]. The new exponential bounds are also extended to conditionally symmetric sub or supermartingales, where the construction of these objects is exemplified later in this section. The relation of some of the exponential bounds derived in this work with some existing bounds is discussed later in this letter. Additional results addressing weak-type inequalities, maximal inequalities and ratio inequalities for conditionally symmetric martingales were derived in [13], [14] and [19].

A. Main Results

Our main results for conditionally symmetric martingales with bounded jumps are introduced in Theorems 1, 3 and 4. Theorems 2 and 5 are existing bounds, for general martingales without the conditional symmetry assumption, that are introduced in connection to the new theorems. Corollaries 1 and 2 provide an extension of the new results to conditionally symmetric sub/ supermartingales with bounded jumps. Our first result is the following theorem:

Theorem 1: Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a discrete-time real-valued and conditionally symmetric martingale. Assume that, for some fixed numbers $d, \sigma > 0$, the following two requirements are satisfied a.s.

$$|X_k - X_{k-1}| \leq d, \qquad \text{Var}(X_k | \mathcal{F}_{k-1}) = \mathbb{E}\big[(X_k - X_{k-1})^2 \,|\, \mathcal{F}_{k-1}\big] \leq \sigma^2 \tag{1}$$

for every $k \in \mathbb{N}$. Then, for every $\alpha \geq 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |X_k - X_0| \geq \alpha n\right) \leq 2 \exp\big(-nE(\gamma, \delta)\big) \tag{2}$$

where

$$\gamma \triangleq \frac{\sigma^2}{d^2}, \quad \delta \triangleq \frac{\alpha}{d} \tag{3}$$